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Convergence Theorem for Difference Approximations of Hyperbolic Quasi-Linear Initial-Boundary Value Problems*

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Abstract. Dissipative difference approximations to multi-dimensional hyperbolic quasi-linear initial-boundary value problems are considered. The difference approximation is assumed to be consistent with the differential problem and its linearization should be stable in l_2. A formal asymptotic expansion to the difference solution is constructed. This expansion includes boundary and initial layers. It is proved that the expansion indeed approximates the difference solution to the required order. As a result, the difference solution converges to the differential one as the mesh size h tends to 0.

Introduction. The convergence of difference schemes is considered to be one of the main problems numerical analysis is concerned with. In this context one often quotes Lax's equivalence theorem that "stability is equivalent to convergence" provided the difference approximation is consistent with a well-posed initial value problem. Although the said theorem is stated in a broad setting of continuous semigroups in Banach spaces, it applies only to linear initial value problems. The nonlinear problems require a more detailed treatment. The result one expects here is of the following kind. Suppose the difference scheme is consistent with a well-posed differential problem and the *linearization* of the scheme around the analytic solution is stable in some norm. Then the convergence should follow. The stability really means that a certain a priori estimate is valid. In order to control nonlinear terms one should bound the maximum norm of the solution. Unfortunately, the usual stability estimates for hyperbolic problems are in the l_2-norm. Strang in [6] used a clever idea to overcome this difficulty. He constructed a high-order approximate solution to the difference scheme $u_{\text{ap}} = \sum_{i=0}^{N} u^{(i)}h^i$, where $u^{(i)}$ are smooth functions of the space-time variables and h is the mesh size. The function $u^{(0)}$ is the solution of the original nonlinear differential problem while $u^{(i)}$ for $i \geqslant 1$ are solutions of the linearized differential problem with forcing terms depending on $u^{(j)}$, $j < i$. The approximate solution u_{ap} satisfies the difference equations up to order $O(h^N)$. Thus, one expects that the difference $v = u - u_{\text{ap}}$ between the exact solution of the scheme and the approximate one will be of order $O(h^N)$ in the l_2-norm. For $N > n/2 + 1$, where n is the space-time dimension, this would imply that $v = O(h^{1+\delta})$ in the

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maximum norm. Thus the l_2-norm of quadratic terms like v^2 is negligible compared with $\|v\|_{l_2}$ and the final bound on $\|v\|_{l_2}$ would follow from the stability estimate. Strang applied this idea to difference approximations of initial value problems for quasi-linear hyperbolic systems, i.e., the problem is considered in the whole space or has periodic boundary conditions. In this paper we study the *initial-boundary* value problems in a half space or in a strip. As a rule, the difference approximation requires more boundary conditions than the differential problem. The additional boundary conditions are often called the artificial ones. When the scheme is dissipative, the situation is somewhat similar to the singular perturbations of hyperbolic systems, with the mesh size h playing the role of the viscosity coefficient. As a result, numerical boundary layers develop as h tends to 0. Therefore, there is no smooth approximate solution u_{ap} as in the case of a Cauchy problem since the smooth functions $u^{(i)}$ would not satisfy the artificial boundary conditions. One can, however, circumvent this difficulty by adding boundary layers to the approximate solution. Namely, let us look for a function

$$u_{\mathrm{ap}}(x,h) = \sum_{i=0}^{N} u_{\mathrm{out}}^{(i)}(x)h^i + \sum_{i=1}^{N} u_{\mathrm{bd}}^{(i)}(x_1/h, x_2, \ldots, x_n)h^i, \tag{0.1}$$

where $u_{\mathrm{out}}^{(i)}$ and $u_{\mathrm{bd}}^{(i)}$ are smooth functions of their arguments so that $u_{\mathrm{ap}}(x,h)$ satisfies the difference equations and boundary conditions up to order $O(h^{N+1})$ (here the boundary is $x_1 = 0$). The first sum in (0.1) is called the outer solution while the second is the inner one or the boundary layer. The coefficients $u_{\mathrm{out}}^{(i)}$ and $u_{\mathrm{bd}}^{(i)}$ could be computed using the technique of singular perturbations. In the case of multi-level difference schemes there are also artificial initial conditions, so that one has to add to u_{ap} an initial layer $\sum_{i=1}^{N} u_{\mathrm{in}}^{(i)}(x_1, x_2, \ldots, x_{n-1}, x_n/h)h^i$, where x_n is the time direction. Such initial layers develop also in Cauchy problems (this is the reason Strang considered only two level schemes). It is indeed essential that the boundary and initial layer are weak, i.e., of order h. Otherwise, it would be impossible to construct the approximate solution, let alone prove the estimate for the difference $v = u - u_{\mathrm{ap}}$. We will see that the weakness of the layers follows from the consistency assumption. The proof of convergence then proceeds as in [6]. The above approach requires a considerable smoothness of the data and of the analytic solution $u^{(0)}$. As mentioned before, N should be greater than $n/2 + 1$. The functions $u_{\mathrm{out}}^{(i)}$ are solutions of linear hyperbolic systems with forcing terms depending on the derivatives $D^\alpha u_{\mathrm{out}}^{(j)}$, $|\alpha| + j \leqslant i + 1$. Since there is a loss of derivative in hyperbolic problems, in order for $u_{\mathrm{out}}^{(N)}$ to be in C^1 the function $u^{(0)}$ should belong to a Sobolev space of order greater than $2N + (n+1)/2 > 3n/2 + 5/2$. An alternative approach to the convergence problem is to derive a linear stability estimate in a discrete Sobolev space of order greater than $n/2$. Actually, with weak boundary layers one may expect one bounded (numerical) derivative in the directions normal to the boundaries and an unlimited number of tangential derivatives (e.g., see [4]). In the framework of the stability theory in [3] such an estimate indeed could be derived, however the proof is lengthy and very technical. The high-order Sobolev norm bounds the maximum norm of the function and thus controls terms like $O(u - u_0)^2$ in the convergence proof. The optimal smoothness requirements would be that $u^{(0)}$ and the data belong to a Sobolev space of order greater than $n/2 + 2$.

It is also essential that the scheme is *dissipative* in the directions normal to the boundaries since otherwise the boundary layers would not decrease exponentially in these directions. For example, the proof is not valid for the leap-frog scheme. Note also that for multi-dimensional problems the only schemes for which a general stability criteria was proved are the dissipative ones (see [3]).

1. The Difference Scheme and the Approximate Solution. Let $u_h(x) \in R^d$ be a grid vector function defined on a uniform mesh Ω_h with a step size h in the domain

$$\Omega = \{x = (x_1, x_2, \ldots, x_n) \in R^n \,|\, 0 \leqslant x_1 < \infty, 0 \leqslant x_n \leqslant T\}.$$

Consider a difference system

$$(1.1) \qquad L(\{E_u^\alpha u_h(x)\}_{\alpha \in \mathscr{A}}, x, h) = 0, \qquad x \in \Omega_h.$$

Here α is a multi-index belonging to a finite set $\mathscr{A}$,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathscr{A}$$
$$= \{\alpha \in Z^n \,|\, 0 \leqslant \alpha_1 \leqslant \alpha_1^* + 1, 0 \geqslant \alpha_n \geqslant -\alpha_n^* - 1, |\alpha_i| \leqslant \alpha_i^* \text{ for } 2 \leqslant i < n\},$$
$$E^\alpha u_h(x) = E_{x_1}^{\alpha_1} \cdot E_{x_2}^{\alpha_2} \cdot \cdots \cdot E_{x_n}^{\alpha_n} u_h(x) = u_h(x + \alpha h)$$

is a shift operator and L is a smooth real vector function of dimension d which depends smoothly on its variables (the precise order of smoothness will be specified later in Remark 1.3). The system in (1.1) is augmented by boundary conditions

$$(1.2) \quad S(\{E^\alpha u_h(x)\}_{\alpha \in \mathscr{A}_{\text{bd}}}, x, h) = 0, \qquad x \in \Omega_{\text{bd},h} = \{x \in \Omega_h \,|\, x_1 = 0\},$$

where

$$\mathscr{A}_{\text{bd}} = \{\alpha \in \mathscr{A} \,|\, 0 \leqslant \alpha_1 \leqslant \alpha_1^*\},$$

and by initial conditions

$$(1.3) \quad E_{x_n}^{\alpha_n} u_h(x) = f_{\alpha_n}(x, h), \qquad 0 \geqslant \alpha_n \geqslant -\alpha_n^*, \; x \in \Omega_{\text{in},h} = \{x \in \Omega \,|\, x_n = 0\}.$$

Note that the specific form of the sets $\mathscr{A}$ and $\mathscr{A}_{\text{bd}}$ does not pose a restriction, since one can always add to L and S dummy variables $E^\alpha u_h(x)$. However, the lower bound $-\alpha_n^* - 1$ of α_n should agree with the number of the initial layers in (1.3), so that the problem (1.1)–(1.3) is solvable. Now let us state the

Consistency Assumption. (i) For smooth functions $u(x)$,

$$(1.4) \qquad L(\{E^\alpha u(x)\}, x, h) = h \cdot \mathscr{L}(u) + O(h^2),$$

where

$$(1.5) \qquad \mathscr{L}(u) = \sum_{j=0}^{n} A_j(u, x) D_{x_j} u + B(u, x).$$

(ii) There exists a smooth function $u^{(0)}$ which belongs to $C^r([0, T], H^{s-r}(\Omega_{\text{in}}))$ for all $0 \leqslant r \leqslant s$ with $s \geqslant 3[n/2 + 2]$ and which satisfies Eq. (1.1) up to order $O(h^2)$ and Eqs. (1.2) and (1.3) up to order $O(h)$. In other words,

$$(1.6) \qquad \sum_{j=1}^{n} A_j(u^{(0)}(x), x) D_{x_j} u^{(0)}(x) + B(u^{(0)}(x), x) = 0, \qquad x \in \Omega,$$

$$(1.7) \qquad S(\{u^{(0)}(x)\}, x) = 0, \qquad x \in \Omega_{\text{bd}} = \{x \in \Omega \,|\, x_1 = 0\},$$

$$(1.8) \quad u^{(0)}(x) = f_{\alpha_n}(x, 0), \qquad 0 \geqslant \alpha_n \geqslant -\alpha_n^*, \; x \in \Omega_{\text{in}} = \{x \in \Omega \,|\, x_n = 0\}.$$

Until additional assumptions are imposed on the matrices $A_j(u, x)$, the condition in (1.4) merely states that the operator $L(\{E^\alpha u\}, x, 0)$ vanishes on constant grid functions.

Let us linearize the problem in (1.1)–(1.3) at the smooth function $u^{(0)}$. Namely, define the matrix functions

$$L_\beta(x, h) = \partial L(\{E^\alpha u(x)\}, x, h)/\partial E^\beta u(x)\,|_{u=u^{(0)}}. \tag{1.9}$$

Then the linearization of L at $u^{(0)}$ is

$$dL[u^{(0)}] = \sum_\alpha L_\alpha(x, h) E^\alpha, \qquad \alpha \in \mathscr{A}. \tag{1.10}$$

Similarly is defined

$$dS[u^{(0)}] = \sum_\alpha S_\alpha(x, h) E^\alpha, \qquad \alpha \in \mathscr{A}_{\text{bd}}. \tag{1.11}$$

The linearized initial-boundary value problem is

$$\begin{aligned} &\text{(i)} && dL[u^{(0)}]v(x) = hF(x), && x \in \Omega_h,\\ &\text{(ii)} && dS[u^{(0)}]v(x) = g(x), && x \in \Omega_{\text{bd},h},\\ &\text{(iii)} && E_{x_n}^{\alpha_n} v(x) = f_{\alpha_n}(x), && 0 \geqslant \alpha_n \geqslant -\alpha_n^*,\ x \in \Omega_{\text{in},h}. \end{aligned} \tag{1.12}$$

There are several definitions of stability. The one used in [3] is

Definition 1.1. The problem in (1.12) with zero initial conditions is stable if there exist constants $K_0 > 0$, $h_0 > 0$ and $\eta_0 \geqslant 0$ such that for any $0 < h < h_0$ and any grid functions $F \in l_2(\Omega_h)$, $g \in l_2(\Omega_{\text{bd},h})$ there exists a unique solution $u \in l_2(\Omega_h)$ which satisfies the estimate

$$\eta \| e^{-\eta x_n} v \|_{\Omega_h}^2 + \sum_{\alpha_1 = 0}^{\alpha_1^*} \| E_{x_1}^{\alpha_1} e^{-\eta x_n} v \|_{\Omega_{\text{bd},h}}^2 \leqslant K_0 \Big(\eta^{-1} \| e^{-\eta x_n} F \|_{\Omega_h}^2 + \| e^{-\eta x_n} g \|_{\Omega_{\text{bd},h}}^2 \Big) \tag{1.13}$$

for all $\eta_0 < \eta \leqslant 1/h$.

Here $\|v\|_{\Omega_h}^2 = \sum_{x \in \Omega_h} |v(x)|^2 h^n$ is the weighted l_2-norm over the space Ω_h and similarly for the norm $\|\cdot\|_{\Omega_{\text{bd},h}}$. With $\eta = 1/h$, estimate (1.13) implies the solvability of the problem in (1.12). Namely, for grid functions w defined on the mesh $\Omega_{\text{in},h}$ the mapping

$$w \to \big(dL^{(0)}[u^{(0)}]w,\ dS^{(0)}[u^{(0)}]w \big) \tag{1.14}$$

is an isomorphism from $l_2(\Omega_{\text{in},h})$ onto $l_2(\Omega_{\text{in},h}) \times l_2(\Omega_{\text{in},h} \cap \Omega_{\text{bd},h})$ with the estimate

$$\sum |w(x)|^2 \leqslant K_0 \Big(\sum |dL^{(0)} w(x)|^2 + \sum |dS^{(0)} w(x)|^2 \Big) \tag{1.15}$$

holding uniformly for all $0 < h < h_0$. Here,

$$dL^{(0)}[u^{(0)}] = \sum_\alpha L_\alpha(x, h) E^\alpha, \qquad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 0) \in \mathscr{A} \tag{1.16}$$

is the restriction of $dL[u^{(0)}]$ to the upper time level and similarly for $dS^{(0)}$. The summation in (1.15) is carried over the appropriate domains of the argument x. Estimate (1.13) was proved in [3] for dissipative difference approximations of *strictly hyperbolic* problems, provided the so-called uniform Kreiss condition is satisfied.

There are well-posed problems for which the Kreiss condition does not hold uniformly. In such a case one may hope that a weaker estimate

$$(1.17)\qquad \eta\|e^{-\eta x_n}v\|^2_{\Omega_h} \leqslant K_0\left(\eta^{-1}\|e^{-\eta x_n}F\|^2_{\Omega_h} + (\eta h)^{-1}\|e^{-\eta x_n}g\|^2_{\Omega_{\mathrm{bd},h}}\right)$$

is valid (e.g., see [1]). Note that like (1.13), the above estimate with $\eta h = 1$ implies the solvability condition (1.15).

We will need also the following

Remark 1.1. Suppose that the problem in (1.12) is stable in the sense of estimates (1.13) or (1.17). Let us perturb the coefficients of dL and dS by order $O(h)$. Then the perturbed problem is also stable in the sense of the same estimates with, possibly, larger constants η_0 and K_0.

Next, we assume that the linear operator $dL[u^{(0)}]$ is dissipative in the directions normal to the boundaries, i.e., in the directions x_1 and x_n. More precisely, define difference operators dL_{bd} and dL_{in} by the equalities

$$(1.18)\qquad (E_{x_1} - I)dL_{\mathrm{bd}} = \sum_{\alpha\in\mathscr{A}} L_\alpha(x,0)E_{x_1}^{\alpha_1}, \qquad x \in \Omega_{\mathrm{bd}},$$

and

$$(1.19)\qquad (E_{x_n} - I)\,dL_{\mathrm{in}} = \sum_{\alpha\in\mathscr{A}} L_\alpha(x,0)E_{x_n}^{\alpha_n}, \qquad x \in \Omega_{\mathrm{in}}.$$

By the consistency assumption in (1.4) the sum $\sum_{\alpha\in\mathscr{A}} L_\alpha(x,0)$ is zero and hence the operators in the right-hand sides of (1.18) and (1.19) are indeed divisible by $(E_{x_1} - 1)$ and $(E_{x_n} - I)$, respectively.

Dissipativity Assumption. The operators dL_{bd} and dL_{in} do not have eigenvalues on the unit circle.

This is the same as to say that the equations

$$(1.20)\qquad \det\left(\sum_{\alpha\in\mathscr{A}} L_\alpha(x,0)z^{\alpha_1}\right) = 0, \qquad x \in \Omega_{\mathrm{bd}},$$

and

$$(1.21)\qquad \det\left(\sum_{\alpha\in\mathscr{A}} L_\alpha(x,0)z^{\alpha_n}\right) = 0, \qquad x \in \Omega_{\mathrm{in}},$$

do not have solutions with $|z| = 1$ but $z = 1$, and that in addition the matrices $A_1(u^{(0)}, x)$ for $x \in \Omega_{\mathrm{bd}}$ and $A_n(u^{(0)}, x)$ for $x \in \Omega_{\mathrm{in}}$ are nonsingular.

Now let us start to construct the approximate solution u_{ap} for the problems (1.1)–(1.3). As mentioned in the introduction,

$$(1.22)\qquad \begin{aligned} u_{\mathrm{ap}}(x,h) = \sum_{i=0}^{N} u_{\mathrm{out}}^{(i)}(x)h^i + \sum_{i=1}^{N} u_{\mathrm{bd}}^{(i)}(x_1/h, x_2,\ldots,x_n)h^i \\ + \sum_{i=1}^{N} u_{\mathrm{in}}^{(i)}(x_1,\ldots,x_{n-1},x_n/h)h^i, \end{aligned}$$

where $u_{\mathrm{out}}^{(0)} = u^{(0)}$. In order to substitute $u_{\mathrm{ap}}(x,h)$ into (1.1) we first expand

$$(1.23)\qquad \begin{aligned} E^\alpha u_{\mathrm{out}}^{(i)}(x) = \left(\sum_{k=0}^{N-i} \frac{(\alpha D)^k}{k!}h^k\right) u_{\mathrm{out}}^{(i)}(x) \\ + \frac{(\alpha D)^{N-i+1}}{(N-i+1)!}u_{\mathrm{out}}^{(i)}(x+\theta h)h^{N-i+1}, \qquad i \geqslant 0, \end{aligned}$$

and

$$E^{\alpha}u_{\text{bd}}^{(i)}(x_1/h, x_{\tan}) = \left(\sum_{k=0}^{N-i} \frac{(\alpha_{\tan}\cdot D_{\tan})^k}{k!} h^k\right) E_{x_1}^{\alpha_1} u_{\text{bd}}^{(i)}(x_1/h, x_{\tan}) \tag{1.24}$$
$$+ \frac{(\alpha_{\tan}\cdot D_{\tan})^{N-i+1}}{(n-i+1)!} h^{N-i+1} E_{x_1}^{\alpha_1} u_{\text{bd}}^{(i)}(x_1/h, x_{\tan}+\theta h), \qquad i > 1,$$

where $\alpha_{\tan}\cdot D_{\tan} = \sum_{j=2}^{n}\alpha_j D_{x_j}$, and use a similar expansion for $u_{\text{in}}^{(i)}$. Then L is expanded around the point $\{E^{\alpha}u_h(x)\} = \{u^{(0)}(x)\}$, $h = 0$, in powers of $E^{\alpha}u_h(x) - u^{(0)}(x)$ and h up to order $N+1$, and $E^{\alpha}u_h(x) - u^{(0)}(x)$ is replaced by an expansion of $E^{\alpha}u_{\text{ap}}(x,h) - u^{(0)}(x)$ corresponding to (1.23)–(1.24). Combining the terms according to the powers of h gives

$$L\left(\{E^{\alpha}u_{\text{ap}}(x,h)\}, x, h\right) = \sum_{i=1}^{N} L^{(i)}(x, x_1/h, x_n/h)\cdot h^i + O(h^{N+1}), \tag{1.25}$$

where

$$|O(h^{N+1})| \leqslant Kh^{N+1}, \tag{1.26}$$

provided the functions $u_{\text{out}}^{(i)}$, $0 \leqslant i \leqslant N$, belong to $C^{N-i+1}(\Omega)$, $u_{\text{bd}}^{(i)}$, $1 < i \leqslant N$, belong to $C^{N-i+1}(\Omega_{\text{bd}})$ uniformly in x_1/h and similarly for $u_{\text{in}}^{(i)}$. Note that by the Consistency Assumption the expansion in (1.25) indeed starts with a first-order term $L^{(1)}h$, which is effected only by the boundary and initial layers $u_{\text{bd}}^{(1)}$ and $u_{\text{in}}^{(1)}$ (see (1.28) below). For $i > 1$ the function $u_{\text{out}}^{(i-1)}$ contributes to $L^{(i)}$ as

$$\sum_{\alpha\in\mathscr{A}} \left(L_{\alpha}(x,0)(\alpha D) + \partial_h L_{\alpha}(x,0)\right) u_{\text{out}}^{(i-1)} = d\mathscr{L}[u^{(0)}]u_{\text{out}}^{(i-1)}$$
$$= \left[\sum_{j=0}^{n} A_j(u^{(0)}(x),x)D_{x_j} + \sum_{j=0}^{n} \partial_u A_j(u^{(0)}(x),x)D_{x_j}u^{(0)}(x) + \partial_u B(u^{(0)}(x),x)\right] u_{\text{out}}^{(i-1)} \tag{1.27}$$

and $u_{\text{out}}^{(k)}$ with $k \geqslant i$ does not enter $L^{(i)}$. On the other hand, for $i \geqslant 1$ the highest term $u_{\text{bd}}^{(k)}$ to enter $L^{(i)}$ is $u_{\text{bd}}^{(i)}$ and its contribution is

$$\sum_{\alpha\in\mathscr{A}} L_{\alpha}(x,0)E_{x_1}^{\alpha_1}u_{\text{bd}}^{(i)}(x_1/h, x_{\tan}). \tag{1.28}$$

The other terms in $L^{(i)}$ will be separated into four parts. The first one is

$$F_{\text{out}}^{(i)} = \sum \operatorname{coef}(u_0)\prod D_x^{\beta}u_{\text{out}}^{(j)}, \qquad j \leqslant i-2, \tag{1.29}$$

where the sum $\sum(|\beta| + j)$ of the indices in the product does not exceed i. Here and elsewhere, coef(u_0) is a generic notation for partial derivatives of L or S at the point $\{E^{\alpha}u_h\} = \{u^{(0)}\}$, $h = 0$. The second part is

$$\tilde{F}_{\text{bd}}^{(i)} = \sum \operatorname{coef}(u_0)\left(\prod D_x^{\beta}u_{\text{out}}^{(j)}\right)\cdot\left(\prod D_{x_{\tan}}^{\gamma_{\tan}}E_{x_1}^{\gamma_1}u_{\text{bd}}^{(k)}\right), \qquad k \leqslant i-1, \tag{1.30}$$

where

$$\sum(|\beta| + j) + \sum\left(|\gamma_{\tan}| + k\right) \leqslant i, \qquad \sum k > 0. \tag{1.31}$$

This part is obviously absent when $i = 1$. The third part $\tilde{F}^{(i)}_{\text{in}}$ is of the same type as (1.30) with $u^{(k)}_{\text{bd}}$ replaced by $u^{(k)}_{\text{in}}$, while the fourth part $R^{(i)}_{\text{bd,in}}$ is a sum of products where both u_{bd} and u_{in} participate. Now, we balance the smooth part $F^{(i)}_{\text{out}}$ with (1.27), i.e.,

$$(1.32)\qquad d\mathscr{L}[u^{(0)}]u^{(i)}_{\text{out}} = -F^{(i+1)}_{\text{out}}, \qquad x \in \Omega,\ i \geqslant 1.$$

In $\tilde{F}^{(i)}_{\text{bd}}$ two scales x_1 and x_1/h are present. Thus, in the smooth coefficients

$$f(x_1, x_{\tan}) = \operatorname{coef}(u_0) \prod D^{\beta} u^{(j)}_{\text{out}}$$

we write $x_1 = h \cdot (x_1/h)$ and expand

$$(1.33)\qquad f(x_1, x_{\tan}) = \sum_{k=0}^{N-i} \frac{D^k_{x_1}}{k!} f(0, x_{\tan}) \left(\frac{x_1}{h}\right)^k \cdot h^k + O\left(x_1^{N-i+1}\right)$$

around $x = (0, x_{\tan})$. A similar expansion is performed for $L_\alpha(x, 0)$ in (1.28). We will see below that $u^{(i)}_{\text{bd}}$ is of the following form,

$$(1.34)\qquad u^{(i)}_{\text{bd}}(x_1/h, x_{\tan}) = \sum_{k,l} e^{-\lambda_{ik} x_1/h} \cdot (x_1/h)^l f_{ikl}(x_{\tan}),$$

where

$$\lambda_{ik} > \delta > 0, \quad l \leqslant i-1, \quad f_{ikl} \in C^{N-i+1}(\Omega_{\text{bd}}).$$

Thus, the contribution of $O(x_1^{N-i+1})$ to L is bounded by

$$(1.35)\qquad h^i \cdot K x_1^{N-i+1} \cdot (x_1/h)^l e^{-\delta x_1/h} \leqslant K_1 h^{N+1}.$$

With the expansion in (1.33) substituted into $\tilde{F}^{(i)}_{\text{bd}}$ and $L_\alpha(x, 0)$, the sum

$$\sum_{i=1}^{N} \left[\sum_{\alpha} L_\alpha(x, 0) E^{\alpha_1}_{x_1} u^{(i)}_{\text{bd}} + \tilde{F}^{(i)}_{\text{bd}} \right] h^i$$

is recombined into

$$(1.36)\qquad \sum_{i=1}^{N} (E_{x_1} - I)\left[dL_{\text{bd}} u^{(i)}_{\text{bd}} + F^{(i)}_{\text{bd}}\right] h^i + O(h^{N+1}),$$

where $F^{(i)}_{\text{bd}}$ has the form of the right-hand side in (1.34). Here we have used (1.18) and the fact that

$$(E_{x_1} - I)^{-1} e^{\lambda x_1/h} \cdot (x_1/h)^l = e^{-\lambda x_1/h} \sum_{k=0}^{l} c_{lk}(\lambda)(x_1/h)^k.$$

Thus, the boundary layer satisfies the equation

$$(1.37)\qquad dL_{\text{bd}} u^{(i)}_{\text{bd}} = -F^{(i)}_{\text{bd}}.$$

Similarly, for the initial layer

$$(1.38)\qquad dL_{\text{in}} u^{(i)}_{\text{in}} = -F^{(i)}_{\text{in}}.$$

We will prove later that under the compatibility conditions, the boundary and initial layers are negligibly small near the space-time corner. More specifically,

$$(1.39)\qquad D^j_{x_n} u^{(i)}_{\text{bd}} = 0 \quad \text{at } x_n = 0 \quad \text{and} \quad D^j_{x_1} u^{(i)}_{\text{in}} = 0 \quad \text{at } x_1 = 0 \quad \text{for } j \leqslant N-i.$$

Then, by (1.34), the interaction of the boundary and initial layers,

$$(1.40)\qquad \sum_{i=1}^{N} R^{(i)}_{\text{bd,in}} h^i = O(h^{N+2}),$$

is a negligible term.

The boundary operator S is treated in a way similar to L. Define

$$(1.41)\quad dS_{\mathrm{out}} = \sum_{\alpha} S_\alpha(x,0), \quad dS_{\mathrm{bd}} = \sum_{\alpha} S_\alpha(x,0) E_{x_1}^{\alpha_1}, \qquad \alpha \in \mathscr{A}_{\mathrm{bd}},\ x_1 = 0.$$

Then one can write

$$(1.42)\qquad S(\{E^\alpha u_{\mathrm{ap}}\}, x, h) = \sum_{i=1}^{N} S^{(i)}(x_{\mathrm{tan}}) h^i + O(h^{N+1}).$$

By (1.7) there is indeed no zero-order term in the expansion. The coefficients $S^{(i)}$ have the form

$$(1.43)\qquad S^{(i)}(x_{\mathrm{tan}}) = dS_{\mathrm{out}} u_{\mathrm{out}}^{(i)}(x) + dS_{\mathrm{bd}} u_{\mathrm{bd}}^{(i)}(x) + g^{(i)}(x), \qquad x_1 = 0,$$

where

$$(1.44)\qquad g^{(i)}(x) = \sum \mathrm{coef}(u_0) \prod D_x^\beta u_{\mathrm{out}}^{(j)} E_{x_1}^{\gamma_1} D_{\mathrm{tan}}^{\gamma_{\mathrm{tan}}} u_{\mathrm{bd}}^{(k)}$$

and the indices in each product are bounded as

$$(1.45)\qquad \sum (|\beta| + j) + \sum (|\gamma_{\mathrm{tan}}| + k) \leqslant i, \qquad j \leqslant i-1,\ k \leqslant i-1.$$

In view of (1.39), the contribution of the initial layer u_{in} to the boundary operator is an $O(h^{N+1})$ term and is absorbed in the remainder of (1.42). Altogether, this remainder is bounded as in (1.26), provided the mentioned smoothness conditions hold for $u_{\mathrm{out}}^{(i)}$, $u_{\mathrm{bd}}^{(i)}$ and $u_{\mathrm{in}}^{(i)}$. Note that unlike $L^{(i)}$ there are no different scales in $S^{(i)}$. The resulting boundary condition

$$(1.46)\qquad dS_{\mathrm{out}} u_{\mathrm{out}}^{(i)}(x) + dS_{\mathrm{bd}} u_{\mathrm{bd}}^{(i)}(x) = -g^{(i)}(x), \qquad x \in \Omega_{\mathrm{bd}},$$

couples together the outer solution $u_{\mathrm{out}}^{(i)}$ with the boundary layer $u_{\mathrm{bd}}^{(i)}$. However, with the aid of Eq. (1.37) one can decouple the boundary condition (1.46). Namely, for a grid function w: $\mathbf{Z}_+ \to R^d$, denote by $\bar{w}(0)$ the vector

$$(1.47)\qquad \bar{w}(0) = (w(0), w(1), \dots, w(\alpha_1^*)).$$

Let $V(x_{\mathrm{tan}})$ be the space of vectors $\bar{w}(0)$ corresponding to the exponentially decreasing solutions of the equation $dL_{\mathrm{bd}} w = 0$ at fixed x_{tan} (and with $E_{x_1} w(j) = w(j+1)$).

Since dL_{bd} is a difference operator in the x_1-direction with coefficients independent of x_1, such a space could easily be constructed. By the dissipativity assumption the space $V(x_{\mathrm{tan}})$ depends smoothly on x_{tan}. Given a grid function F in the x_1-direction, denote by $dL_{\mathrm{bd}}^{-1}(x_{\mathrm{tan}})F$ the solution w of the problem

$$(1.48)\qquad dL_{\mathrm{bd}} w = F, \qquad \bar{w}(0) \perp V(x_{\mathrm{tan}}).$$

Thus, the vector $dS_{\mathrm{bd}} u_{\mathrm{bd}}^{(i)}(x)$ in (1.46) is equal to $dS_{\mathrm{bd}} \cdot dL_{\mathrm{bd}}^{-1}(x_{\mathrm{tan}})(-F_{\mathrm{bd}}^{(i)})$ modulo the space

$$(1.49)\qquad W(x_{\mathrm{tan}}) = dS_{\mathrm{bd}} \cdot V(x_{\mathrm{tan}}).$$

Let $P_{\mathrm{bd}}(x_{\mathrm{tan}})$ be a projector which acts on the space of the vectors $g^{(i)}(x)$ and whose kernel is the subspace $W(x_{\mathrm{tan}})$. Apply P_{bd} to (1.46). The resulting boundary condition is

$$(1.50)\qquad \begin{aligned} &P_{\mathrm{bd}}(x_{\mathrm{tan}})\, dS_{\mathrm{out}} u_{\mathrm{out}}^{(i)}(x) \\ &\quad = P_{\mathrm{bd}}(x_{\mathrm{tan}}) \cdot (-g^{(i)}(x) + dS_{\mathrm{bd}} \cdot dL_{\mathrm{bd}}^{-1}(x_{\mathrm{tan}}) F_{\mathrm{bd}}^{(i)}), \qquad x = (0, x_{\mathrm{tan}}). \end{aligned}$$

The initial conditions are treated in the same manner as the boundary ones. The functions $u_{\text{out}}^{(i)}(x)$ and $u_{\text{in}}^{(i)}(x)$ satisfy the initial conditions

$$(1.51)\qquad u_{\text{out}}^{(i)}(x)+E_{x_n}^{\alpha_n}u_{\text{in}}^{(i)}(x)=f_{\alpha_n}^{(i)}(x),\qquad x\in\Omega_{\text{in}},\ 0\geqslant\alpha_n\geqslant-\alpha_n^*,$$

where

$$(1.52)\qquad f_{\alpha_n}^{(i)}(x)=\frac{1}{i!}D_h^i f_{\alpha_n}(x,0)+\sum_{j=1}^{i}\frac{1}{j!}D_{x_n}^j u_{\text{out}}^{(i-j)}(x).$$

Thus, the approximate solution u_{ap} satisfies the initial conditions

$$(1.53)\qquad E_{x_n}^{\alpha_n}u_{\text{ap}}(x,h)=f_{\alpha_n}(x,h)+O(h^{N+1}),\qquad x\in\Omega_{\text{in}},\ 0\geqslant\alpha_n\geqslant-\alpha_n^*$$

with $O(h^{N+1})$ bounded by Kh^{N+1} as in (1.26). The separated initial conditions for $u_{\text{out}}^{(i)}$ are

$$(1.54)\qquad \begin{aligned} &P_{\text{in}}(x)\cdot\left(u_{\text{out}}^{(i)}(x),\dots,u_{\text{out}}^{(i)}(x)\right)\\ &\quad=P_{\text{in}}(x)\left[\left(f_{-\alpha_n^*}^{(i)}(x),\dots,f_0^{(i)}(x)\right)+dL_{\text{in}}^{-1}(x)F_{\text{in}}^{(i)}\right],\qquad x\in\Omega_{\text{in}}.\end{aligned}$$

Note that the stability of the problem (1.12) as stated in (1.13) or in (1.17), together with the Dissipativity Assumption, implies that the characteristic equation for dL_{in} has solutions only in the disc $|z|<1$. Thus, the space of the decreasing solutions of the equations $dL_{\text{in}}\cdot w=0$ has dimension $\alpha_n^*\times d$ and the image of P_{in} has dimension d. Since constant grid functions are not eigenfunctions of dL_{in}, Eq. (1.54) uniquely defines the vector $u_{\text{out}}^{(i)}(x)$, $x\in\Omega_{\text{in}}$, for all values of $f_{\alpha_n}^{(1)}(x)$ and $F_{\text{in}}^{(1)}$. Then (1.51) provides the necessary initial conditions for $u_{\text{in}}^{(i)}$ in order to solve Eq. (1.38).

The functions $u_{\text{out}}^{(i)}$, $u_{\text{bd}}^{(i)}$ and $u_{\text{in}}^{(i)}$ are computed in the following order. First one solves the initial-boundary value problem in (1.32), (1.50), (1.54) for $u_{\text{out}}^{(1)}$. With $u_{\text{out}}^{(1)}(x)$ known at the boundary Ω_{bd}, one obtains from (1.46) the necessary boundary conditions to solve Eq. (1.37) for $u_{\text{bd}}^{(1)}$. Similarly one computes $u_{\text{in}}^{(1)}$. Then the same loop is repeated for $i=2,3,\dots,N$. Obviously, one should assume that the initial-boundary value problem

$$(1.55)\qquad \begin{aligned} &\text{(i)}\quad d\mathscr{L}[u^{(0)}]v=f, && x\in\Omega,\\ &\text{(ii)}\quad P_{\text{bd}}\,dS_{\text{out}}v=g, && x\in\Omega_{\text{bd}},\\ &\text{(iii)}\quad v=f, && x\in\Omega_{\text{in}}\end{aligned}$$

is well posed. The usual definition of well-posedness for a hyperbolic initial-boundary value problem requires the a priori estimate

$$(1.56)\qquad \sup_{\tau_1\leqslant x_n\leqslant\tau_2}\|v(\cdot,x_n)\|_{\Omega_{\text{in}}}\leqslant K\left(\|F\|_{\Omega(\tau_1,\tau_2)}+\|v(\cdot,\tau_1)\|_{\Omega_{\text{in}}}\right)$$

for all solutions of (1.55) with $g=0$ and all subintervals $[\tau_1,\tau_2]\subset[0,T]$, and a similar estimate for the adjoint problem. A more strict definition of Kreiss [2] requires

$$(1.57)\qquad \begin{aligned} &\sup_{\tau_1\leqslant x_n\leqslant\tau_2}\|v(\cdot,x_n)\|_{\Omega_{\text{in}}}+\|v(0,\cdot)\|_{\Omega_{\text{bd}}(\tau_1,\tau_2)}\\ &\quad\leqslant K\left(\|F\|_{\Omega(\tau_1,\tau_2)}+\|g\|_{\Omega_{\text{bd}}(\tau_1,\tau_2)}+\|v(\cdot,\tau_1)\|_{\Omega_{\text{in}}}\right)\end{aligned}$$

for the problem in (1.55) and a similar estimate for the adjoint problem. Here and above, $\Omega(\tau_1, \tau_2) = \{x \in \Omega \mid \tau_1 \leqslant x_n \leqslant \tau_2\}$, and similarly for $\Omega_{\text{bd}}(\tau_1, \tau_2)$. Our next assumption is that the map

$$dS_{\text{bd}}\colon V(x_{\tan}) \to W(x_{\tan}) \text{ is an isomorphism.} \tag{1.58}$$

Thus, one can solve (1.37) and (1.46) uniquely for $u_{\text{bd}}^{(i)}$ provided $u_{\text{out}}^{(i)}$ satisfies (1.50). The simplest way to assure both (1.57) and (1.58) as well as the stability in (1.13) is by imposing all the conditions of Theorem 1.3 in [3]. Namely,

(i) the operator $d\mathscr{L}[u^{(0)}]$ is strictly hyperbolic with x_n being the time variable,

(ii) the matrix $A_1(u^{(0)}(x), x)$ is nonsingular at $x \in \Omega_{\text{bd}}$,

(iii) the difference problem in (1.12) is solvable as stated in (1.15),

(iv) the difference operator is dissipative as stated in [3] (Assumption 1.4),

(v) the uniform Kreiss condition (UKC) holds for problem (1.12) (see (1.34) in [3]).

As shown in [3, Lemma 1.1], conditions (i), (ii) and (v) above together with the dissipativity assumption for dL_{bd} imply the Kreiss condition for the problem in (1.55) and the isomorphism in (1.58). Hence the problem in (1.55) is well posed in the sense of estimate (1.57). There are, however, difference schemes for which the conditions (i), (ii), (iv), and (v) are not fulfilled and for which estimate (1.17) could be proved by an energy method. In such a case the conditions in (1.56) and (1.58) should be imposed independently.

The construction of the approximate solution requires that

$$u_{\text{out}}^{(i)} \in C^{N-i+1}(\Omega), \quad u_{\text{bd}}^{(i)}(x_1, \cdot) \in C^{N-i+1}(\Omega_{\text{bd}})$$

and

$$u_{\text{in}}^{(i)}(\cdot, x_n) \in C^{N-i+1}(\Omega_{\text{in}}).$$

The natural spaces for the hyperbolic problem in (1.55) are, however, the Sobolev spaces H^s. Since there is a loss of derivative in hyperbolic problems, the appropriate smoothness conditions for u_{out}, u_{bd}, and u_{in} are

$$u_{\text{out}}^{(i)} \in C^r([0, T], H^{s-2i-r}(\Omega_{\text{in}})), \qquad 0 \leqslant r \leqslant s - 2i, \tag{1.59}$$

$$u_{\text{bd}}^{(i)}(x_1, \cdot) \in C^r([0, T], H^{s-2i-r-1/2}(\Omega_{\text{bd}} \cap \Omega_{\text{in}})), \qquad 0 \leqslant r \leqslant s - 2i - 1 \tag{1.60}$$

and

$$u_{\text{in}}^{(i)}(\cdot, x_n) \in H^{s-2i}(\Omega_{\text{in}}), \tag{1.61}$$

where

$$s > 2N + (n+1)/2. \tag{1.62}$$

Remark 1.2. If the problem in (1.55) is Kreiss well posed, i.e., estimate (1.57) holds, then $u_{\text{out}}^{(i)}(0, \cdot) \in H^{s-2i}(\Omega_{\text{bd}})$ and $u_{\text{bd}}^{(i)}(x_1, \cdot) \in H^{s-2i}(\Omega_{\text{bd}})$.

For $u_{\text{out}}^{(i)}$ to be of the required smoothness, the problem in (1.32) should satisfy the compatibility conditions of order $s - 2i - 1$ at the time-space corner $x_1 = x_n = 0$ (see [5]). Since one does not want the boundary and initial layers to interfere with these conditions, one has to request that

$$D_{x_1}^j u_{\text{in}}^{(i)} = 0 \quad \text{at } x_1 = 0 \quad \text{and} \quad D_{x_n}^j u_{\text{bd}}^{(i)} = 0 \quad \text{at } x_n = 0 \text{ for } j \leqslant s - 2i - 1. \tag{1.63}$$

This is obviously stronger than the assumption in (1.39). The resulting compatibility condition for the problem in (1.1)–(1.3) could be stated as follows.

Compatibility Condition. Let us substitute in (1.1)–(1.3) a formal expansion $u_h = \sum_{i=0}^{N} u_{\text{out}}^{(i)}(x)h^i$ with $u_{\text{out}}^{(0)} = u^{(0)}$. Define $F_{\text{out}}^{(i)}$ as in (1.29), $f_{\alpha_n}^{(i)}$ as in (1.52), and $g^{(i)}$ as in (1.44) with $u_{\text{bd}}^{(k)} = 0$. With the aid of the equations in (1.32) and the initial conditions in (1.54) (with $F_{\text{in}}^{(i)} = 0$) express the derivatives $D_x^\alpha u_{\text{out}}^{(i)}$, $0 \leqslant |\alpha| \leqslant s - 2i - 1$, at the corner $x_1 = x_n = 0$ in terms of the original data f in (1.3),

$$D_x^\alpha u_{\text{out}}^{(i)} = \sum \text{coef}(u^{(0)}) \prod D_x^\beta u^{(0)} D_x^\gamma D_h^k f_l(x, 0), \tag{1.64}$$

where in the above products the sum of the indices $\Sigma(|\beta| + |\gamma| + k) \leqslant \alpha + i$. Substitute the above expressions into $D_{x_n}^j(dS_{\text{out}} u_{\text{out}}^{(i)} + g^{(i)})$ and $D_{x_1}^j(u_{\text{out}}^{(i)} - f_{\alpha_n}^{(i)})$, $0 \leqslant j \leqslant s - 2i - 1$. Then the resulting similar expressions in terms of coef$(u^{(0)})$, $D_x^\beta u^{(0)}$ and $D_x^\gamma D_h^k f_l(x, 0)$ should vanish at $x_1 = x_n = 0$ for all $1 \leqslant i \leqslant N$.

The above deliberation can be summarized in the following

THEOREM 1.1. *Let the difference problem in* (1.1)–(1.3) *satisfy the Consistency Assumption with* $s > 2N + (n+1)/2$, *the Dissipativity Assumption and the Compatibility Condition. Also let the reduced differential problem in* (1.55) *be well posed in the sense of* (1.56) *and the boundary operator* dS_{bd} *satisfy the assumption in* (1.58). *Finally, let the functions* $f_{\alpha_n}(x, h)$ *in* (1.3) *belong to* $C^i([0, h_0], H^{s-2i}(\Omega_{\text{in}}))$ *for* $0 \leqslant i \leqslant N + 1$. *Then there exists an approximate solution* $u_{\text{ap}}(x, h)$ *as in* (1.22) *such that* $u_{\text{out}}^{(i)}$, $u_{\text{bd}}^{(i)}$, $u_{\text{in}}^{(i)}$ *belong to the spaces shown in* (1.59)–(1.61), $u_{\text{in}}^{(i)}$, $u_{\text{bd}}^{(i)}$ *vanish at the time-space corner as in* (1.63) *and the derivatives* $D^\alpha u_{\text{out}}^{(i)}$ *for* $|\alpha| \leqslant s - 2i - 1$ *at* $x_1 = x_0 = 0$ *coincide with the ones in* (1.64). *This approximate solution satisfies*

$$\begin{aligned} &\text{(i)} && L\big(\{E^\alpha u_{\text{ap}}(x,h)\}, x, h\big) = O_L(h^{N+1}), && x \in \Omega_h, \\ &\text{(ii)} && S\big(\{E^\alpha u_{\text{ap}}(x,h)\}, x, h\big) = O_S(h^{N+1}), && x \in \Omega_{\text{bd},h}, \\ &\text{(iii)} && E_{x_n}^{\alpha_n} u_{\text{ap}}(x,h) - f_{\alpha_n}(x,h) = O_f(h^{N+1}), && x \in \Omega_{\text{in},h}, \end{aligned} \tag{1.65}$$

where

$$\|O_L(h^{N+1})\|_{\Omega_h} + \|O_S(h^{N+1})\|_{\Omega_{\text{bd},h}} + \|O_f(h^{N+1})\|_{\Omega_{\text{in},h}} \leqslant Kh^{N+1} \tag{1.66}$$

and the constant K is independent of h.

Proof. We will prove the smoothness conditions in (1.59)–(1.61) and the degeneracy of the initial boundary layers in (1.63) by induction in i. Consider the problem in (1.32), (1.50), (1.54) for $i = i_0$. Note that the function $F_{\text{out}}^{(i_0+1)}$ in (1.29) belongs to $H^{s-2i_0}(\Omega)$. Indeed $D_x^\beta u_{\text{out}}^{(j)} \in H^{s-2j-|\beta|}(\Omega) \subset H^{s-2i_0}(\Omega)$, since $2j + |\beta| = (j + |\beta|) + j \leqslant (i_0 + 1) + (i_0 - 1)$, while $H^{s-2i_0}(\Omega)$ is a Banach algebra for $s - 2i_0 > n/2$. Similarly, for the function $g^{(i_0)}$ in (1.44), $D_x^\beta u_{\text{out}}^{(j)}$ belongs to $H^{s-2i_0+1}(\Omega)$ and its restriction to Ω_{bd} as well as the second factor $E_{x_1}^{\gamma_1} D_{\text{tan}}^{\gamma_{\text{tan}}} u_{\text{bd}}^{(k)}(0, \cdot)$ lie in $H^{s-2i_0+1/2}(\Omega_{\text{bd}})$. Since the last space for $s - 2i_0 > n/2$ is a Banach algebra, the function $g^{(i_0)}$ belongs to $H^{s-2i_0+1/2}(\Omega_{\text{bd}})$. Recall that the function $F_{\text{bd}}^{(i_0)}$ has the form displayed at the right-hand side of (1.34) and the corresponding coefficients $f_{ikl}(x_{\text{tan}})$ belong to $H^{s-2i_0+1/2}(\Omega_{\text{bd}})$. Thus for $i = i_0$ the right-hand side in (1.50) belongs to $H^{s-2i_0+1/2}(\Omega_{\text{bd}})$. Finally the functions $f_{\alpha_n}^{(i_0)}$ belong to $H^{s-2i_0}(\Omega_{\text{in}})$. Clearly, the data in the problem (1.32), (1.50), (1.54) is smooth enough for the purposes of the compatibility conditions of order $s - 2i_0 - 1$. By the induction hypothesis, (1.63) holds for $i \leqslant i_0 - 1$. Thus the initial and boundary layers $u_{\text{in}}^{(i)}$ and

$u_{\mathrm{bd}}^{(i)}$ do not affect the partial derivatives $D_x^\alpha u_{\mathrm{out}}^{(i_0)}$, $|\alpha| \leqslant s - 2i_0 - 1$, at the corner $x_1 = x_n = 0$. Hence the later ones coincide with $D_x^\alpha u_{\mathrm{out}}^{(i_0)}$ in (1.64). As a result, the initial-boundary value problem in (1.32), (1.50), (1.54) for $i = i_0$ satisfies the usual compatibility conditions of order $s - 2i_0 - 1$ at the time-space corner. As it follows from Theorem 5.1 in [5], the solution $u_{\mathrm{out}}^{(i_0)}$ of the above problem belongs to the spaces in (1.59) with $i = i_0$. The function $u_{\mathrm{bd}}^{(i_0)}$ is computed by solving Eq. (1.37) with boundary conditions in (1.46). Since $F_{\mathrm{bd}}^{(i_0)}(x_1, \cdot) \in H^{s-2i_0+1/2}(\Omega_{\mathrm{bd}})$ belongs to $C^r([0,T], H^{s-2i_0-1/2-r}(\Omega_{\mathrm{bd}} \cap \Omega_{\mathrm{in}}))$ for all $0 \leqslant r \leqslant s - 2i_0 - 1$, and so does the restriction $u_{\mathrm{out}}^{(i_0)}(0, \cdot)$, the function $u_{\mathrm{bd}}^{(i_0)}$ therefore belongs to the spaces in (1.60). By the Compatibility Condition, $D_{x_n}^j(dS_{\mathrm{out}} u_{\mathrm{out}}^{(i_0)} + g^{(i_0)})$ vanishes at $x_1 = x_0 = 0$ for $j \leqslant s - 2i_0 - 1$. Since the derivatives $D_{x_n}^j u_{\mathrm{bd}}^{(i)}$ vanish at $x_n = 0$ for $j \leqslant s - 2i - 1$ and $i \leqslant i_0 - 1$, so do the derivatives $D_{x_n}^j F_{\mathrm{bd}}^{(i_0)}$ for $j \leqslant s - 2i_0 - 1$. Therefore, (1.63) holds also for $u_{\mathrm{bd}}^{(i_0)}$. The function $u_{\mathrm{in}}^{(i_0)}$ is treated in a similar way. The formulas in (1.65) follow from the construction of the approximate solution. With $s > 2N + (n+1)/2$ the function $u_{\mathrm{out}}^{(i)}$ belongs to $C^{N-i+1}(\Omega)$,

$$u_{\mathrm{bd}}^{(i)}(x_1, \cdot) \in C^{N-i+1}(\Omega_{\mathrm{bd}}) \quad \text{and} \quad u_{\mathrm{in}}^{(i)}(x_n, \cdot) \in C^{N-i+1}(\Omega_{\mathrm{bd}}).$$

Hence the truncation errors in (1.65) are bounded by Kh^{N+1} in the maximum norm. With the integral formula of the remainder in the Taylor expansions (1.23), (1.24) and (1.33), one is able to prove that the bound in (1.66) holds also in the L_2-norm. This observation concludes the proof of the theorem. □

Remark 1.3. In the proof above we were not concerned with the smoothness of L and S as functions of $\{E^\alpha u_h(x)\}$, x and h. It could be shown that the sufficient smoothness requirements are that L and S lie in the space H^s with respect to the totality of their variables, where s is as in (1.62).

2. The Convergence to the Asymptotic Solution. In this section we shall prove that the formal asymptotic solution $u_{\mathrm{ap}}(x)$ indeed approximates the true solution of u_h of the problem (1.1)–(1.3). The specific construction of u_{ap} and the smoothness of the coefficients as in (1.59)–(1.61) are not needed any more. The precise statement of the result is as follows.

THEOREM 2.1. *Let $u_{\mathrm{ap}}(x,h)$ be a family of grid functions defined on the grids Ω_h such that the difference $u_{\mathrm{ap}}(x,h) - u^{(0)}(x)$ is $O(h)$ in the $l_\infty(\Omega_h)$-norm, where $u^{(0)}(x)$ is a function defined on Ω. Assume that $u_{\mathrm{ap}}(x,h)$ satisfies Eqs.* (1.65) *with the truncation errors bounded as in* (1.66), *where $N > n/2 + 1$. Finally, let the linearized problem in* (1.12) *be stable in the sense of* (1.17). *Then for $h < h_0$ there exists a solution u_h of* (1.1)–(1.3) *such that*

$$\|u_h - u_{\mathrm{ap}}(x,h)\|_{\Omega_h} \leqslant Kh^N, \tag{2.1}$$

where h_0 and K are some positive constants. If $\bar{u}_h$ is another solution of (1.1)–(1.3) *and $\bar{u}_h - u^{(0)}$ is sufficiently small in the $l_\infty(\Omega_h)$-norm, then necessarily $\bar{u}_h = u_h$.*

Proof. If u_k is a solution of (1.1)–(1.3), the difference $v = u_h - u_{\mathrm{ap}}(x,h)$ satisfies the equations

$$\begin{aligned} &\text{(a)} \quad dL[u_{\mathrm{ap}}]v(x) = hF(x), \qquad x \in \Omega_h, \\ &\text{(b)} \quad dS[u_{\mathrm{ap}}]v(x) = g(x), \qquad x \in \Omega_{\mathrm{bd},h}, \\ &\text{(c)} \quad E_{x_n}^{\alpha_n} v(x) = f_{\alpha_n}(x), \qquad 0 \geqslant \alpha_n \geqslant -\alpha_n^*,\ x \in \Omega_{\mathrm{in},h}, \end{aligned} \tag{2.2}$$

where $hF(x) = -O_L(h^{N+1}) + O_1(v^2)$, $g(x) = -O_S(h^{N+1}) + O_2(v^2)$ and $f_{\alpha_n}(x) = -O_f(h^{N+1})$. The quadratic terms O_1 and O_2 are bounded by

$$\|O_1(v^2)\|_{\Omega_h} + h^{1/2}\|O_2(v^2)\|_{\Omega_{\text{bd},h}} \leqslant K_1 \sup_{x\in\Omega_h} |v(x)| \cdot \|v\|_{\Omega_h}. \tag{2.3}$$

(We assume that L and S are C^2 functions with respect to the variables $E^\alpha u$.) Since $u_{\text{ap}}(x, h) - u^{(o)}(x) = O(h)$, estimate (1.17) for the problem in (1.12) implies the estimate

$$\|v\|_{\Omega_h} \leqslant K_2\Big(\|F\|_{\Omega_h} + h^{-1/2}\|g\|_{\Omega_{\text{bd},h}} + h^{-1/2}\sum_{\alpha_n}\|f_{\alpha_n}\|_{\Omega_{\text{in},h}}\Big) \tag{2.4}$$

for the problem in (2.2). Thus, in view of (1.66),

$$\|v\|_{\Omega_h} \leqslant K_3 h^N + K_1K_2h^{-1} \sup_{x\in\Omega_h} |v(x)| \cdot \|v\|_{\Omega_h}, \tag{2.5}$$

where $K_3 = K_2K$ and K is the constant in (1.66). We wish to prove that for h sufficiently small

$$\|v\|_{\Omega_h} \leqslant 2K_3h^N. \tag{2.6}$$

Clearly, $\|v\|_{\Omega_h} = O(h^N)$ would imply $h^{-1}\sup_x|v(x)| = h^{-1}O(h^{N-n/2}) = o(1)$, so that the second term in the right-hand side of (2.5) is negligible compared with $\|v\|_{\Omega_h}$. Let

$$\Omega_h^k = \{x = (x_-, x_n) \,|\, x_- \in \Omega_{\text{in},h}, x_n = jh, -\alpha_n^* \leqslant j \leqslant k\}.$$

We will prove by induction in k that for $h < h_3$

$$\|v\|_{\Omega_h^k} \leqslant 2K_3h^N, \tag{2.7}$$

where h_3 is to be defined later. For $k = 0$ this estimate follows from (1.66). Since the difference scheme may be implicit, in order to construct u_h at $x_n = (k+1)h$, one has to apply the implicit map theorem. Namely, consider the equations

$$\begin{aligned} &\text{(a)} \quad L(\{E^\alpha(u_{\text{ap}} + v)\}, x, h) - L(\{E^\alpha u_{\text{ap}}\}, x, h) + O_F = 0, \\ &\text{(b)} \quad S(\{E^\alpha(u_{\text{ap}} + v)\}, x, h) - S(\{E^\alpha u_{\text{ap}}\}, x, h) + O_S = 0 \end{aligned} \tag{2.8}$$

at the time level $x_n = (k+1)h$. The left-hand side of (2.8) could be thought of as a map which depends on the grid functions

$$v(x_-, (k+1)h), \ldots, v(x_-, (k-\alpha_n^*)h), \qquad x_- \in \Omega_{\text{in},h},$$

and on O_F, O_S at $x_n = (k+1)h$ with the norms

$$\|v\|^2_{l_2(\Omega_{\text{in},h})} = \sum |v(x_-)|^2, \qquad x_- \in \Omega_{\text{in},h} \tag{2.9}$$

(i.e., without the weight h^{n-1} as in the $\|\cdot\|^2_{\Omega_{\text{in},h}}$-norm). The values of the map are pairs of grid functions with the same norm as in (2.9). Clearly, this map is continuously differentiable and its differential with respect to $v(\cdot, (k+1)h)$ at the zero point is given by the pair $(dL^{(0)}[u_{\text{ap}}], dS^{(0)}[u_{\text{ap}}])$ (see the definition in (1.16)). Since $u_{\text{ap}} - u^{(0)} = O(h)$, the above differential, like the map in (1.14), is an isomorphism and estimate (1.15) holds. Thus the equations in (2.8) could be solved with respect to $v(x_-, (k+1)h)$ and the solution is bounded by

$$\begin{aligned} &\|v(\cdot, (k+1)h)\|_{l_2(\Omega_{\text{in},h})} \\ &\leqslant K_4\Big(\sum_{j=k+\alpha_n^*}^{k} \|v(\cdot, jh)\|_{l_2(\Omega_{\text{in},h})} + \|O_F\|_{l_2(\Omega_{\text{in},h})} + \|O_S\|_{l_2(\Omega_{\text{in},h})}\Big) \\ &\leqslant K_4h^{-n/2}\Big((1+\alpha_n^*)^{1/2}\|v\|_{\Omega_h^k} + \|O_F\|_{\Omega_h} + \|O_S\|_{\Omega_h}\Big) \leqslant K_5h^{N-n/2}, \end{aligned} \tag{2.10}$$

provided the last bound is sufficiently small. Hence,

$$\sup_{x \in \Omega_h^{k+1}} |v(x)| \leqslant \max(K_5, 2K_3) h^{N-n/2} = K_6 h^{N-n/2}. \tag{2.11}$$

Note that estimate (2.5) is valid also when Ω_h is replaced by Ω_h^k. Hence, for $K_1 K_2 K_6 h^{N-n/2-1} < 1/2$ we obtain $\|v\|_{\Omega_h^{k+1}} \leqslant 2K_3 h^N$. Since the constants K_i are independent of h and k, estimate (2.7), and hence (2.1), are valid for all h bounded by some h_3. For the local uniqueness of the solution u_h in $l_\infty(\Omega_h)$ one has to prove that the map in (1.14) is an isomorphism also in the maximum norm. Indeed, as shown in [3, Theorem 1.1], the solvability of the problem in (1.12) is equivalent to a certain algebraic coercivity condition for the pair of operations $(dL^{(0)}, dS^{(0)})$. For a complex vector $z = (z_1, z_2, \ldots, z_{n-1})$ define

$$dL_z^{(0)} = \sum_\alpha L_\alpha(x, h)(zE)^\alpha,$$

where the above sum is carried over the same set of indices α as in (1.16) and $(zE)^\alpha = (z_1 E_{x_1})^{\alpha_1} \cdots (z_{n-1} E_{x_{n-1}})^{\alpha_{n-1}}$, and in a similar way define $dS_z^{(0)}$. Note that the pair $(dL_z^{(0)}, dS_z^{(0)})$ satisfies the same coercivity conditions as $(dL^{(0)}, dS^{(0)})$, provided $(1+\varepsilon)^{-1} \leqslant |z_i| \leqslant (1+\varepsilon)$, $i = 1, 2, \ldots, n-1$, and ε is sufficiently small. As a result, for such z,

$$\sum_x |z^{x/h} w(x)|^2 \leqslant K\left(\sum_x |z^{x/h} dL^{(0)} w(x)|^2 + \sum_{x_1 = 0} |z^{x/h} dS^{(0)} w(x)|^2\right). \tag{2.12}$$

If the supports of $dL^{(0)}w$ and $dS^{(0)}w$ lie in the cone $x \geqslant 0$, the above estimate with $z_i = (1+\varepsilon)^{-1}$, $i = 1, \ldots, n-1$, implies

$$|w(0)| \leqslant K(\|dL^{(0)}w\|_\infty + \|dS^{(0)}w\|_\infty), \tag{2.13}$$

where $\|\cdot\|_\infty$ is the l_∞-norm. The other cones in the half space $x_1 \geqslant 0$ are treated in a similar way, i.e., if $x_i \leqslant 0$ then choose $z_i = 1 + \varepsilon$. In the general case, the functions $F = dL^{(0)}w$ and $g = dS^{(0)}w$ are split into a sum of 2^{n-2} terms with supports in corresponding cones so that (2.13) follows. Since the point $x_2 = 0, \ldots, x_{n-1} = 0$ has no preference in the grid $\Omega_{\mathrm{bd},h} \cap \Omega_{\mathrm{in},h}$, the same bound as in (2.13) holds also for $\|w(x_1 = 0, \cdot)\|_\infty$. Now let $\overline{w}(x) = w(x)$ for $x_1 \geqslant 0$ and $\overline{w}(x) = 0$ for $x_1 < 0$. With $dL^{(0)}\overline{w}$ defined in the whole space, we obtain

$$\begin{aligned} \|\overline{w}\|_\infty &\leqslant K\|dL^{(0)}\overline{w}\|_\infty \leqslant K\big(\|dL^{(0)}w\|_\infty + \|w(x_1 = 0, \cdot)\|_\infty\big) \\ &\leqslant K\big(\|dL^{(0)}w\|_\infty + \|dS^{(0)}w\|_\infty\big). \end{aligned} \tag{2.14}$$

This estimate concludes the proof of the theorem. □

COROLLARY 2.1. *Let the conditions of Theorem* 1.1 *hold and let* $u_{\mathrm{ap}}^{(N_1)}$ *be the part of the expansion in* (1.22) *with powers of* h *up to* (*and including*) h^{N_1}. *If* $N_1 \leqslant N - 1$, *then*

$$\left\|u_h - u_{\mathrm{ap}}^{(N_1)}\right\|_{\Omega_h} \leqslant K h^{N_1+1}, \tag{2.15}$$

and if $N_1 \leqslant N - n/2 - 1$, *then*

$$\left\|u_h - u_{\mathrm{ap}}^{(N_1)}\right\|_{l_\infty(\Omega_h)} \leqslant K h^{N_1+1}. \tag{2.16}$$

In particular, if the difference problem in (1.1)–(1.3) *is an rth order approximation of* (1.6)–(1.8), *then* $u^{(0)} = u_{\mathrm{ap}}^{(r-1)}$ *and hence*

$$\|u_h - u^{(0)}\|_{l_\infty(\Omega_h)} \leqslant Kh^r, \tag{2.17}$$

provided the conditions of Theorem 1.1 *hold with* $N \geqslant r + n/2$.

Thus, estimate (2.17) establishes the precise rate with which the difference solution u_h converges to the analytical solution $u^{(0)}$.

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