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Error Estimates for Spatially Discrete Approximations of Semilinear Parabolic Equations with Nonsmooth Initial Data

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Dedicated to Professor J. A. Nitsche on the occasion of his sixtieth birthday, September 2, 1986.

Abstract. We consider time-continuous spatially discrete approximations by the Galerkin finite element method of initial-boundary value problems for semilinear parabolic equations with nonsmooth or incompatible initial data. We find that the numerical solution enjoys a gain in accuracy at positive time of essentially two orders relative to the initial regularity, as a result of the smoothing property of the parabolic evolution operator. For higher-order elements the restriction to two orders is in contrast to known optimal order results in the linear case.

1. Introduction. We consider continuous in time spatially discrete approximate solutions by Galerkin finite element methods of the semilinear initial-boundary value problem

$$\begin{aligned} &u_t - \Delta u = f(u) &&\text{in } \Omega \times I,\ I = (0, t^*], \\ &u = 0 &&\text{on } \partial\Omega \times I, \\ &u(0) = v &&\text{in } \Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in R^d, $d = 1$, 2 or 3, with a sufficiently smooth boundary $\partial\Omega$, and f is a smooth function on R for which we assume provisionally that

$$|f(y)|, |f'(y)| \leqslant B \quad \text{for } y \in R. \tag{1.2}$$

Such an assumption is normally reasonable only if the solution of (1.1) is known a priori to be bounded; see the discussion at the beginning of Section 3.

For spatial discretization of (1.1) let $S_h \subset H_0^1 = H_0^1(\Omega)$ be a family of finite-dimensional spaces parametrized by a small positive parameter h, and let the semidiscrete solution $u_h\colon \bar{I} \to S_h$ be defined by

$$\begin{aligned} &(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f(u_h), \chi) \quad \text{for } \chi \in S_h, \\ &u_h(0) = v_h \in S_h, \end{aligned} \tag{1.3}$$

where $(\cdot, \cdot)$ denotes the standard inner product in $L_2 = L_2(\Omega)$.

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In order to discuss the error in (1.3), we assume that S_h is such that the corresponding linear elliptic problem admits an $O(h^r)$ error estimate in L_2. More precisely, we assume that the elliptic projection P_1, i.e., the orthogonal projection onto S_h with respect to the Dirichlet inner product $(\nabla v, \nabla w)$, satisfies for some $r \geqslant 2$ and some constant M,

$$\|P_1 v - v\| \leqslant Mh^r \|v\|_{H^r} \quad \text{for } v \in H_0^1 \cap H^r, \tag{1.4}$$

where $\|\cdot\|$ is the usual norm in L_2 and $H^r = H^r(\Omega)$ is the standard Sobolev space of order r. It is well known that if u is sufficiently smooth on the closed interval $\bar{I}$, and if the discrete initial data v_h are suitably chosen, then (cf. Wheeler [26])

$$\|u_h(t) - u(t)\| \leqslant C(u, B, M) h^r \quad \text{for } t \in \bar{I}. \tag{1.5}$$

To guarantee that u is smooth enough for this result, both smoothness of v and compatibility conditions between v and the differential equation at $\partial\Omega$ for $t = 0$ are necessary. For instance, in the linear homogeneous case ($f \equiv 0$ in (1.1)) it was shown in Bramble, Schatz, Thomée and Wahlbin [4] that, with $w_h(t)$ and $w(t)$ denoting the solutions in this case, we have

$$\|w_h(t) - w(t)\| \leqslant Ch^r \|v\|_{H^r} \quad \text{for } v \in D\big((-\Delta)^{r/2}\big),\ t \in \bar{I}, \tag{1.6}$$

which thus requires $\Delta^j v|_{\partial\Omega} = 0$ for $j < r/2$. (An analogous result for the semilinear problem is contained in our Theorem 3.1 below.) The conditions on v thus required for optimal order error estimates in $L_\infty(I, L_2)$ are not always satisfied in practice and it is therefore of interest to analyze the error for nonsmooth or incompatible data. Note that the solution of (1.1) will always be smooth for positive time. For the linear homogeneous equation this may be expressed by saying that the solution operator $E(t)$ of the initial value problem is an analytic semigroup and that

$$\|E(t)v\|_{\dot{H}^\beta} \leqslant Ct^{-(\beta-\alpha)/2} \|v\|_{\dot{H}^\alpha}, \quad \text{where } \|v\|_{\dot{H}^\alpha} = \big\|(-\Delta)^{\alpha/2} v\big\|. \tag{1.7}$$

For the linear homogeneous equation the nonsmooth data situation has been investigated in Blair [2], Helfrich [9], Fujita and Mizutani [6], Bramble, Schatz, Thomée and Wahlbin [4] and later papers (cf. Thomée [23]). In this case it may be shown by use of the smoothing property (1.7) that if $E_h(t)$ is the solution operator for the initial value problem for the linear homogeneous equation, and if v_h is chosen as $P_0 v$, the L_2-projection of v onto S_h, then

$$\begin{aligned} \|E_h(t)P_0 v - E(t)v\| &= \|w_h(t) - w(t)\| \\ &\leqslant C(M) h^{\alpha+\sigma} t^{-\sigma/2} \|v\|_{\dot{H}^\alpha} \quad \text{for } 0 \leqslant \alpha \leqslant \alpha + \sigma \leqslant r. \end{aligned} \tag{1.8}$$

In particular, optimal order convergence is attained for t positive, even if v is only in L_2. A similar result showing $O(h^r)$ convergence for positive time without initial regularity holds also for the linear inhomogeneous problem; see (1.14) below.

The purpose of this paper is to investigate to what extent results such as the above carry over to the semilinear situation. We begin by stating and proving the following result.

THEOREM 1.1. *Assume that* (1.2) *holds and let u be a solution of* (1.1) *with* $\|v\| \leqslant K$. *Assume further that* (1.4) (*and thus* (1.8)) *is satisfied and let u_h be the solution of* (1.3) *with $v_h = P_0 v$. Then there exists a constant $C = C(B, K, M, t^*)$ such that*

$$\|u_h(t) - u(t)\| \leqslant Ch^2\big(t^{-1} + \big|\log(h^2/t)\big|\big) \quad \textit{for } t \in I. \tag{1.9}$$

Proof. Simple energy arguments show that $u(t)$ and $u_h(t)$ are bounded in L_2 so that (1.9) trivially holds for $t \leqslant h^2$. With our above notation we have, by Duhamel's principle, for the solutions of (1.1) and (1.3) that

$$u(t) = E(t)v + \int_0^t E(t-s)f(u(s))\,ds$$

and

$$u_h(t) = E_h(t)v_h + \int_0^t E_h(t-s)P_0 f(u_h(s))\,ds.$$

Hence, with $F_h(t) = E_h(t)P_0 - E(t)$ the error operator for the linear homogeneous equation, the error $e = u_h - u$ of the semilinear problem satisfies

$$\begin{aligned} e(t) = F_h(t)v &+ \int_0^t E_h(t-s)P_0[f(u_h(s)) - f(u(s))]\,ds \\ &+ \int_0^t F_h(t-s)f(u(s))\,ds. \end{aligned} \tag{1.10}$$

Using the cases $\sigma = 2$ and 0, $\alpha = 0$ of (1.8), we thus find for $t \geqslant h^2$ that

$$\begin{aligned} \|e(t)\| &\leqslant Ch^2t^{-1} + C\left(\int_0^{h^2} + \int_{h^2}^t\right)\|e(s)\|\,ds \\ &\quad + \left(\int_0^{t-h^2} + \int_{t-h^2}^t\right)\|F_h(t-s)f(u(s))\|\,ds \\ &\leqslant Ch^2t^{-1} + Ch^2 + C\int_{h^2}^t \|e(s)\|\,ds + Ch^2\int_0^{t-h^2} \frac{ds}{t-s} + Ch^2 \\ &\leqslant Ch^2t^{-1} + Ch^2\log(t/h^2) + C\int_{h^2}^t \|e(s)\|\,ds. \end{aligned} \tag{1.11}$$

Letting $\varphi(t) = \int_{h^2}^t \|e(s)\|\,ds$, we conclude that

$$\begin{aligned} &\varphi'(t) - C\varphi(t) \leqslant Ch^2t^{-1} + Ch^2\log(t/h^2) \quad \text{for } h^2 \leqslant t \leqslant t^*, \\ &\varphi(h^2) = 0, \end{aligned}$$

whence

$$\begin{aligned} \varphi(t) &\leqslant C\int_{h^2}^t e^{C(t-s)}\left(h^2s^{-1} + h^2\log(s/h^2)\right)ds \\ &\leqslant Ch^2\log(t/h^2). \end{aligned}$$

Inserting this into (1.11) completes the proof of the theorem.

The above result shows that for $r = 2$ the error in the semilinear case is essentially of the same order as for the linear homogeneous equation. For $r > 2$, however, the result of Theorem 1.1 is weaker than the case $\alpha = 0$ of (1.8). The reason why the above argument fails to yield order of convergence higher than second is the lack of integrability of the right-hand side of (1.8) for $\sigma > 2$. In Section 6 we shall see that, in fact, Theorem 1.1 is essentially sharp in the sense that an estimate of the form

$$\|u_h(t_0) - u(t_0)\| \leqslant C(B, M, t_0)h^\sigma, \qquad |u(x,t)| \leqslant B, \tag{1.12}$$

cannot hold for any $\sigma > 2$ and $t_0 > 0$, regardless of the value of r. Note that the requirement that u is bounded is more stringent than boundedness in L_2 of initial data. We next give a preliminary example, based on Fourier series, to indicate this and begin by remarking that our discussion of (1.1) applies equally well to systems of equations of this form.

Consider the following system with periodic boundary conditions and $u = (u_1, u_2)$,

$$(1.13)\qquad \begin{aligned} &u_{1,t} = u_{1,xx} + f(u_2), \qquad f(y) = 4y^2 \quad \text{for } |y| \leqslant 1,\\ &u_{2,t} = u_{2,xx} \quad \text{on } [-\pi, \pi] \times (0, \infty),\\ &u_1(\cdot, 0) = 0, \qquad u_2(\cdot, 0) = v_2. \end{aligned}$$

Taking $v_2(x) = \cos(nx)$, we have

$$\begin{aligned} u_2(x,t) &= \exp(-n^2 t)\cos(nx),\\ u_1(x,t) &= \frac{1 - \exp(-2n^2 t)}{n^2}\left[1 + \exp(-2n^2 t)\cos(2nx)\right], \end{aligned}$$

so that by nonlinear interaction ("aliasing") an initial high Fourier mode has resulted in a low, indeed constant, mode.

To approximate this problem, let $h = 1/n$ with n a positive integer, and set $S_h = \text{span}\{1, \cos x, \sin x, \ldots, \cos(n-1)x, \sin(n-1)x\}$. Since $P_0 v \equiv 0$, the Galerkin solution u_h vanishes identically and hence, at any positive time t_0,

$$\|u_h(t_0) - u(t_0)\| = \|u(t_0)\| \approx \frac{\sqrt{2\pi}}{n^2} = \sqrt{2\pi}\, h^2 \quad \text{for large } n.$$

Since u is bounded independently of n, this contradicts (1.12) for $\sigma > 2$.

We next elucidate to what extent the restriction $\sigma \leqslant 2$ in (1.12) is a truly nonlinear phenomenon. We thank Professor Jim Douglas, Jr. for interesting discussions which helped clarify this point. In the linear case the guiding principle is that the solution has to be sufficiently smooth near the time of interest in order to guarantee optimal order error there, whereas what roughness and consequent bad approximation went on before that time is dampened out and does not particularly matter. To be precise, we quote the following result from Thomée [22], [23, Chapter 3, Theorem 5] for the linear problem $u_t = \Delta u + g(x,t)$, $u = 0$ on $\partial\Omega$, $u(0) = v$: For any $t_0 > 0$, $\delta > 0$,

$$(1.14)\qquad \begin{aligned} &\|u_h(t_0) - u(t_0)\|\\ &\quad \leqslant h^r C(t_0, \delta)\left\{\|v\| + \int_0^{t_0} \|g(t)\|\, dt + \int_{t_0-\delta}^{t_0} \left[\|u(t)\|_{H^r} + \|u_t(t)\|_{H^r}\right] dt\right\}. \end{aligned}$$

As a linear analogue of the counterexample (1.13) above one may naturally take $g(x,t) = 0$ (since $f(0) = 0$) or $g(x,t) = 4\exp(-2t/h^2)\cos^2(x/h)$. In both cases, by (1.14), the finite element approximation will be of optimal order for positive time. Thus the limit to second-order accuracy in (1.12) appears to be a genuinely nonlinear effect.

We remark that a practical limitation in our result is that the L_2-projection and the term $(f(u_h), \chi)$ in (1.3) are assumed to be evaluated exactly; cf. Wahlbin [25].

Results similar to the above, concerning the discretization in time of equations such as (1.1) and (1.3), will be presented in Crouzeix and Thomée [5].

Investigations related to the present work can be found in Hale, Lin and Raugel [7] where a case of our Theorem 4.1 is derived, and in Heywood and Rannacher [10] where the Navier-Stokes equations with incompatible initial data are considered. The latter authors obtain estimates for positive time of order h^m, $m = 3, 4, 5$, and remark that it has to be left as an open question whether the limitation of the smoothing results to approximation order $m \leqslant 5$ is inherent to the problem or to the method of proof. The sharpness of our Theorem 1.1 is certainly suggestive in this regard.

We now give an outline of the rest of the paper. In Section 2 we study the existence and regularity of the solution of (1.1) with particular emphasis on the dependence of the regularity of the solution at positive time upon the smoothness and compatibility of the initial data. The main result, Theorem 2.2, is an analogue of the estimate (1.7) for the linear homogeneous problem and is proved in the Appendix.

In Section 3 we consider the error in the semidiscrete solution in the case that the initial data have some amount of smoothness and compatibility, but not enough for (1.5) to hold. Generalizing Theorem 1.1, we show that the convergence rate in L_2 at positive time is almost two powers of h higher than the order of regularity of the initial data.

In Section 4 we prove that the gain in the L_2 convergence rate of almost $O(h^2)$ relative to the regularity of the initial data carries over to the gradient of the error up to the optimal order. In Section 5 the analogous results are derived for the maximum-norm of the error and its gradient.

In Section 6, finally, we present a scalar counterexample in a standard family of piecewise polynomial approximating spaces to show that Theorem 1.1 cannot be substantially improved.

2. Existence and Regularity of the Exact Solution. As in the introduction, let $\Omega \subset R^d$, $d \leqslant 3$, be a bounded domain with smooth boundary and $I = (0, t^*]$ a finite time interval; consider the semilinear parabolic problem

$$\begin{aligned} &u_t - \Delta u = f(u) && \text{in } \Omega \times I, \\ &u = 0 && \text{on } \partial\Omega \times I, \\ &u(\cdot, 0) = v && \text{in } \Omega. \end{aligned} \tag{2.1}$$

We assume now that f is a smooth, possibly unbounded function and that v is bounded on Ω. Our purpose here is to present a regularity estimate for the solution of (2.1) that we shall need in the sequel. The outline of this section follows Chapter 1 of Larsson [12], where more details can be found.

Our argument will be based on well-known properties of the corresponding homogeneous linear problem

$$\begin{aligned} &w_t - \Delta w = 0 && \text{in } \Omega \times (0, \infty), \\ &w = 0 && \text{on } \partial\Omega \times (0, \infty), \\ &w(\cdot, 0) = v && \text{in } \Omega. \end{aligned} \tag{2.2}$$

Denoting as before the solution operator of this problem by $E(t)$, we recall the smoothing property

$$\|E(t)v\|_{\dot H^\beta} \leqslant Ct^{-(\beta-\alpha)/2}\|v\|_{\dot H^\alpha}, \qquad 0 \leqslant \alpha \leqslant \beta,\ \|v\|_{\dot H^\alpha} = \left\|(-\Delta)^{\alpha/2}v\right\|, \tag{2.3}$$

where the norms and the corresponding Hilbert spaces are defined by means of the fractional powers of $-\Delta$, considered as an unbounded operator on L_2. It is well known that, except for small α, the spaces $\dot{H}^\alpha$ involve boundary conditions. We have, for example, $\dot{H}^1 = H_0^1 = \{v \in H^1: v = 0 \text{ on } \partial\Omega\}$ and $\dot{H}^2 = H^2 \cap H_0^1$.

We shall now consider the existence of a solution of the nonlinear problem (2.1). Our aim is to allow nonsmooth initial values, i.e., we do not want to make restrictive assumptions on regularity or boundary compatibility. (For existence of a solution in the case of smooth data, cf. Amann [1].) It is clear that any classical solution of (2.1) also satisfies the integral equation

$$u(t) = E(t)v + \int_0^t E(t-s)f(u(s))\,ds, \qquad t \in I. \tag{2.4}$$

With the above purpose in mind, we shall assume that $v \in L_\infty$, and we shall solve (2.4) in $L_\infty(I, L_\infty)$. It does not seem possible to solve (2.4) with arbitrary initial values in L_2 without restricting the nonlinearity f. It should also be remarked that our solution concept could not be replaced by $C(\bar{I}, L_\infty)$, since the semigroup $\{E(t)\}_{t \geqslant 0}$ is not strongly continuous on L_∞, as is easily seen by taking for example a discontinuous initial function. However, $\{E(t)\}_{t \geqslant 0}$ is bounded on $L_\infty(\Omega)$,

$$\|E(t)v\|_{L_\infty} \leqslant \|v\|_{L_\infty}, \qquad t \geqslant 0. \tag{2.5}$$

This follows from the maximum principle for weak solutions of linear parabolic equations; see, e.g., Ladyženskaja, Solonnikov and Ural'ceva [11, Theorem III.7.2].

Using (2.5) and standard contraction mapping arguments (cf., e.g., Smoller [20]), we obtain for each $v \in L_\infty$ a unique local solution u of (2.4). Moreover, if we have an a priori bound in the L_∞-norm on the whole interval I, we obtain a global solution (cf. again [20]). We state this result as the first part of the following theorem.

THEOREM 2.1. *Let $v \in L_\infty$ and assume that a priori $\|u(t)\|_{L_\infty} \leqslant B$, $t \in I$, for any possible solution u of* (2.4). *Then* (2.4) *has a unique solution $u \in L_\infty(I, L_\infty)$. Moreover, $u \in C(\bar{I}, L_2) \cap C^\infty(\bar{\Omega} \times I)$ and satisfies the differential equation and boundary condition in* (2.1).

The second part of the theorem states that u is a classical solution of (2.1) for positive time. This can be proved by a standard bootstrapping technique using the Schauder estimates for linear parabolic equations (see, e.g., [11, Theorem IV.5.2]) and the smoothness of f and $\partial\Omega$. The initial value, however, may in general be attained only in the L_2 sense. With u a priori bounded, we may assume that $f(u)$ and any number of its derivatives are bounded on R. We shall use B to denote also such bounds below.

Our main object in this section is to study the regularity of the solution u as measured by the fractional-order Sobolev spaces $H^\alpha = H^\alpha(\Omega)$, defined by interpolation between L_2 and H^m, m integer. Our aim is to obtain an estimate similar to the smoothing property (2.3) for the homogeneous linear problem (2.2). This is achieved by first estimating the time derivatives $u^{(m)} = D_t^m u$, where $D_t = \partial/\partial t$, in the $\dot{H}^\alpha$ norms for $0 \leqslant \alpha \leqslant 2$ and then estimating spatial derivatives by using the equation satisfied by $u^{(m)}$ and a regularity estimate for linear elliptic problems. Thus, we shall need to differentiate Eq. (2.1) with respect to time, which is allowed in view of

Theorem 2.1. Hence, for $m \geqslant 0$ we have

$$u_t^{(m)} - \Delta u^{(m)} = D_t^m f(u), \qquad t \in I, \tag{2.6}$$

so that, for $0 < \tau \leqslant t \leqslant t^*$,

$$u^{(m)}(t) = E(t-\tau)u^{(m)}(\tau) + \int_\tau^t E(t-s) D_s^m f(u(s))\, ds, \tag{2.7}$$

where $u^{(m)}(\tau)$ may have no limit as $\tau \to 0$. For $m \geqslant 1$ we have here

$$D_t^m f(u) = \sum_{j=1}^m \sum_l c_l f^{(j)}(u) u^{(l_1)} \cdots u^{(l_j)}, \tag{2.8}$$

where the inner sum is to be taken over the set of all multi-indices $l = (l_1, \ldots, l_j)$ with $l_i \geqslant 1$, $1 \leqslant i \leqslant j$, and $\Sigma_1^j l_i = m$, and where the c_l are combinatorial coefficients.

In order for u to have a certain regularity on $\bar{I}$ it is necessary not only that the initial value v be regular enough, but also that v be compatible with the nonlinearity f. In order to describe this, for given $v \in L_\infty$, we define v_j recursively by the following procedure: First put $v_0 = v$. Suppose, then, that v_j has been defined for some $j \geqslant 0$. If $v_j \notin \dot{H}^2$, then stop. Otherwise, i.e., if $v_s \in \dot{H}^2$ for $s \leqslant j$, put

$$v_{j+1} = \Delta v_j + \sum_{i=1}^j \sum_l c_l f^{(i)}(v) v_{l_1} \cdots v_{l_i}.$$

Since

$$\begin{aligned}\left\| f^{(i)}(v) v_{l_1} \cdots v_{l_i} \right\| &\leqslant \| f^{(i)}(v) \|_{L_\infty} \| v_{l_1} \|_{L_\infty} \cdots \| v_{l_i} \|_{L_\infty} \\ &\leqslant C \| v_{l_1} \|_{\dot{H}^2} \cdots \| v_{l_i} \|_{\dot{H}^2},\end{aligned}$$

by Sobolev's inequality ($d \leqslant 3$), this defines v_{j+1} as an element of L_2.

Let $\alpha \geqslant 0$ be a real number and let k be the integer part of $\alpha/2$, so that $0 \leqslant \alpha - 2k < 2$. We define the set $\mathscr{F}_\alpha$ by

$$\mathscr{F}_\alpha = \left\{ v \in L_\infty \colon v_j \in \dot{H}^2 \text{ if } 0 \leqslant j < k \text{ and } v_k \in \dot{H}^{\alpha - 2k} \right\}.$$

This definition is motivated by the fact that if $u \in C^k(\bar{I}, L_2)$ then $u^{(j)}(0) = v_j$ for $0 \leqslant j \leqslant k$ and $v_j \in \dot{H}^2$ for $0 \leqslant j < k$. We shall think of $\mathscr{F}_\alpha$ as the set of all bounded initial values that are regular and compatible of order α. We note that $v \in \mathscr{F}_\alpha$ implies that $v_j = 0$ on $\partial\Omega$ for $0 \leqslant j < k$.

For $v \in \mathscr{F}_\alpha$ we define

$$F_\alpha(v) = \max\left\{ \|v\|_{L_\infty}, \|v_j\|_{\dot{H}^2}, 0 \leqslant j < k, \|v_k\|_{\dot{H}^{\alpha-2k}} \right\}.$$

This makes F_α a nonlinear functional that measures initial regularity and compatibility in a convenient way. Note that for $\alpha < 2$, $F_\alpha(v) = \max\{\|v\|_{L_\infty}, \|v\|_{\dot{H}^\alpha}\}$. We now state the main result in this section.

THEOREM 2.2. *Let $u \in L_\infty(I, L_\infty)$ be a solution of* (2.4) *with $\|u(t)\|_{L_\infty} \leqslant B$, $t \in I$. Let $0 \leqslant \beta - \alpha \leqslant 5$ and $2j \leqslant \beta$ and assume that $v \in \mathscr{F}_\alpha$ with $F_\alpha(v) \leqslant K$. Then there is a constant $C = C(B, K, t^*)$ such that*

$$\|u^{(j)}(t)\|_{H^{\beta-2j}} \leqslant C t^{-(\beta-\alpha)/2}, \qquad t \in I. \tag{2.9}$$

The case $\beta - \alpha \leqslant 4$ of Theorem 2.2 was proved in Larsson [12]. Because of the nonlinear character of the problem it becomes more difficult to prove the regularity estimate (2.9) as the difference between β and α increases. Since $\beta - \alpha \leqslant 5$ is sufficient for our applications, we shall content ourselves with the present version of the theorem, although it might be possible to squeeze a little more out of our method of proof. The proof of Theorem 2.2 is elementary but rather technical and is relegated to the Appendix.

3. A Basic Error Estimate. In this section we shall prove our basic error estimate for the semidiscrete Galerkin solution of (2.1), i.e., the solution u_h: $\bar{I} = [0, t^*] \to S_h$ of

$$(3.1)\qquad \begin{aligned} &(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f(u_h), \chi) \quad \text{for } \chi \in S_h,\ t \in I, \\ &u_h(0) = v_h = P_0 v. \end{aligned}$$

With the situation of Theorem 2.1 in mind, i.e., the case that the solution sought is a priori uniformly bounded on $\bar{I}$, we shall assume that f, f' and sometimes also f'' are bounded on R. If this is not the case originally, $f(u)$ may be modified for $|u| > B$ so that this becomes true, where B is an upper bound for the exact solution which we shall also use to denote the bounds on f. Such a modification will change the Galerkin formulation (3.1), and thereby also possibly the semidiscrete solution, as the latter is in general not known to be bounded, uniformly in h, but the exact solution remains the same. Some situations where this modification is known not to be necessary are given in [24].

We assume in this section that $S_h \subset H_0^1(\Omega)$ and that, with P_1: $H_0^1(\Omega) \to S_h$ the elliptic projection defined by $(\nabla(P_1 v - v), \nabla\chi) = 0$ for $\chi \in S_h$, there is an $r \geqslant 2$ such that

$$(3.2)\qquad \|P_1 v - v\| \leqslant M h^s \|v\|_{H^s} \quad \text{for } 1 \leqslant s \leqslant r,\ v \in H_0^1(\Omega) \cap H^s(\Omega).$$

For straight-edged simplicial partitions of a convex smooth domain, and with h the maximal diameter of an element, one has $r = 2$ regardless of the polynomial degree used. For isoparametric elements of degree $p - 1$, typically $r = p$, but the assumption that $S_h \subset H_0^1(\Omega)$, although in principle possible to satisfy, is seldom fulfilled in practice; cf. Remark 3.1 below.

THEOREM 3.1. *Let u be the solution of* (2.1) *with* $F_\alpha(v) \leqslant K$ *for some* $\alpha \geqslant 0$. *Assume that* (3.2) *holds and let further* $0 \leqslant \sigma < 2$ *and* $1 \leqslant \alpha + \sigma \leqslant r$. *Then there exists a constant* $C = C(B, K, M, t^*, \alpha, \sigma)$ *such that, if u_h is the solution of* (3.1), *we have*

$$\|u_h(t) - u(t)\| \leqslant C h^{\alpha+\sigma} t^{-\sigma/2} \quad \textit{for } t \in I.$$

Proof. We begin by noting that the argument of the proof of Theorem 1.1, using a superposition of the estimate (1.8) for the linear homogeneous problem, may not be applied in general. In fact, in order to deal with the term $F_h(t-s) f(u(s))$ in (1.10), this would require $f(u(s))$ to be in some $\dot{H}^\alpha$ space and, in particular, if $\alpha > 1/2$, would demand $f(0) = 0$, which is not always true. We shall therefore give a direct proof which does not depend on (1.8).

Let $T = (-\Delta)^{-1}$: $L_2 \to \dot{H}^2$ and T_h: $L_2 \to S_h$ be the approximation defined by

$$(3.3)\qquad (\nabla T_h f, \nabla \chi) = (f, \chi) \quad \text{for } \chi \in S_h.$$

As is well known and easy to check, the operator T_h is bounded, symmetric, positive semidefinite on L_2 and positive definite on S_h. The elliptic projection satisfies $P_1 v = T_h(-\Delta)v$. Application of T to (2.1) yields

$$Tu_t + u = Tf(u) \quad \text{for } t \geqslant 0,$$

and the semidiscrete problem (3.1) may similarly be written

$$\begin{aligned} {}_t u_{h,t} + u_h &= T_h f(u_h) \quad \text{for } t \geqslant 0, \\ u_h(0) &= v_h. \end{aligned} \tag{3.4}$$

Let $e = u_h - u$ be the error. We have

$$\begin{aligned} T_h e_t + e &= T_h u_{h,t} + u_h - T_h u_t - u \\ &= T_h f(u_h) - Tf(u) + (T - T_h)u_t \\ &= T_h(f(u_h) - f(u)) + (T - T_h)(f(u) - u_t) \end{aligned}$$

or

$$T_h e_t + e = T_h(\omega e) + \rho, \tag{3.5}$$

where

$$\omega = \int_0^1 f'(yu + (1-y)u_h)\,dy,$$

which is bounded by our assumption that f' is bounded and where

$$\rho = -(T_h - T)\Delta u = (P_1 - I)u.$$

Multiplication of (3.5) by e_t yields

$$\begin{aligned} (T_h e_t, e_t) + \frac{1}{2}\frac{d}{dt}\|e\|^2 &= (T_h(\omega e), e_t) + (\rho, e_t) \\ &= (T_h(\omega e), e_t) + \frac{d}{dt}(\rho, e) - (\rho_t, e). \end{aligned}$$

Since T_h is positive semidefinite, we have the Cauchy inequality

$$|(T_h v, w)| \leqslant (T_h v, v)^{1/2}(T_h w, w)^{1/2} \tag{3.6}$$

and hence, by the geometric-arithmetic mean inequality,

$$(T_h e_t, e_t) + \frac{1}{2}\frac{d}{dt}\|e\|^2 \leqslant \frac{1}{2}(T_h e_t, e_t) + \frac{1}{2}(T_h(\omega e), \omega e) + \frac{d}{dt}(\rho, e) - (\rho_t, e)$$

or, employing the boundedness of T_h and ω,

$$\frac{d}{dt}\|e\|^2 \leqslant C\|e\|^2 + 2\frac{d}{dt}(\rho, e) - 2(\rho_t, e).$$

Multiplication by t^2 now gives, using also that $t \leqslant t^*$ and again the geometric-arithmetic mean inequality,

$$\begin{aligned} \frac{d}{dt}\left(t^2\|e\|^2\right) &\leqslant 2t\|e\|^2 + Ct^2\|e\|^2 + 2\frac{d}{dt}(t^2(\rho, e)) - 4t(\rho, e) - 2t^2(\rho_t, e) \\ &\leqslant 2\frac{d}{dt}(t^2(\rho, e)) + C\left(t\|\rho\|^2 + t^3\|\rho_t\|^2 + t\|e\|^2\right), \end{aligned}$$

whence, by integration and a trivial kickback argument,

$$t^2\|e\|^2 \leqslant Ct^2\|\rho\|^2 + C\int_0^t \left(s\|\rho\|^2 + s^3\|\rho_t\|^2\right) ds + C\int_0^t s\|e\|^2\,ds. \tag{3.7}$$

In order to estimate the last integral, we return to the error equation (3.5), which we now multiply by e to obtain

$$\frac{1}{2}\frac{d}{dt}(T_h e, e) + \|e\|^2 = (T_h(\omega e), e) + (\rho, e)$$

or, after multiplication by $2t$ and manipulation similar to that above,

$$\frac{d}{dt}(t(T_h e, e)) + 2t\|e\|^2 \leqslant 2t(T_h(\omega e), e) + 2t(\rho, e) + (T_h e, e).$$

Here, by (3.6), for ε suitable, since T_h and ω are bounded,

$$(T_h(\omega e), e) \leqslant \varepsilon(T_h(\omega e), \omega e) + \frac{1}{4\varepsilon}(T_h e, e) \leqslant \frac{1}{4}\|e\|^2 + C(T_h e, e),$$

so that

$$\frac{d}{dt}(t(T_h e, e)) + 2t\|e\|^2 \leqslant t\|e\|^2 + C\left(t\|\rho\|^2 + (T_h e, e)\right)$$

and hence, by integration,

$$\int_0^t s\|e\|^2\,ds \leqslant C\int_0^t s\|\rho\|^2\,ds + C\int_0^t (T_h e, e)\,ds. \tag{3.8}$$

We shall now estimate the last integral. We set $E(t) = \int_0^t e(s)\,ds$ and integrate (3.5) to obtain

$$T_h(e(t) - e(0)) + E(t) = T_h\int_0^t \omega e\,ds + \int_0^t \rho\,ds. \tag{3.9}$$

We note that $T_h e(0) = 0$, since

$$(T_h e(0), \varphi) = (P_0 v - v, T_h\varphi) = 0 \quad \text{for } \varphi \in L_2(\Omega).$$

Hence, multiplying (3.9) by $E'(t) = e(t)$, we obtain

$$\begin{aligned}(T_h e, e) + \frac{1}{2}\frac{d}{dt}\|E\|^2 &= \left(T_h\int_0^t \omega e\,ds, e\right) + \left(\int_0^t \rho\,ds, e\right)\\ &\leqslant \frac{1}{2}(T_h e, e) + \frac{1}{2}\left(T_h\int_0^t \omega e\,ds, \int_0^t \omega e\,ds\right) + \int_0^t \|\rho\|\,ds\|e\|,\end{aligned}$$

or, by integration, since $E(0) = 0$,

$$\int_0^t (T_h e, e)\,ds \leqslant C\int_0^t \left(\int_0^s \|e(\tau)\|\,d\tau\right)^2 ds + 2\int_0^t \|e\|\left(\int_0^s \|\rho(\tau)\|\,d\tau\right) ds.$$

Altogether, using (3.7) and (3.8), we find

$$\begin{aligned}t^2\|e\|^2 \leqslant C\biggl\{t^2\|\rho\|^2 &+ \int_0^t \left(s\|\rho\|^2 + s^3\|\rho_t\|^2\right) ds\\ &+ \int_0^t \left(\int_0^s \|e(\tau)\|\,d\tau\right)^2 ds + \int_0^t \|e\|\left(\int_0^s \|\rho(\tau)\|\,d\tau\right) ds\biggr\}.\end{aligned}$$

Now by our assumptions on u we have by (3.2) and Theorem 2.2

$$\|\rho(t)\| = \|(P_1 - I)u(t)\| \leqslant Ch^{\alpha+\sigma}\|u(t)\|_{H^{\alpha+\sigma}} \leqslant Ch^{\alpha+\sigma}t^{-\sigma/2},$$

and similarly

$$\|\rho_t(t)\| \leqslant Ch^{\alpha+\sigma}\|u_t(t)\|_{H^{\alpha+\sigma}} \leqslant Ch^{\alpha+\sigma}t^{-1-\sigma/2},$$

so that, since $\sigma < 2$,

$$t^2\|e\|^2 \leqslant C\left\{h^{2(\alpha+\sigma)}t^{2-\sigma} + \int_0^t \left(\int_0^s \|e\|\,d\tau\right)^2 ds + h^{\alpha+\sigma}\int_0^t s^{1-\sigma/2}\|e\|\right\} ds.$$

Setting $\varphi(t) = t^{\sigma/2}\|e(t)\|$, this shows

$$\varphi(t)^2 \leqslant C\left\{h^{2(\alpha+\sigma)} + t^{-(2-\sigma)}\int_0^t \left(\int_0^s \tau^{-\sigma/2}\varphi(\tau)\,d\tau\right)^2 ds \right.$$
$$\left. + h^{\alpha+\sigma}t^{-(2-\sigma)}\int_0^t s^{1-\sigma}\varphi(s)\,ds\right\}.$$

With $\psi(t) = \max_{0\leqslant s\leqslant t}\varphi(s)$, and choosing $t_0 = t_0(t)$ such that $\varphi(t_0) = \psi(t)$, we have

$$\varphi(t)^2 \leqslant \varphi(t_0)^2 \leqslant C\left\{h^{2(\alpha+\sigma)} + t_0^{-(2-\sigma)}\int_0^{t_0} s^{2-\sigma}\psi(s)^2\,ds + h^{\alpha+\sigma}\psi(t)\right\}$$

or

$$\psi(t)^2 \leqslant C\left\{h^{2(\alpha+\sigma)} + \int_0^t \psi(s)^2\,ds\right\}.$$

Gronwall's lemma now shows

$$t^{\sigma/2}\|e(t)\| \leqslant \psi(t) \leqslant Ch^{\alpha+\sigma},$$

which completes the proof.

Remark 3.1. As is well known, for $r > 2$ there are difficulties connected with the construction of finite element spaces S_h for which the functions satisfy homogeneous Dirichlet boundary conditions, that is, such that $S_h \subset H_0^1(\Omega)$. To deal with this difficulty, a variety of different methods have been proposed (cf. Nitsche [14] and Bramble [3]) for which the approximate solution operator T_h is not defined by (3.3) but in some other way, and with the properties that T_h: $L_2 \to S_h \subset L_2$ is selfadjoint, positive semidefinite, positive definite on S_h and such that

$$\|T_h f - Tf\| \leqslant Ch^s\|u\|_{H^{s-2}} \quad \text{for } 2 \leqslant s \leqslant r.$$

Defining now the semidiscrete problem by (3.4), the above proof of Theorem 3.1 extends immediately to this case with the only change that in the formulation of the theorem $\alpha + \sigma$ is required to lie in the interval $[2, r]$; cf. [4] for the linear parabolic case.

4. An Estimate for the Gradient of the Error. In this and the following section we shall derive some further error estimates for the semidiscrete problem (3.1) in which the regularity and compatibility assumptions are the same as those of our basic error estimate of Theorem 3.1. The first of these states that the convergence order of Theorem 3.1, i.e., $O(h^{\alpha+\sigma})$ for $v \in \mathcal{F}_\alpha$, $0 \leqslant \sigma < 2$ and $t > 0$, is maintained for the gradient of the error, provided $\alpha + \sigma$ is bounded by the optimal convergence order for gradients.

In addition to (3.2) we assume in this section that the functions in S_h vanish outside a subdomain $\Omega_h \subseteq \Omega$ (with equality possible) and that for some r' with $r \leqslant r' < r + 1$

(4.1) $\quad \|P_1 v - v\|_{H^1(\Omega_h)} \leqslant Mh^{s-1}\|v\|_{H^s(\Omega)}, \qquad 1 \leqslant s \leqslant r' \text{ for } v \in H_0^1(\Omega) \cap H^s(\Omega).$

(Recall that we generally omit the domain in our notation if it is the whole of Ω.) For instance, for straight-edged triangular partitions in the plane and continuous piecewise polynomials of quadratic or higher degree, $r' = 5/2$ (Strang and Fix [21, p. 195]), whereas for isoparametric elements of degree $p-1$ one has $r' = p$. Note that (4.1) would not hold with Ω_h replaced by Ω in these cases.

THEOREM 4.1. *Let u be the solution of* (2.1) *with $F_\alpha(v) \leqslant K$, $\alpha \geqslant 0$. Assume that* (4.1) *holds and let $0 \leqslant \sigma < 2$, $1 \leqslant \alpha + \sigma \leqslant r' - 1$. Then there exists a constant $C = C(B, K, M, t^*, \alpha, \sigma)$ such that for u_h, the solution of* (3.1),

$$\|\nabla(u_h(t) - u(t))\|_{\Omega_h} \leqslant Ch^{\alpha+\sigma}t^{-1/2-\sigma/2} \quad \textit{for } t \in I.$$

Proof. In addition to our earlier notation $e = u_h - u$, $\rho = P_1 u - u$, and

$$\omega = \int_0^1 f'(yu + (1-y)u_h)\, dy,$$

set $\theta = u_h - P_1 u$. Then

$$(4.2) \qquad \begin{aligned} (\theta_t, \chi) + (\nabla\theta, \nabla\chi) &= (f(u_h) - f(u), \chi) - (P_1 u_t - u_t, \chi) \\ &= (\omega e - \rho_t, \chi) \quad \text{for } \chi \in S_h. \end{aligned}$$

Choosing $\chi = \theta_t$, we have

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 = (\omega e - \rho_t, \theta_t) \leqslant \|\theta_t\|^2 + C\left(\|e\|^2 + \|\rho_t\|^2\right),$$

and hence

$$\frac{d}{dt}\left(t^3\|\nabla\theta\|^2\right) \leqslant Ct^3\left(\|e\|^2 + \|\rho_t\|^2\right) + 3t^2\|\nabla\theta\|^2,$$

or, after integration,

$$(4.3) \qquad t^3\|\nabla\theta\|^2 \leqslant C\int_0^t s^3\left(\|e\|^2 + \|\rho_t\|^2\right) ds + 3\int_0^t s^2\|\nabla\theta\|^2\, ds.$$

To estimate the last integral, we choose instead $\chi = \theta$ in (4.2) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 = (\omega e - \rho_t, \theta),$$

or, using that $\theta = e - \rho$ and $t \leqslant t^*$,

$$\begin{aligned} \frac{1}{2}\frac{d}{d}\left(t^2\|\theta\|^2\right) + t^2\|\nabla\theta\|^2 &\leqslant Ct\|\theta\|^2 + Ct^3\left(\|e\|^2 + \|\rho_t\|^2\right) \\ &\leqslant C\left(t\|e\|^2 + t\|\rho\|^2 + t^3\|\rho_t\|^2\right), \end{aligned}$$

so that

$$\int_0^t s^2\|\nabla\theta\|^2\, ds \leqslant C\int_0^t \left(s\|e\|^2 + s\|\rho\|^2 + s^3\|\rho_t\|^2\right) ds.$$

Combining this with (4.3), we have

$$t^3\|\nabla\theta\|^2 \leqslant C\int_0^t \left(s\|e\|^2 + s\|\rho\|^2 + s^3\|\rho_t\|^2\right) ds.$$

We now use Theorem 3.1 to estimate $e(s)$, and (3.2) and Theorem 2.2 to bound ρ and ρ_t. It results that $t^3\|\nabla\theta\|^2 \leqslant Ch^{2(\alpha+\sigma)}t^{2-\sigma}$ or,

$$(4.4) \qquad \|\nabla\theta\| \leqslant Ch^{\alpha+\sigma}t^{-1/2-\sigma/2}.$$

Since also, by (4.1) and Theorem 2.2,

$$\|\nabla\rho\|_{\Omega_h} \leqslant Ch^{\alpha+\sigma}t^{-1/2-\sigma/2},$$

this completes the proof.

We note that the full range $1 \leqslant \alpha + \sigma \leqslant r$ is allowed in (4.4).

5. Maximum-Norm Error Estimates. In this section we shall show that the estimates of Theorems 3.1 and 4.1 carry over, with the appropriate changes, to the case that the error is measured in the maximum norm. Again, we shall assume that certain error estimates hold for the elliptic problem. More precisely, we suppose that the elliptic projection P_1 satisfies, in addition to (3.2) and (4.1), for some $r \geqslant 2$, $\delta \geqslant 0$,

$$\|P_1v - v\|_{L_\infty(\Omega)} \leqslant M\left(\log\frac{1}{h}\right)^{\delta} h^s\|v\|_{W^s_\infty} \quad \text{for } 0 \leqslant s \leqslant r,$$

and, again with $\Omega_h \subseteq \Omega$, cf. (4.1),

$$\|\nabla(P_1v - v)\|_{L_\infty(\Omega_h)} \leqslant M\left(\log\frac{1}{h}\right)^{\delta} h^{s-1}\|v\|_{W^s_\infty} \quad \text{for } 1 \leqslant s \leqslant r.$$

For notational convenience below we set $\Omega_h^0 = \Omega$, $\Omega_h^1 = \Omega_h$, so that the above assumptions combine into

$$(5.1)\quad \|P_1v - v\|_{W^j_\infty(\Omega_h^j)} \leqslant M\left(\log\frac{1}{h}\right)^{\delta} h^{s-j}\|v\|_{W^s_\infty(\Omega)} \quad \text{for } j \leqslant s \leqslant r,\ j = 0,1.$$

For many finite element spaces based on quasiuniform triangulations, isoparametric or not, this estimate is an easy consequence of the almost best approximation result over Ω_h of Schatz and Wahlbin [18]; cf. also Nitsche [15]. For straight-edged simplicial partitions, for instance, (5.1) holds with $r = 2$, and for isoparametric elements of degree $p - 1$, if $\Omega_h \subseteq \Omega$, with $r = p$. (For $j = 0$ and $r = 2$, one has $\delta = 1$, cf. Haverkamp [8], while for $j = 0$ and $r > 2$, $\delta = 0$ [18]. For $j = 1$, one may conjecture that $\delta = 0$ also in the case $r = 2$; cf. Rannacher and Scott [17] for polygonal domains.

We shall also assume that with Δ_h: $S_h \to S_h$ the discrete analogue of the Laplacian defined by

$$(5.2)\qquad (\Delta_h v_h, \chi) = -(\nabla v_h, \nabla\chi) \quad \text{for } \chi \in S_h,$$

we have

$$(5.3)\qquad \|\chi\|_{\dot H^\beta} \leqslant M\left\|(-\Delta_h)^{\beta/2}\chi\right\| \quad \text{for } \chi \in S_h, \text{if } -1 \leqslant \beta \leqslant 1.$$

In many situations this may be verified as follows: For $\beta = 1$ this is immediate (with $M = 1$) by the definition of $-\Delta_h$. For $\beta = -1$ we have, for $v \in \dot H^1$,

$$(\chi, v) = \left((-\Delta_h)^{-1/2}\chi, (-\Delta_h)^{1/2}P_0v\right) \leqslant \left\|(-\Delta_h)^{-1/2}\chi\right\| \|\nabla P_0 v\|.$$

Assuming an inverse property and a suitable low-order approximation property of S_h, we find that P_0 is bounded in $\dot H^1$ and hence

$$\|\chi\|_{\dot H^{-1}} = \sup\frac{(\chi, v)}{\|v\|_{\dot H^1}} \leqslant M\left\|(-\Delta_h)^{1/2}\chi\right\|.$$

For $-1 < \beta < 1$, (5.3) now follows by Hilbert scale interpolation on the two norms on S_h occurring.

Note that with the notation (5.2) the semidiscrete solution satisfies

$$u_{h,t} - \Delta_h u_h = P_0 f(u_h). \tag{5.4}$$

The following is our result in this section.

THEOREM 5.1. *Let u be the solution of* (2.1) *with $F_\alpha(v) \leqslant K$, $\alpha \geqslant 0$, and assume that* (3.2), (4.1), (5.1) *and* (5.3) *hold. Let $0 \leqslant \sigma < 2$, $\alpha + \sigma + j \leqslant r$ and $\varepsilon > 0$. Then there is a constant $C = C(B, K, M, t^*, \alpha, \sigma, \varepsilon)$ such that for the solution u_h of* (3.1),

$$\|u_h(t) - u(t)\|_{W_\infty^j(\Omega_h^j)} \leqslant C\left(\log\frac{1}{h}\right)^{\nu} h^{\alpha+\sigma} t^{-\sigma/2-d/4-j/2-\varepsilon} \quad \textit{for } j = 0, 1,$$

where $\nu = \delta$ if $\alpha + \sigma + j = r$ and $\delta > 0$, and $\nu = 0$ otherwise.

We shall need the following lemma for the semigroup $E_h(t) = \exp(t\Delta_h)$ on S_h, the solution operator for the semidiscrete problem (3.1) (or (5.4)) with $f \equiv 0$.

LEMMA 5.2. *Given $\varepsilon > 0$, there exists a constant $C = C(M, \varepsilon)$ such that*

$$\|E_h(t)v_h\|_{W_\infty^j} \leqslant Ct^{-d/4-\varepsilon}\|v_h\|_{\dot H^j} \quad \textit{for } j = 0, 1,\ v_h \in S_h.$$

Proof. Setting $w_h(t) = E_h(t)v$, we have

$$\|u_h\|_{W_\infty^j} = \|T_h u_{h,t}\|_{W_\infty^j(\Omega_h)} \leqslant \|Tu_{h,t}\|_{W_\infty^j(\Omega)} + \|(T_h - T)u_{h,t}\|_{W_\infty^j(\Omega_h)}.$$

Here, by Sobolev's inequality and elliptic regularity,

$$\|Tu_{h,t}\|_{W_\infty^j} \leqslant C\|Tu_{h,t}\|_{H^{d/2+j+\varepsilon}} \leqslant C\|u_{h,t}\|_{H^{d/2+j-2+\varepsilon}}. \tag{5.5}$$

By our assumption (5.1) we have for any $\varepsilon > 0$

$$\|P_1 v - v\|_{W_\infty^j(\Omega_h^j)} \leqslant M\left(\log\frac{1}{h}\right)^{\delta} h^{\varepsilon}\|v\|_{W_\infty^{j+\varepsilon}} \leqslant C\|v\|_{W_\infty^{j+\varepsilon}}.$$

Hence, since $T_h - T = (I - P_1)T$, we obtain, using (5.5),

$$\|(T_h - T)u_{h,t}\|_{W_\infty^j(\Omega_h)} \leqslant C\|Tu_{h,t}\|_{W_\infty^{j+\varepsilon}} \leqslant C\|u_{h,t}\|_{H^{d/2+j-2+2\varepsilon}}.$$

Using (5.3), we have, except for $d = 1$, $j = 0$,

$$\begin{aligned}\|u_{h,t}\|_{H^{d/2+j-2+2\varepsilon}} &= \|\Delta_h E_h(t)v_h\|_{H^{d/2+j-2+2\varepsilon}} \\ &\leqslant C\left\|(-\Delta_h)^{d/4+\varepsilon}E_h(t)(-\Delta_h)^{j/2}v_h\right\| \\ &\leqslant \sup_{\lambda\geqslant 0}|\lambda^{d/4+\varepsilon}e^{-t\lambda}|\,\left\|(-\Delta_h)^{j/2}v_h\right\| \leqslant Ct^{-d/4-\varepsilon}\|v_h\|_{\dot H^j}.\end{aligned}$$

For the remaining case $d = 1$, $j = 0$ we have easily

$$\|u_h\|_{L_\infty} \leqslant C\|u_h\|_{H^{1/2+2\varepsilon}} \leqslant C\left\|(-\Delta_h)^{1/4+\varepsilon}E_h(t)v_h\right\| \leqslant Ct^{-1/4-\varepsilon}\|v_h\|.$$

This completes the proof of the lemma.

Note that the estimate of Lemma 5.2 is in terms of v_h in L_2 based spaces and hence not a stability estimate for $E_h(t)$ in L_∞ such as in Schatz, Thomée and Wahlbin [19] or Nitsche and Wheeler [16].

We are now ready for the proof of the theorem. Letting $\rho = P_1 v - v$, we have, with $\varepsilon > 0$ such that $\alpha + \sigma + j + \varepsilon \leqslant r$, if $\delta > 0$ and $\alpha + \sigma + j < r$,

$$\|\rho(t)\|_{W^j_\infty(\Omega^j_h)} \leqslant C\left(\log \frac{1}{h}\right)^{\delta} h^{\alpha+\sigma+\varepsilon} \|u(t)\|_{W^{\alpha+\sigma+\varepsilon+j}_\infty}$$
$$\leqslant Ch^{\alpha+\sigma}\|u(t)\|_{H^{\alpha+\sigma+d/2+j+2\varepsilon}} \leqslant Ch^{\alpha+\sigma}t^{-\sigma/2-d/4-j/2-\varepsilon}.$$

In the case $\alpha + \sigma + j = r$ and $\delta > 0$ the estimate contains the factor $(\log 1/h)^{\delta}$.

It remains to estimate $\theta = u_h - P_1 u$. Using (4.2) and (5.4), we find easily

$$\theta_t - \Delta_h \theta = P_0(\omega e - \rho_t),$$

and hence, with $\beta > 0$,

$$(t^{\beta}\theta)_t - \Delta_h(t^{\beta}\theta) = t^{\beta}P_0(\omega e - \rho_t) + \beta t^{\beta-1}\theta$$

and, by Duhamel's principle,

$$t^{\beta}\theta = \int_0^t E_h(t-s)\left[s^{\beta}P_0(\omega e - \rho_t) + \beta t^{\beta-1}\theta\right] ds.$$

Thus by Lemma 5.2, noting also that $\theta = e - \rho$,

$$t^{\beta}\|\theta\|_{L_\infty} \leqslant C\int_0^t (t-s)^{-d/4-\varepsilon}\left[s^{\beta}\|\rho_t\| + s^{\beta-1}(\|e\| + \|\rho\|)\right] ds.$$

Using that $\|\rho(t)\| \leqslant Ch^{\alpha+\sigma}t^{-\sigma/2}$ and the corresponding estimate for ρ_t, and also Theorem 3.1, the above yields (for β large enough so that the integrand is integrable)

$$t^{\beta}\|\theta\|_{L_\infty} \leqslant Ch^{\alpha+\sigma}\int_0^t (t-s)^{-d/4-\varepsilon}s^{\beta-\sigma/2-1}\, ds \leqslant Ch^{\alpha+\sigma}t^{\beta-\sigma/2-d/4-\varepsilon},$$

from which our theorem follows if $j = 0$.

For $j = 1$ we use again Lemma 5.2 to obtain

$$t^{\beta}\|\theta\|_{W^1_\infty} \leqslant C\int_0^t (t-s)^{-d/4-\varepsilon}\left[s^{\beta}\left(\|\omega e\|_{H^1} + \|\rho_t\|_{H^1}\right) + s^{\beta-1}\left(\|e\|_{H^1} + \|\rho\|_{H^1}\right)\right] ds. \tag{5.6}$$

Here,

$$\|\omega e\|_{H^1} \leqslant C\left(\|e\|_{H^1} + \|e\|_{L_\infty}\|\omega\|_{H^1}\right),$$

and since

$$\nabla\omega = \int_0^t f''(yu_h + (1-y)u)(y\nabla u_h + (1-y)\nabla u)\, dy,$$

we have by obvious energy estimates,

$$\|\omega(t)\|_{H^1} \leqslant C\left(\|u_h(t)\|_{H^1} + c\|u(t)\|_{H^1}\right) \leqslant Ct^{-1/2},$$

so that, by Theorem 4.1 and our result for $j = 0$,

$$\|\omega e\|_{H^1} \leqslant C\left(h^{\alpha+\sigma}t^{-\sigma/2-1/2} + h^{\alpha+\sigma}t^{-\sigma/2-d/4-1/2-\varepsilon}\right)$$
$$\leqslant Ch^{\alpha+\sigma}t^{-\sigma/2-d/4-1/2-\varepsilon}.$$

This result is now used in (5.6), and the proof is concluded as for the case $j = 0$, using once more Theorem 4.1 to estimate $\|e\|_{H^1}$.

6. A Scalar Counterexample in Spline Spaces. Let u and u_h be the solutions of (2.1) and (3.1), respectively, and assume that $\|u\|_{L_\infty(I, L_\infty)} \leqslant B$ and that S_h is such that (3.2) holds. Then by Theorem 1.1 (or the case $\alpha = 0$ of Theorem 3.1) we have for any $\sigma < 2$,

$$\|u_h(t) - u(t)\| \leqslant C(B, M, t_0, \sigma) h^\sigma \quad \text{for } 0 < t_0 \leqslant t \leqslant t^*. \tag{6.1}$$

In this section we shall show the sharpness of this result in the sense that (6.1) cannot hold for any $\sigma > 2$. In fact, as is shown below, the restriction $\sigma \leqslant 2$ is valid even if the L_2-norm on the left is replaced by a weaker (negative) norm. Our example here is similar to the system in the introduction, using trigonometric polynomials as approximating functions, but we feel that it is of interest to exhibit an example with a scalar equation and a family of standard finite element spaces.

Thus, consider the problem

$$\begin{aligned} &u_t = u_{xx} + u^2, \qquad x \in [0, \pi],\ t > 0,\\ &u(0, t) = u(\pi, t) = 0,\\ &u(\cdot, 0) = v, \end{aligned} \tag{6.2}$$

and let $S_h = \{\chi;\ \chi \in C^k[0, \pi],\ \chi|_{J_j} \in \Pi_{r-1}\}$, where $h = \pi/n$, n integer, $J_j = (jh, (j+1)h)$, $0 \leqslant k < r - 1$. Our interest here is in the case $r > 2$.

We shall construct solutions $u = u(h; x, t)$ of (6.2) which contradict (6.1) with $\sigma > 2$. For this purpose, let

$$\psi(y) = \sum_{j=1}^{r+1} \psi_j \sin(jy)$$

be a not identically vanishing function, where the ψ_j are chosen so that ψ is orthogonal to Π_{r-1} on $[0, \pi]$, or

$$\int_0^\pi \psi(y) y^l \, dy = 0, \qquad l = 0, \ldots, r-1.$$

This is possible since we have more unknowns than equations. Let ψ_J denote the first nonvanishing coefficient and normalize so that $\psi_J = 1$. Now choose initial data for (6.2) as

$$v(x) = v(h, x) = \psi(nx), \qquad nh = \pi.$$

Note that, independently of n,

$$\|v\|_\infty \leqslant \sum_{j=J}^{r+1} |\psi_j| \equiv K. \tag{6.3}$$

It follows, by comparison with the initial value problems $z_t = z^2$, $t > 0$, $z(0) = \pm K$, that there is a $t^* > 0$ and B such that $\|u(t)\|_\infty \leqslant B$ for $t \in \bar{I}$, uniformly in n. Hence the conditions of Theorem 2.2 are satisfied.

We remark that since there is a uniform bound for u, we may regard u^2 is altered to a function $f(u)$ with $f(u) = u^2$ for $|u| \leqslant B$ and with f and f' bounded on R, thus satisfying the assumptions of Theorem 3.1 (and Theorem 1.1).

By construction, the L_2-projection of v into S_h is zero, so that the semidiscrete solution vanishes and the error is identical with the exact solution of (6.2).

Letting $c_j = c_j(g) = \int_0^\pi g(x)\sin(jx)\,dx$ be the jth Fourier sine coefficient, we define the Hilbert scale

$$\|g\|_s = \left(\sum_{j=1}^{\infty} j^{2s}c_j^2\right)^{1/2}, \qquad s \in R.$$

We shall prove that, given t_0 with $0 < t_0 < t^*$, there exist positive constants c_0 and h_0 such that with v as above and s arbitrary,

$$\|u(t)\|_s \geqslant c_0 h^2 \quad \text{for } t_0 \leqslant t \leqslant t^*,\ h \leqslant h_0. \tag{6.4}$$

This shows, in particular, that (6.1) cannot hold for any $\sigma > 2$.

Let $c = c(u; t) = \int_0^\pi u(x,t)\sin x\,dx$ denote the first Fourier sine coefficient of u. To prove (6.4), it is clearly enough to show that

$$c(u;t) \geqslant c_0 n^{-2}, \qquad t_0 \leqslant t \leqslant t^*,\ n \geqslant n_0. \tag{6.5}$$

The remainder of this section is devoted to this.

We introduce the auxiliary functions w and $\tilde{u}$ as the solutions of

$$\begin{aligned} &w_t = w_{xx} \quad \text{in } [0,\pi]\times I,\\ &w(x,t) = 0 \quad \text{for } x = 0,\pi,\ t\in I,\\ &w(\cdot,0) = v, \end{aligned}$$

and, with $w_+ = \max(0,w)$,

$$\begin{aligned} &\tilde{u}_t = \tilde{u}_{xx} + (w_+)^2 \quad \text{in } [0,\pi]\times I,\\ &\tilde{u}(x,t) = 0 \quad \text{for } x = 0,\pi,\ t\in I,\\ &\tilde{u}(\cdot,0) = v. \end{aligned}$$

Since $u^2 \geqslant 0$, we have by comparison that $u(x,t) \geqslant w(x,t)$ and hence also $u(x,t)^2 \geqslant w_+(x,t)^2$, so that again by comparison $u(x,t) \geqslant \tilde{u}(x,t)$. Hence, letting $\tilde{c} = c(\tilde{u},t)$ be the first Fourier sine coefficient of $\tilde{u}$, we have $c \geqslant \tilde{c}$, so that it suffices now to show that $\tilde{c} \geqslant c_0 n^{-2}$. By construction, $\tilde{c}(0) = \int_0^\pi \psi(nx)\sin x\,dx = 0$ for $n > 1$, and hence $\tilde{c}$ satisfies the initial value problem

$$\begin{aligned} &\tilde{c}' + \tilde{c} = \varphi(t) \equiv \int_0^\pi w_+(x,t)^2 \sin x\,dx, \qquad t > 0,\\ &\tilde{c}(0) = 0. \end{aligned}$$

We shall show that there are constants c_1 and k_1 such that

$$\varphi(t) \geqslant c_1 e^{-2n^2J^2t} \quad \text{for } n^2t \geqslant k_1. \tag{6.6}$$

From this it will follow, since $\varphi \geqslant 0$, that

$$\begin{aligned} \tilde{c}(t) &= \int_0^t e^{-(t-s)}\varphi(s)\,ds \geqslant c_1 e^{-t}\int_{k_1/n^2}^t e^{-(2n^2J^2-1)s}\,ds\\ &= \frac{c_1}{2n^2J^2} e^{-2k_1J^2} e^{-t}(1 + o(1)) \quad \text{for } n \text{ large}, \end{aligned}$$

which concludes the proof.

It remains to show (6.6). For this we note that

$$w(x,t) = e^{-J^2n^2t}\sin(nJx) + \sum_{j=J+1}^{r+1} \psi_j e^{-j^2n^2t}.$$

Denoting the first term on the right by $\tilde{w}$, it is obvious that

$$\int_0^\pi \tilde{w}_+(x,t)^2 \sin x\,dx \geqslant c_2 e^{-2J^2n^2t}.$$

Since $\tilde{w}_+ \leqslant w_+ + |w - \tilde{w}|$ we have that $\tilde{w}_+^2 \leqslant 2w_+^2 + 2|w - \tilde{w}|^2$ and hence, cf. (6.3),

$$w_+^2 \geqslant \frac{1}{2}\tilde{w}_+^2 - |w - \tilde{w}|^2 \geqslant \frac{1}{2}\tilde{w}_+^2 - (K-1)^2 e^{-2(J+1)^2n^2t},$$

so that

$$\begin{aligned}\varphi(t) = \int_0^\pi w_+(x,t)^2\sin x\,dx &\geqslant \frac{c_2}{2}e^{-2J^2n^2t} - 2(K-1)^2 e^{-2(J+1)^2n^2t} \\ &= e^{-2J^2n^2t}\left[\frac{c_2}{2} - 2(K-1)^2e^{-(4J+2)n^2t}\right] \geqslant \frac{c_2}{4}e^{-2J^2n^2t} \quad \text{for } n^2t \geqslant k_1.\end{aligned}$$

This proves (6.6) and completes the proof of (6.5), thus establishing a contradiction to (6.1) for $\sigma > 2$.

Appendix. In this appendix we present a proof of Theorem 2.2. The proof is an expanded version of an argument in Larsson [12], where the case $\beta - \alpha \leqslant 4$ of (2.2) was treated. The main step of the proof consists in estimating certain time derivatives of the solution u in the $\dot{H}^\sigma$-norms for $0 \leqslant \sigma \leqslant 2$. This is done in the following lemma. In the sequel we shall not explicitly indicate the dependence on B and t^* of various constants such as C in (2.9).

LEMMA A.1. (a) *Let $k \geqslant 0$ be an integer and suppose that $2k \leqslant \alpha \leqslant 2(k+1)$. Then $v \in \mathcal{F}_\alpha$ implies that $u^{(j)} \in C(\bar{I}, \dot{H}^2)$ for $0 \leqslant j < k$ and $u^{(k)} \in C(\bar{I}, \dot{H}^{\alpha-2k})$ with $u^{(j)}(0) = v_j$ for $0 \leqslant j \leqslant k$.*

(b) *Let $\alpha \geqslant 0$ and assume that $v \in \mathcal{F}_\alpha$ with $F_\alpha(v) \leqslant K$. Let k be the integer part of $\alpha/2$, so that $2k \leqslant \alpha < 2(k+1)$, and suppose that β satisfies $0 \leqslant \beta - \alpha \leqslant 5$ and $2(k+i) \leqslant \beta \leqslant 2(k+i+1)$ with $i = 0, 1, 2, 3$. Then there is a constant $C = C(K)$ such that*

$$\|u^{(k+i)}(t)\|_{\dot{H}^{\beta-2(k+i)}} \leqslant Ct^{-(\beta-\alpha)/2}, \qquad t \in I. \tag{A.1}$$

Our argument will be based on Eqs. (2.6) and (2.7), and in order to handle the term $D_t^m f(u)$, we shall need estimates of the expressions on the right-hand side of (2.8) in various norms. We collect these estimates in Lemma A.2 below. One difficulty that arises in this context is that the spaces $\dot{H}^\alpha$ generally involve boundary conditions. Since we do not want to assume that $f(u)$ satisfies any boundary conditions, $f(u)$ will, in general, belong to $\dot{H}^\alpha$ only for $0 \leqslant \alpha < 1/2$.

LEMMA A.2. *Let $j \geqslant 1$ and $k \geqslant 2$ be integers and let $3/2 < \lambda \leqslant 2$. For each $K \geqslant 0$ there is $C = C(K)$ such that:*

(a) *$\|u(t)\|_{L_\infty} \leqslant K$ implies*

$$\|f(u)\|_{\dot{H}^\alpha} \leqslant C(1 + \|u\|_{\dot{H}^\alpha}) \quad \text{for } 0 \leqslant \alpha < 1/2, \tag{A.2}$$

$$\begin{aligned}(\mathrm{A}.3)\qquad &\|f^{(j)}(u)u^{(l_1)}\cdots u^{(l_j)}\|_{\dot H^\alpha}\\ &\quad\leqslant C(1+\|u\|_{\dot H^\alpha})\|u^{(l_1)}\|_{\dot H^\lambda}\cdots\|u^{(l_j)}\|_{\dot H^\lambda}\quad \textit{for } 0\leqslant\alpha<1/2,\end{aligned}$$

$$(\mathrm{A}.4)\qquad \|f(u)\|_{H^k}\leqslant C(1+\|u\|_{H^k})$$

and, for $0\leqslant\alpha\leqslant k$,

$$\begin{aligned}(\mathrm{A}.5)\qquad &\|f^{(j)}(u)u^{(l_1)}\cdots u^{(l_j)}\|_{H^\alpha}\\ &\quad\leqslant C(1+\|u\|_{H^k})\|u^{(l_1)}\|_{H^k}\cdots\|u^{(l_{j-1})}\|_{H^k}\|u^{(l_j)}\|_{H^\alpha}.\end{aligned}$$

(b) $\|u(t)\|_{\dot H^2}\leqslant K$ *implies*

$$\begin{aligned}(\mathrm{A}.6)\qquad &\|f^{(j)}(u)u^{(l_1)}\cdots u^{(l_j)}\|_{\dot H^\alpha}\\ &\quad\leqslant C\|u^{(l_1)}\|_{\dot H^\lambda}\cdots\|u^{(l_{j-1})}\|_{\dot H^\lambda}\|u^{(l_j)}\|_{\dot H^\alpha}\quad \textit{for } 0\leqslant\alpha<1/2.\end{aligned}$$

Proof. For $0\leqslant\alpha<1/2$ we have $\dot H^\alpha=H^\alpha$ with equivalent norms. Hence,

$$\|u\|_{\dot H^\alpha}\simeq\left[\|u\|^2+\int_\Omega\int_\Omega|u(x)-u(y)|^2/|x-y|^{d+2\alpha}\,dx\,dy\right]^{1/2}$$

for $0<\alpha<1/2$, since the latter expression is equivalent to the H^α-norm. Now (A.2) follows by a simple computation. In the same way we find

$$\|uv\|_{\dot H^\alpha}\leqslant C\|u\|_{C^\gamma}\|v\|_{\dot H^\alpha}$$

for $0\leqslant\alpha<\gamma<1/2$, and (A.6) follows. Together with Sobolev's inequality

$$\|u\|_{C^\gamma}\leqslant C\|u\|_{\dot H^\lambda},\qquad \gamma<\lambda-d/2,$$

and (A.2), this also shows (A.3). A proof of (A.4) can be found in Moser [13, p. 273]. For the proof of (A.5) we recall that, since $k\geqslant 2>d/2$, the H^k-norm is multiplicative,

$$\|uv\|_{H^k}\leqslant C\|u\|_{H^k}\|v\|_{H^k},$$

and that

$$\|uv\|\leqslant C\|u\|_{H^k}\|v\|.$$

Hence,

$$\|uv\|_{H^\alpha}\leqslant C\|u\|_{H^k}\|v\|_{H^\alpha},\qquad 0\leqslant\alpha\leqslant k,$$

by interpolation of the linear operator $v\to uv$ mapping H^k into H^k and L_2 into L_2. This proves (A.5) and the proof is complete.

We shall frequently use the following well-known generalization of Gronwall's lemma, cf. Amann [1].

LEMMA A.3. *Let* $t^*>0$, $0\leqslant\alpha$, $\beta<1$ *and* $A,B\geqslant 0$. *Then there is a positive constant* $C=C(t^*,B,\alpha,\beta)$ *such that*

$$\varphi(t)\leqslant At^{-\alpha}+B\int_0^t(t-s)^{-\beta}\varphi(s)\,ds,\qquad 0<t\leqslant t^*,$$

implies

$$\varphi(t)\leqslant CAt^{-\alpha},\qquad 0<t\leqslant t^*.$$

We are now prepared for the proof of Lemma A.1.

Proof of Lemma A.1. During the course of this proof we shall use the notation $\|u\|_\alpha = \|u\|_{\dot H^\alpha}$. We begin by proving part (a) by induction on k and, at the same time, the particular case $i = 0$ of the estimate (A.1). Then, for fixed α, we shall show that (A.1) holds also for $i = 1, 2$ and 3.

Thus, let $k = 0$, so that $0 \leqslant \alpha \leqslant \beta \leqslant 2$, and assume that $v \in \mathscr{F}_\alpha$. By (2.4) and (2.3) we have for $0 \leqslant \alpha \leqslant \beta < 2$

$$\begin{aligned}\|u(t)\|_\beta &\leqslant \|E(t)v\|_\beta + \int_0^t \|E(t-s)f(u(s))\|_\beta\,ds\\ &\leqslant Ct^{-(\beta-\alpha)/2}\|v\|_\alpha + C\int_0^t (t-s)^{-\beta/2}\|f(u(s))\|\,ds\\ &\leqslant Ct^{-(\beta-\alpha)/2}(1+\|v\|_\alpha).\end{aligned}$$

If $\beta = 2$, we take $0 < \delta < 1/2$ and use (A.2) to get

$$\begin{aligned}\|u(t)\|_2 &\leqslant Ct^{-(2-\alpha)/2}\|v\|_\alpha + C\int_0^t (t-s)^{-(2-\delta)/2}\|f(u(s))\|_\delta\,ds\\ &\leqslant Ct^{-(2-\alpha)/2}\|v\|_\alpha + C\int_0^t (t-s)^{-(2-\delta)/2}(1+\|u(s)\|_\delta)\,ds\\ &\leqslant Ct^{-(2-\alpha)/2}(1+\|v\|_\alpha) + C\int_0^t (t-s)^{-(2-\delta)/2}s^{-\delta/2}\,ds\|v\|\\ &\leqslant Ct^{-(2-\alpha)/2}(1+\|v\|_\alpha),\end{aligned}$$

where we have also used the result just obtained for $\beta < 2$. Altogether, we may conclude that for $0 \leqslant \alpha \leqslant \beta \leqslant 2$ we have

$$\|u(t)\|_\beta \leqslant Ct^{-(\beta-\alpha)/2}(1+\|v\|_\alpha) \leqslant Ct^{-(\beta-\alpha)/2} \tag{A.7}$$

if $F_\alpha(v) \leqslant K$. This is the desired estimate (A.1) for $i = k = 0$.

To show that $u \in C(\bar I, \dot H^\alpha)$, we take $0 \leqslant \delta < 1/2$ such that $0 \leqslant \alpha - \delta < 2$ and obtain similarly

$$\begin{aligned}\|u(t)-v\|_\alpha &\leqslant \|(E(t)-I)v\|_\alpha + C\int_0^t (t-s)^{-(\alpha-\delta)/2}\|f(u(s))\|_\delta\,ds\\ &\leqslant \left\|(E(t)-I)(-\Delta)^{\alpha/2}v\right\| + C\int_0^t (t-s)^{-(\alpha-\delta)/2}(1+s^{-\delta/2})\,ds.\end{aligned}$$

Hence $u(t) \to v$ in $\dot H^\alpha$ as $t \to 0$, if $v \in \mathscr{F}_\alpha$. We have thus proved part (a) for $k = 0$.

For the induction step we let $m \geqslant 1$ and assume that (a) has been proved for $0 \leqslant k \leqslant m-1$. Let $2m \leqslant \alpha \leqslant \beta \leqslant 2(m+1)$ and suppose that $v \in \mathscr{F}_\alpha$. By our induction hypothesis we have $u^{(j)} \in C(\bar I, \dot H^2)$ for $0 \leqslant j \leqslant m-1$, so that in view of (A.6), and since $\sum_1^j l_i = m$ and $l_i \geqslant 1$,

$$\begin{aligned}\|D_t^m f(u)\|_\delta &\leqslant C\sum_{j=1}^m \sum_l \|f^{(j)}(u)u^{(l_1)}\cdots u^{(l_j)}\|_\delta\\ &\leqslant C(1+\|u^{(m)}\|_\delta), \qquad 0 \leqslant \delta < 1/2.\end{aligned}$$

Since

$$u^{(m)} = \Delta u^{(m-1)} + \sum_{j=1}^{m-1}\sum_l c_l f^{(j)}(u)u^{(l_1)}\cdots u^{(l_j)},$$

it also follows that $u^{(m)}(t) \to v_m$ in L_2 as $t \to 0$, because $u^{(j)}(t) \to v_j$ in $\dot{H}^2$, and hence also in L_∞, for $0 \leqslant j \leqslant m-1$. Therefore, we may take $\tau = 0$ in (2.6),

$$u^{(m)}(t) = E(t)v_m + \int_0^t E(t-s)D_s^m f(u(s))\,ds.$$

With $\rho = \alpha - 2m$ the same argument as in the case $k = 0$ now shows that $u^{(m)} \in C(\bar{I}, \dot{H}^\rho)$ and that

$$\text{(A.8)} \quad \|u^{(m)}(t)\|_\sigma \leqslant Ct^{-(\sigma-\rho)/2}\left(1 + \|v_m\|_\rho\right) \leqslant Ct^{-(\sigma-\rho)/2}, \qquad 0 \leqslant \rho \leqslant \sigma \leqslant 2.$$

This contains the desired case of (A.1) for $\sigma = \beta - 2m$ and the induction step is complete.

Having thus proved (A.1) for $i = 0$ and arbitrary α, we shall now fix α, and hence also k, and prove (A.1) for $i = 1, 2$ and 3. The case $0 \leqslant \alpha < 2$, i.e., $k = 0$, is slightly particular and we shall treat it separately.

Consider thus the case $i = 1$, $k = 0$. We shall show

$$\text{(A.9)} \qquad \|u_t(t)\|_\sigma \leqslant C(t-\tau)^{-(\sigma-\rho)/2}\|u_t(\tau)\|_\rho, \qquad 0 \leqslant \rho \leqslant \sigma \leqslant 2.$$

Assume that this has been done. Observing that

$$\|u_t\| = \|\Delta u + f(u)\| \leqslant C(1 + \|u\|_2)$$

and using also (A.7), we have in particular, with $\rho = 0$,

$$\text{(A.10)} \qquad \begin{aligned} \|u_t(t)\|_\sigma &\leqslant C(t/2)^{-\sigma/2}\|u_t(t/2)\| \\ &\leqslant Ct^{-\sigma/2}\left(1 + \|u(t/2)\|_2\right) \leqslant Ct^{-\sigma/2}, \qquad 0 \leqslant \sigma \leqslant 2, \end{aligned}$$

which contains the desired case of (A.1) for $\sigma = \beta - 2$.

For the proof of (A.9) we consider first $0 \leqslant \rho \leqslant \sigma < 2$. Using the trivial estimate

$$\|f'(u)u_t\| \leqslant C\|u_t\| \leqslant C\|u_t\|_\sigma,$$

we obtain from (2.7)

$$\|u_t(t)\|_\sigma \leqslant C(t-\tau)^{-(\sigma-\rho)/2}\|u_t(\tau)\|_\rho + C\int_\tau^t (t-s)^{-\sigma/2}\|u_t(s)\|_\sigma\,ds,$$

and an application of Gronwall's lemma shows (A.9) for $0 \leqslant \rho \leqslant \sigma < 2$. For the remaining case $\sigma = 2$ we take $0 < \delta < 1/2$ and $3/2 < \lambda < 2$ and note that by (A.3), (A.7) and the case of (A.9) just proved,

$$\begin{aligned} \|u_t(t)\|_2 &\leqslant C(t-\tau)^{-(2-\rho)/2}\|u_t(\tau)\|_\rho \\ &\quad + C\int_\tau^t (t-s)^{-(2-\sigma)/2}(s-\tau)^{-(\delta+\max\{0,\lambda-\rho\})/2}\,ds\|u(\tau)\|\,\|u_t(\tau)\|_\rho \\ &\leqslant C(t-\tau)^{-(2-\rho)/2}\|u_t(\tau)\|_\rho, \end{aligned}$$

which completes the proof of (A.9).

For the case $i = 2$, $k = 0$ we set $\sigma = \beta - 4$. We must prove that

$$\text{(A.11)} \qquad \|u_{tt}(t)\|_\sigma \leqslant Ct^{-(4+\sigma-\alpha)/2}$$

for $0 \leqslant \sigma \leqslant 2$, $\max\{0, \sigma - 1\} \leqslant \alpha < 2$ (the lower bound on α is due to the restriction $\beta - \alpha \leqslant 5$). As in the previous case, we consider first $0 \leqslant \sigma < 2$. For this we note that

$$\|u_{tt}\| = \|\Delta u_t + f'(u)u_t\| \leqslant C\|u_t\|_2$$

and that for $3/2 < \lambda < 2$

$$\begin{aligned}\|D_s^2 f(u(s))\| &\leqslant \|f'(u)u_{tt}\| + \|f''(u)u_t^2\| \\ &\leqslant C\|u_{tt}\| + C\|u_t\|_\lambda \|u_t\| \\ &\leqslant C\|u_{tt}(s)\| + C(s-\tau)^{-\lambda/2}\|u_t(\tau)\|^2.\end{aligned}$$

Now the same Gronwall argument as above shows

$$\text{(A.12)} \qquad \|u_{tt}(t)\|_\sigma \leqslant C(t-\tau)^{-\sigma/2}\left(\|u_t(\tau)\|_2 + \|u_t(\tau)\|^2\right), \qquad 0 \leqslant \sigma < 2,$$

and, in view of (A.9), the desired case of (A.11) follows.

For $\sigma = 2$ we choose $0 < \delta < 1/2$ and $3/2 < \lambda < 2$ and (A.3) yields

$$\begin{aligned}\|D_s^2 f(u(s))\|_\delta &\leqslant C(1 + \|u\|_\delta)\left(\|u_{tt}\|_\lambda + \|u_t\|_\lambda^2\right) \\ &\leqslant C(s-\tau)^{-\lambda/2}\left(\|u_t(\tau)\|_2 + \|u_t(\tau)\|_\lambda \|u_t(\tau)\|\right),\end{aligned}$$

where we have also used (A.12) and (A.9) and the fact that $\alpha \geqslant 1$ implies $\|u(s)\|_\delta \leqslant C$ by (A.7). Thus an argument similar to the second part of the proof of (A.9) now shows

$$\begin{aligned}\|u_{tt}(t)\|_2 \leqslant\ & C(t-\tau)^{-1}\|u_t(\tau)\| \\ & + C(t-\tau)^{-(\lambda-\delta)/2}\|u_t(\tau)\|_\lambda \|u_t(\tau)\|.\end{aligned}$$

Taking, e.g., $\lambda = 3/2 + \delta/2$ and using (A.9), this proves the remaining case of (A.11), which completes the proof of (A.1) for $i = 2$, $k = 0$.

For $i = 3$, $k = 0$ we let $\sigma = \beta - 6$. We need only consider $0 \leqslant \sigma \leqslant 1$ and $1 \leqslant \alpha < 2$, in view of the restriction $\beta - \alpha \leqslant 5$. Again take $3/2 < \lambda < 2$. Then, by (A.9) and (A.12),

$$\begin{aligned}\|D_s^3 f(u(s))\| &= \|f'(u)u_{ttt} + 3f''(u)u_{tt}u_t + f'''(u)u_t^3\| \\ &\leqslant C\left\{\|u_{ttt}\| + \|u_{tt}\|_\lambda \|u_t\| + \|u_t\|_\lambda^2 \|u_t\|\right\} \\ &\leqslant C\|u_{ttt}(s)\| + C(s-\tau)^{-\lambda/2}\left\{\|u_t(\tau)\|_2 \|u_t(\tau)\| + \|u_t(\tau)\|_\lambda \|u_t(\tau)\|^2\right\},\end{aligned}$$

and our usual Gronwall argument leads to

$$\|u_{ttt}(t)\|_\sigma \leqslant C(t-\tau)^{-\lambda/2}\left\{\|u_t(\tau)\|_2 \|u_t(\tau)\| + \|u_t(\tau)\|_\lambda \|u_t(\tau)\|^2\right\},$$

which shows

$$\|u_{ttt}(t)\|_\sigma \leqslant C(t-\tau)^{-(6+\sigma-\alpha)/2},$$

since $\alpha \geqslant 1$. This completes the proof of (A.1) for $0 \leqslant \alpha < 2$.

Now let $2k \leqslant \alpha < 2(k+1)$ for some $k \geqslant 1$. We consider first $i = 1$. We shall show

$$\text{(A.13)} \qquad \|u^{(k+1)}(t)\|_\sigma \leqslant C(t-\tau)^{-\sigma/2}\left(1 + \|u^{(k)}(\tau)\|_2\right), \qquad 0 \leqslant \sigma \leqslant 2.$$

With $\rho = \alpha - 2k$ this shows, in view of (A.8),

$$\|u^{(k+1)}(t)\|_\sigma \leqslant Ct^{-(\sigma+2-\rho)/2}, \qquad 0 \leqslant \sigma \leqslant 2,$$

which contains the desired case of (A.1) for $\sigma = \beta - 2(k+1)$.

For the proof of (A.13) we apply (A.6) to get

$$\|D_t^{k+1}f(u)\|_\delta \leqslant C\sum_{j=1}^{k+1}\sum_{l}\|u^{(l_1)}\|_2\cdots\|u^{(l_{j-1})}\|_2\|u^{(l_j)}\|_\delta$$

for $0 \leqslant \delta < 1/2$. Since $\|u^{(j)}(t)\|_2 \leqslant C$ for $j < k$ and $\|u^{(k)}(t)\| \leqslant C$, most of the factors on the right-hand side are bounded. In fact, taking the restriction $\sum_1^j l_i = k+1$ into account, we find that for $k \geqslant 2$

$$\|D_t^{k+1}f(u)\|_\delta \leqslant C\{1 + \|u^{(k+1)}\|_\delta + \|u^{(k)}\|_\delta\}$$

and for $k = 1$

$$\|D_t^{2}f(u)\|_\delta \leqslant C\{\|u^{(2)}\|_\delta + \|u^{(1)}\|_2\|u^{(1)}\|_\delta\}.$$

Similarly,

$$\|u^{(k+1)}\| = \|\Delta u^{(k)} + D_t^k f(u)\| \leqslant C\{1 + \|u^{(k)}\|_2\}.$$

Using these inequalities, we can prove (A.13) by the same procedure as in the proof of (A.9).

For $i = 2$ we argue similarly. For $0 \leqslant \delta < 1/2$ we have

$$\|D_t^{k+2}f(u)\|_\delta \leqslant C\sum_{j=1}^{k+2}\sum_{l}\|u^{(l_1)}\|_2\cdots\|u^{(l_{j-1})}\|_2\|u^{(l_j)}\|_\delta$$

and since $\sum_1^j l_i = k+2$, this gives for $k \geqslant 3$

$$\|D_t^{k+2}f(u)\|_\delta \leqslant C\{1 + \|u^{(k+2)}\|_\delta + \|u^{(k+1)}\|_\delta + \|u^{(k)}\|_\delta\},$$

for $k = 2$

$$\|D_t^{4}f(u)\|_\delta \leqslant C\{1 + \|u^{(4)}\|_\delta + \|u^{(3)}\|_\delta + (1 + \|u^{(2)}\|_2)\|u^{(2)}\|_\delta\},$$

and for $k = 1$

$$\|D_t^{3}f(u)\|_\delta \leqslant C\{\|u^{(3)}\|_\delta + \|u^{(1)}\|_2\|u^{(2)}\|_\delta + \|u^{(1)}\|_2^2\|u^{(1)}\|_\delta\}.$$

Similarly,

$$\|u^{(k+2)}\| \leqslant C\{1 + \|u^{(k)}\|_2\},$$

and in the same way as for u_{tt} we obtain the desired estimate for $u^{(k+2)}$.

Finally, for $i = 3$ we have

$$\|D_t^{k+3}f(u)\| \leqslant C\sum_{j=1}^{k+3}\sum_{l}\|u^{(l_1)}\|_2\cdots\|u^{(l_{j-1})}\|_2\|u^{(l_j)}\|.$$

Similarly to the above, we find that for $k \geqslant 4$

$$\|D_t^{k+3}f(u)\| \leqslant C\{1 + \|u^{(k+3)}\| + \|u^{(k+2)}\| + \|u^{(k+1)}\|\},$$

for $k = 3$

$$\|D_t^{6}f(u)\| \leqslant C\{1 + \|u^{(6)}\| + \|u^{(5)}\| + \|u^{(4)}\| + \|u^{(3)}\|_2\},$$

for $k = 2$

$$\|D_t^{5}f(u)\| \leqslant C\{1 + \|u^{(5)}\| + \|u^{(4)}\| + \|u^{(4)}\|_2 + \|u^{(2)}\|_2\|u^{(2)}\|\},$$

and for $k = 1$

$$\|D_t^{4}f(u)\| \leqslant C\{1 + \|u^{(4)}\| + \|u^{(3)}\|_2 + \|u^{(2)}\|_2\|u^{(2)}\| + \|u^{(1)}\|_2^2\|u^{(2)}\|_2 + \|u^{(1)}\|_2^3\}.$$

Using these inequalities and the previous estimates, we can prove the correct estimate for $u^{(k+3)}$. This completes the proof of Lemma A.1.

Our next lemma will be the main step in converting the special case of Lemma A.1 into the general case of Theorem 2.2.

LEMMA A.4. *Let $\alpha \geqslant 0$ and let $m \geqslant 1$ be an integer. Suppose that*

$$\|u^{(i)}(t)\|_{H^{\sigma-2i}} \leqslant Ct^{-\max\{0,\sigma-\alpha\}/2}, \qquad t \in I, \tag{A.14}$$

for $0 \leqslant \sigma \leqslant 2m$ and $0 \leqslant 2i \leqslant \sigma$. Then

$$\|D_t^i f(u(t))\|_{H^{\sigma-2(i+1)}} \leqslant Ct^{-(\sigma-\alpha)/2}, \qquad t \in I,$$

for $2m \leqslant \sigma \leqslant 2(m+1)$, $0 \leqslant \sigma - \alpha \leqslant 5$ and $0 \leqslant i \leqslant m-1$.

Proof. The idea of the proof is to replace the expression $\sigma - 2(i+1)$ by an integer $k \geqslant \max\{2, \sigma - 2(i+1)\}$. Then we can apply (A.4) and (A.5). For $i = 0$ we take $k = 2m$ and obtain, by (A.4) and (A.14),

$$\|f(u)\|_{H^{\sigma-2}} \leqslant \|f(u)\|_{H^k} \leqslant C(1 + \|u\|_{H^k}) \leqslant Ct^{-\max\{0,k-\alpha\}/2} \leqslant Ct^{-(\sigma-\alpha)/2}.$$

Similarly, for $i = 1$ and $i = 2$ we set

$$k = \begin{cases} \max\{2, 2(m-i)-1\}, & \text{if } 2m \leqslant \sigma < 2m+1, \\ 2(m-i), & \text{if } 2m+1 \leqslant \sigma \leqslant 2(m+1), \end{cases}$$

and (A.5) and (A.14) yield

$$\begin{aligned} \|D_t f(u)\|_{H^{\sigma-4}} &= \|f'(u)u_t\|_{H^{\sigma-4}} \leqslant C(1 + \|u\|_{H^k})\|u_t\|_{H^{\sigma-4}} \\ &\leqslant Ct^{-\max\{0,k-\alpha\}/2 - \max\{0,\sigma-4-\alpha\}/2} \leqslant Ct^{-(\sigma-\alpha)/2} \end{aligned}$$

and

$$\begin{aligned} \|D_t^2 f(u)\|_{H^{\sigma-6}} &\leqslant \|f'(u)u_{tt}\|_{H^{\sigma-6}} + \|f''(u)u_t^2\|_{H^{\sigma-6}} \\ &\leqslant C(1 + \|u\|_{H^k})\left(\|u_{tt}\|_{H^{\sigma-6}} + \|u_t\|_{H^k}\|u_t\|_{H^{\sigma-6}}\right) \\ &\leqslant Ct^{-\max\{0,k-\alpha\}/2}\left(t^{-\max\{0,\sigma-6-\alpha\}/2} + t^{-\max\{0,k-\alpha\}/2 - \max\{0,\sigma-6-\alpha\}/2}\right) \\ &\leqslant Ct^{-(\sigma-\alpha)/2}. \end{aligned}$$

Finally, for $i \geqslant 3$ we may simply take $k = 2(m-i)$,

$$\begin{aligned} &\|D_t^i f(u)\|_{H^{\sigma-2(i+1)}} \\ &\quad \leqslant C(1 + \|u\|_{H^{2(m-i)}}) \sum_{n=1}^{i} \sum_{l} \|u^{(l_1)}\|_{H^{2(m-i)}} \cdots \|u^{(l_n)}\|_{H^{2(m-i)}}. \end{aligned} \tag{A.15}$$

Here, all the factors are of the type $\|u^{(j)}\|_{H^{2(m-i)}}$ with $0 \leqslant j \leqslant i$ and, since $2(m+j-i) \leqslant 2m$, (A.14) gives

$$\|u^{(j)}\|_{H^{2(m-i)}} = \|u^{(j)}\|_{H^{2(m+j-i)-2j}} \leqslant Ct^{-\max\{0,2(m+j-i)-\alpha\}/2}.$$

For $j \leqslant i-3$ we have here $2(m+j-i) - \alpha \leqslant 2m - 6 - \alpha \leqslant \sigma - 6 - \alpha \leqslant -1$, so that all factors in (A.15) involving time derivatives of order less than or equal to $i-3$ are bounded. Taking the restrictions $\sum_1^n l_j = i$ and $l_j \geqslant 1$ into account, we verify that the remaining negative powers of t are no worse than $t^{-(\sigma-\alpha)/2}$, and the proof is complete.

At last we are ready to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. We first note that the theorem holds under the additional assumption that $0 \leqslant \beta - 2j \leqslant 2$. Since $\dot{H}^\sigma \subset H^\sigma$, this follows from (A.1). In order to prove the general case from this particular case, we shall use the inequality

$$\text{(A.16)} \qquad \|u^{(j)}\|_{H^{\beta-2j}} \leqslant C\left\{\|u^{(j+m)}\|_{H^{\beta-2(j+m)}} + \sum_{i=j}^{j+m-1} \|D_t^i f(u)\|_{H^{\beta-2(i+1)}}\right\},$$

valid for $0 \leqslant 2j \leqslant \beta$ and where m is to be chosen such that $0 \leqslant \beta - 2(j+m) \leqslant 2$. We obtain (A.16) easily by repeated use of (2.5) and the elliptic regularity estimate

$$\|u\|_{H^\sigma} \leqslant C\|\Delta u\|_{H^{\sigma-2}}, \qquad u \in H^\sigma \cap H_0^1,\ \sigma \geqslant 2.$$

We first claim that the theorem holds for $2k \leqslant \alpha \leqslant \beta \leqslant 2(k+1)$, where k can be any nonnegative integer. The proof will proceed by induction on k.

If $k = 0$ we have $0 \leqslant \alpha \leqslant \beta \leqslant 2$ and our claim follows from (A.1). For the induction step we let $k \geqslant 1$ and assume that the claim has been proved for $2(k-1) \leqslant \alpha \leqslant \beta \leqslant 2k$. Then let $2k \leqslant \alpha \leqslant \beta \leqslant 2(k+1)$ and $2j \leqslant \beta$. If $j = k$ or $k+1$, then $\beta - 2j \leqslant 2$ and (A.1) applies. Otherwise $j \leqslant k-1$ and we take $m = k - j$ in (A.16), i.e.,

$$\|u^{(j)}\|_{H^{\beta-2j}} \leqslant C\left\{\|u^{(k)}\|_{H^{\beta-2k}} + \sum_{i=j}^{k-1} \|D_t^i f(u)\|_{H^{\beta-2(i+1)}}\right\},$$

where $0 \leqslant \beta - 2k \leqslant 2$, so that (A.1) can be applied to the first term, giving the correct negative power of t. For the remaining terms we note that, by our induction hypothesis,

$$\|u^{(i)}(t)\|_{H^{\sigma-2i}} \leqslant \|u^{(i)}(t)\|_{H^{2(k-i)}} \leqslant C$$

for $\sigma \leqslant 2k$, $2i \leqslant \sigma$. Hence, by Lemma A.4 with $m = k$, we have

$$\|D_t^i f(u(t))\|_{H^{\sigma-2(i+1)}} \leqslant Ct^{-(\sigma-\alpha)/2},$$

for $\alpha \leqslant \beta \leqslant 2(k+1)$, $0 \leqslant i \leqslant k-1$, which completes the induction step.

Assuming next that $2k \leqslant \alpha \leqslant 2(k+1)$ for some $k \geqslant 0$, we thus know that the claim holds for $\alpha \leqslant \beta \leqslant 2(k+1)$ and we proceed to consider the general case $\alpha \leqslant \beta \leqslant \alpha + 5$. Subject to the latter condition, it is clear that β can be no larger than $2k+7$. In a first step we consider $2(k+1) \leqslant \beta \leqslant 2(k+2)$, $2j \leqslant \beta$ and argue analogously to the above. Thus, if $j \leqslant k$, we take $m = k + 1 - j$ in (A.16),

$$\|u^{(j)}\|_{H^{\beta-2j}} \leqslant C\left\{\|u^{(k+1)}\|_{H^{\beta-2(k+1)}} + \sum_{i=j}^{k} \|D_t^i f(u)\|_{H^{\beta-2(i+1)}}\right\},$$

where the first term is taken care of by (A.1). For the remaining terms we recall that we have already proved

$$\|u^{(i)}(t)\|_{H^{\sigma-2i}} \leqslant Ct^{-\max\{0,\sigma-\alpha\}/2}$$

for $0 \leqslant \sigma \leqslant 2(k+1)$, $2i \leqslant \sigma$, and by application of Lemma A.4 with $m = k+1$ we obtain the desired result for $2(k+1) \leqslant \beta \leqslant 2(k+2)$. Repeating this argument twice more, we reach $2(k+2) \leqslant \beta \leqslant 2(k+3)$ and $2(k+3) \leqslant \beta \leqslant 2k+7$, and Theorem 2.2 is proved.

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