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The Boundary Element Numerical Method for Two-Dimensional Linear Quadratic Elliptic Problems: (I) Neumann Control*

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Abstract. For two-dimensional distributed control systems governed by the Laplace equation, the boundary element method is an efficient numerical method to solve problems whose quadratic cost involves boundary integrals only. In this paper we formulate a duality-boundary integral equation scheme and use piecewise constant boundary elements to approximate the problem. This method involves discretization of the boundary curve only and it can conveniently handle the compatibility constraint due to the Neumann data. Convergence and optimal error estimates $\mathcal{O}(h)$ have been proved. Numerical data for the case of a disk are computed to illustrate the theory.

1. Introduction. In this paper we apply the boundary element method (BEM) to compute boundary controls of two-dimensional linear quadratic problems governed by the Laplace equation.

The Laplace equation models many physical processes such as equilibrium heat conduction, perfect incompressible irrotational flow, elastostatics, etc. A reasonable distributed control model is the following: Find an optimal flux (the Neumann data) on the boundary so that the corresponding observation on the boundary (the Dirichlet data) can be close to a given profile. To be specific, let us consider the following class of linear quadratic problems:

$$\operatorname*{Inf}_{u\in L^2(\Gamma)} J(y,u) = \|y(\cdot\,;u) - z_d\|^2 + \langle Nu, u\rangle \tag{1.1}$$

governed by the Laplace equation with Neumann control,

$$\begin{cases} \Delta y(x;u) = 0 & \text{in } \Omega, \\ \dfrac{\partial y}{\partial \nu}(x;u) = u(x) & \text{on } \Gamma, \end{cases} \tag{1.2}$$

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where

$\| \ \|$, $\langle \ , \ \rangle$ are, respectively, the norm and the inner product in $L^2(\Gamma)$, Ω is a bounded *convex* open domain in $\mathbf{R}^2$, *with* $(\textit{diameter}\ \Omega) < 1$, $x = (x_1, x_2) \in \Omega$,

Γ is the boundary of Ω, C^∞ *smooth*, *with nonzero curvature everywhere*,

(1.3) $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is the Laplacian,

$z_d \in L^2(\Gamma)$ is given,

N is a positive definite symmetric operator on $L^2(\Gamma)$ satisfying

$$\langle Nu, u \rangle \geqslant \alpha \| u^2 \| \quad \text{for some } \alpha > 0, \forall\, u \in L^2(\Gamma),$$

ν is the unit exterior normal on Γ,

$u \in L^2(\Gamma)$ is the distributed control.

The above is similar to a problem mentioned by Lions in [6, p. 81] (where instead of the governing equation $\Delta y = 0$ he used $(-\Delta + a_0)y = f$ for some $a_0 > 0$ to assume the useful positive definiteness). It is easy to see that the theory in [6] applies, and problem (1.1), (1.2) has a unique optimal control $\hat{u} \in L^2(\Gamma)$ minimizing J and satisfying the compatibility condition

$$\int_\Gamma \hat{u}(x)\, d\sigma = 0 \tag{1.4}$$

with corresponding state

$$\hat{y} = y(\cdot, \hat{u}) \in H^{3/2}(\Omega), \tag{1.5}$$

where in the above and throughout the rest of the paper, H^r denotes the Sobolev space of order r.

We wish to develop numerical methods to treat the above. The problem is a two-dimensional PDE. For multi-dimensional problems, generally speaking, the amount of calculations grows exponentially with space dimension n. The associated numerical difficulty can often be awesome, and the number of operations is also burdensome for most computing hardware.

The most commonly used numerical methods to solve PDEs are finite differences and finite elements. The former are relatively easy to use but work best when the domain has only straight edges as boundary. The latter involve extensive quadratures but are advantageous for domains with curved boundary. Both methods require careful *discretizations of the entire domain*. The efforts and labor involved in programming and testing computer codes are also proportionally large.

Let us examine the *special setting* of our problem (1.1), (1.2): In (1.1) the cost functional involves only the state y and the control u on the boundary Γ; in the state equation $\Delta y = 0$, there is no distributed forcing term. Accordingly, *can we approximate the optimal control and state on the boundary* Γ *only*, *without discretizing the entire domain* Ω? If affirmative, this would give us numerical solutions of $\hat{u}$ and $\hat{y}$ on the boundary Γ, normally the most *vital* information we wish to obtain.

A satisfactory answer to the above is provided by BEM, the boundary element method. Indeed, using BEM we are fortunate here to avoid the "curse of higher dimensionality."

The boundary element method is essentially a collocation method for solving integral equations only on the boundary of the domain in which the PDE is posed. Thus a reduction of spatial domain has resulted. In the past ten years, BEM has attracted the attention of applied mathematicians and engineers, and the method has been found very effective for many problems [1]. Nevertheless, to our knowledge, not many applications of BEM have been made to distributed parameter control problems. We hope to initiate a series of papers on the applications of BEM and computations of general multi-dimensional distributed parameter control problems.

The outline of our paper is as follows.

In Section 2 we first formulate the boundary integral equation approach. The primal problem is solved by the duality method through the use of a Lagrange multiplier. In Section 3 we introduce the BEM numerical scheme and discretize the dual problem. In Section 4 we prove the convergence of solutions. Optimal error estimates $\mathcal{O}(h)$ for the optimal control and state and $\mathcal{O}(h^2)$ for the cost are obtained with piecewise constant boundary elements. In Sections 5 and 6 we present a numerical example and discussions. Numerical data indicate a convergence rate $\mathcal{O}(h^{1.93})$ for the cost and a superconvergence rate $\mathcal{O}(h^{1.93})$ for the optimal control and state, confirming the theory.

2. The Duality Scheme Based on the Boundary Integral Equation Formulation. Let y satisfy (1.2) with Neumann data u. It is clear that a solution y of (1.2) exists if and only if u satisfies the compatibility condition (1.4).

Let $v(\xi \,|\, x)$ be the fundamental solution of the Laplace equation in $\mathbf{R}^2$ satisfying

$$(2.1)\qquad -\Delta_\xi v(\xi \,|\, x) = -\left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}\right) v(\xi \,|\, x) = \delta(\xi - x), \qquad \xi \in \mathbf{R}^2.$$

It is known [9] that

$$v(\xi \,|\, x) = -\frac{1}{2\pi} \ln|x - \xi|,$$

where $|\;|$ is the length norm in $\mathbf{R}^2$:

$$|x - \xi| = \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2\right]^{1/2}.$$

For $x \in \Gamma$, upon using the double layer property, it is well known that

$$(2.2)\qquad \begin{aligned} \frac{1}{2} y(x) &= -\frac{1}{2\pi} \int_\Gamma u(\xi) \ln|x - \xi| \, d\sigma_\xi \\ &\quad + \frac{1}{2\pi} \int_\Gamma \frac{1}{|x - \xi|} \left(\frac{\partial}{\partial \nu_\xi} |x - \xi|\right) y(\xi) \, d\sigma_\xi, \qquad x \in \Gamma, \end{aligned}$$

as Γ is assumed to be smooth. Thus we get a Fredholm boundary integral equation of the second kind,

$$(2.3)\qquad y - Ky = Lu,$$

for the Dirichlet data $y|_\Gamma$ of y (provided that u is given), where K and L are integral operators defined by

$$(2.4) \qquad (Ky)(x) = \frac{1}{\pi}\int_\Gamma \frac{1}{|x-\xi|}\left(\frac{\partial}{\partial \nu_\xi}|x-\xi|\right) y(\xi)\, d\sigma_\xi, \qquad x \in \Gamma,$$

$$(2.5) \qquad (Lu)(x) = -\frac{1}{\pi}\int_\Gamma u(\xi) \ln|x-\xi|\, d\sigma_\xi, \qquad x \in \Gamma.$$

Their properties are indicated in the following two lemmas.

LEMMA 2.1. *Let s be the arc length parameter on* Γ. *Then*

(i)

$$(2.6) \qquad (Ky)(x(s)) = \frac{1}{\pi}\int_0^l \left[\frac{\partial}{\partial s}\theta_x(\xi(s))\right] y(\xi(s))\, ds,$$

where l *is the total arc length of* Γ *and* $\theta_x(\xi)$ *is the counterclockwise angle formed between* $\overline{x\xi}$ *and the tangent to* Γ *at* x; *cf. Figure* 1.

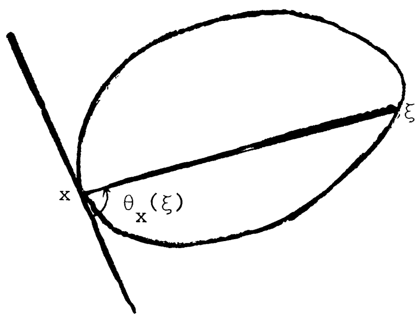

FIGURE 1

The angle function $\theta_x(\xi)$

Consequently, K *and its adjoint* K^* *are bounded linear Hilbert-Schmidt operators on* $L^2(\Gamma)$ *with* C^∞*-smooth kernels* $\partial\theta_x(\xi(s))/\partial s$ *and* $\partial\theta_{\xi(s)}(x)/\partial s$, *respectively, and*

$$(2.7) \qquad K, K^*\colon H^r(\Gamma) \to C^\infty(\Gamma) \quad \forall r \geq 0.$$

(ii) *The entire spectrum of* K *consists only of* 0 (*in the continuous spectrum*) *and a simple eigenvalue* 1 *with corresponding eigenfunction* 1, *the constant. The restricted operators* $I - K$, $I - K^*$ *are invertible mappings,*

$$(2.8) \qquad I - K\colon [\ker(I-K)]^\perp \to [\ker(I-K^*)]^\perp,$$

$$(2.9) \qquad I - K^*\colon [\ker(I-K^*)]^\perp \to [\ker(I-K)]^\perp$$

with bounded inverses defined on their ranges.

Proof. See [2], [5], [8]. □

LEMMA 2.2. *The operator* L *is a positive definite weakly singular* (*hence compact*) *operator on* $L^2(\Gamma)$ *satisfying*

$$(2.10) \qquad L\colon H^r(\Gamma) \to H^{r+1}(\Gamma) \quad \text{for any } r \geq 0.$$

Furthermore, there exists $\gamma_1 > 1$ *such that*

$$(2.11) \qquad \gamma_1^{-1}\|g\|_{L^2(\Gamma)} \leq \|Lg\|_{H^1(\Gamma)} \leq \gamma_1\|g\|_{L^2(\Gamma)}$$

for all $g \in L^2(\Gamma)$.

Proof. See [5, Theorem 2].

Having at disposal the boundary integral equation (2.3), we now consider solving the linear quadratic problems (1.1) and (1.2). The differential constraints (1.2) are now replaced by the integral constraint (2.3). Thus, one now considers the following *primal* problem:

$$\begin{cases} \text{Min } J(y,u) \\ (y,u) \quad \text{subject to} \\ (I-K)y = Lu, \\ u \perp 1. \end{cases} \tag{2.12}$$

Here u must be chosen to be orthogonal to 1, and as $c \in \ker(I-K)$, the solution y could differ by any constant c. This is rather inconvenient.

The *dual* scheme does not have any such disadvantage, as can be seen below.

Let $\lambda \in L^2(\Gamma)$ be a Lagrange multiplier. Consider

$$\min_{y,\,u \in L^2(\Gamma)} J_\lambda(y,u) = \|y - z_d\|^2 + \langle Nu, u\rangle + \langle \lambda, (I-K)y - Lu\rangle. \tag{2.13}$$

A simple variational analysis gives the unique minimizing solution $(\hat{y}_\lambda, \hat{u}_\lambda)$ of (2.13) satisfying

$$\begin{cases} 2(\hat{y}_\lambda - z_d) + (I-K)^*\lambda = 0, \\ 2N\hat{u}_\lambda - L^*\lambda = 0. \end{cases} \tag{2.14}$$

Thus,

$$\begin{cases} \hat{y}_\lambda = z_d - \frac{1}{2}(I-K)^*\lambda = z_d - \frac{1}{2}(I-K^*)\lambda, \\ \hat{u}_\lambda = \frac{1}{2}N^{-1}L^*\lambda = \frac{1}{2}N^{-1}L\lambda. \end{cases} \tag{2.15}$$

Substituting (2.15) into J_λ in (2.13), we get

$$\begin{aligned} J_\lambda \equiv J_\lambda(\hat{y}_\lambda, \hat{u}_\lambda) &= -\tfrac{1}{4}a(\lambda,\lambda) + \tfrac{1}{2}\theta(\lambda) \\ &= -\tfrac{1}{4}\|(I-K^*)\lambda\|^2 - \tfrac{1}{4}\langle N^{-1}L\lambda, L\lambda\rangle + \langle \lambda, (I-K)z_d\rangle, \end{aligned}$$

where a is a bilinear form defined on $L^2(\Gamma) \times L^2(\Gamma)$ by

$$a(w_1,w_2) \equiv \langle (I-K^*)w_1, (I-K^*)w_2\rangle + \langle N^{-1}Lw_1, Lw_2\rangle$$

and θ is a linear form on $L^2(\Gamma)$ defined by

$$\theta(w) = 2\langle w, (I-K)z_d\rangle.$$

Now the dual problem becomes

$$\max_{\lambda \in L^2(\Gamma)} J_\lambda. \tag{2.16}$$

LEMMA 2.3. *There exists $\gamma_2 > 1$ such that for all $w, w_1, w_2 \in L^2(\Gamma)$*

$$\gamma_2^{-1}\|w\|^2 \leqslant a(w,w) \leqslant \gamma_2\|w\|^2 \tag{2.17}$$

and

$$|a(w_1,w_2)| \leqslant \gamma_2\|w_1\|\,\|w_2\|. \tag{2.18}$$

Proof. Let F be any finite-dimensional subspace of $L^2(\Gamma)$. Then by (1.3) and (2.11) there exists $c_1 > 1$ such that

$$c_1^{-1}\|w\|^2 \leqslant \langle N^{-1}Lw, Lw\rangle \leqslant c_1\|w\|^2 \quad \forall w \in F. \tag{2.19}$$

This constant c_1 depends on F only.

Since $I - K^*$ is invertible with bounded inverse on $[\ker(I-K)]^\perp$, there exists $c_2 > 1$ such that

$$\begin{aligned} c_2^{-1}\|w\|^2 &\leqslant \langle (I-K)^*w, (I-K)^*w\rangle \\ &\leqslant c_2\|w\|^2 \quad \forall w \in [\ker(I-K^*)]^\perp. \end{aligned} \tag{2.20}$$

We can choose $F = \ker(I - K^*)$. Combining (2.19) and (2.20), we conclude (2.17). The rest is easy. □

THEOREM 2.4. *The dual problem* (2.16) *has a unique maximizing solution* λ^*. *Furthermore*, $\hat{y}_{\lambda^*}$ *and* $\hat{u}_{\lambda^*}$ *in* (2.15) *corresponding to* λ^* *are the unique optimal state and control for the problem* (1.1) (1.2).

Proof. The only thing we need to worry about here is whether $\hat{u}_{\lambda^*}$ satisfies the orthogonality constraint $\langle \hat{u}_{\lambda^*}, 1\rangle = 0$ in (2.12). The rest follows from the standard minimax duality theory in mathematical programming.

Since λ^* exists and solves (2.16), by calculus of variations, λ^* satisfies

$$-\tfrac{1}{2}\left[(I-K)(I-K^*) + LN^{-1}L\right]\lambda^* + (I-K)z_d = 0. \tag{2.21}$$

The bracketed operator above is invertible by Lemma 2.3, so

$$\lambda^* = 2\left[(I-K)(I-K^*) + LN^{-1}L\right]^{-1}(I-K)z_d. \tag{2.22}$$

Let e be the natural layer of Γ [5] defined by

$$\int_\Gamma e(x)\,d\sigma = 1, \qquad Le = E = \text{constant} > 0 \text{ on } \Gamma.$$

Then [5],

$$e \in C^\infty(\Gamma), \quad e > 0 \quad \text{and} \quad e \in \ker(I - K^*). \tag{2.23}$$

Thus, $\hat{u}_{\lambda^*}$ is orthogonal to 1 if and only if

$$\langle \hat{u}_{\lambda^*}, 1\rangle = \frac{1}{E}\langle \hat{u}_{\lambda^*}, E\rangle = \frac{1}{E}\langle \hat{u}_{\lambda^*}, Le\rangle = \frac{1}{E}\langle L\hat{u}_{\lambda^*}, e\rangle = 0,$$

i.e., if and only if $L\hat{u}_{\lambda^*}$ is orthogonal to e.

But by (2.15), $\hat{u}_{\lambda^*} = \frac{1}{2}N^{-1}L\lambda^*$, so

$$\begin{aligned} 2\langle L\hat{u}_{\lambda^*}, e\rangle &= \langle LN^{-1}L\lambda^*, e\rangle \\ &= \langle -(I-K)(I-K^*)\lambda^* + 2(I-K)z_d, e\rangle \quad (\text{by (2.21)}) \\ &= -\langle (I-K^*)\lambda^*, (I-K^*)e\rangle + 2\langle z_d, (I-K^*)e\rangle \\ &= 0, \quad \text{by (2.23)}. \ \square \end{aligned}$$

Another useful by-product is the regularity of solutions below.

THEOREM 2.5. *Assume that*

(2.24) $$z_d \in H^r(\Gamma) \quad \text{for some } r \geqslant 0$$

and

(2.25) $$N^{-1}\colon H^s(\Gamma) \to H^s(\Gamma) \quad \forall s\colon 0 \leqslant s \leqslant r+1.$$

Then we have the regularity of the optimal multipliers, state and control:

$$\lambda^* \in H^r(\Gamma), \quad \hat{y}_{\lambda^*} \in H^{r+5/2}(\Omega), \quad \hat{u}_{\lambda^*} \in H^{r+1}(\Gamma).$$

Proof. We first verify that $\lambda^* \in H^r(\Gamma)$. From (2.21),

(2.26) $$\lambda^* = (K + K^*)\lambda^* - KK^*\lambda^* + LN^{-1}L\lambda^* + 2(I - K)z_d.$$

From the smoothing properties (2.7), (2.10) as well as (2.25), we see that the right-hand side of (2.26) lies in $H^s(\Gamma)$ for $s = \min(2, r)$.

If $s = r$, we have verified that $\lambda^* \in H^r(\Gamma)$. If $s < r$, then $s = 2$. We apply the smoothing properties of K and L again to the right side of (2.26) and get

$$\lambda^* \in H^s(\Gamma), \qquad s = \min(4, r).$$

Continuing this indefinitely, we conclude $\lambda^* \in H^r(\Gamma)$.

Since $\hat{u}_{\lambda^*} = \frac{1}{2}N^{-1}L\lambda^*$, by (2.10) and (2.25) we have $\hat{u}_{\lambda^*} \in H^{r+1}(\Gamma)$.

Since $\hat{y}_{\lambda^*}$ is the solution of (1.2) corresponding to $\hat{u}_{\lambda^*}$, we get $\hat{y}_{\lambda^*} \in H^{r+5/2}(\Omega)$. □

3. The Numerical Algorithm with Piecewise Constant Boundary Elements. We briefly introduce the boundary element numerical scheme with piecewise constant boundary elements. For a detailed account, the reader is referred to [4].

BEM is a collocation method to approximate the integral equation (2.3). Let us divide Γ into meshes $\tilde{\Gamma}_h = \{\Gamma_1, \Gamma_2, \dots, \Gamma_{n(h)}\}$, $h = \max_{1 \leqslant i \leqslant n(h)}(\text{length } \Gamma_i)$, as shown in Figure 2. On each mesh curve Γ_i, let us choose the midpoint x_i as the nodal point. We assume that the mesh is uniform, i.e., there exists $c > 0$ for all h such that

$$h < c \min_{1 \leqslant i \leqslant n(h)} (\text{length } \Gamma_i).$$

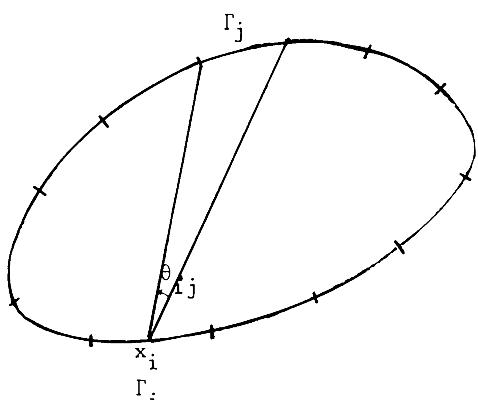

FIGURE 2
Piecewise constant boundary elements

Let us approximate y and u in (2.4) by piecewise constant functions y_h and u_h, where

$$\begin{cases} y_h(x) = y_h(x_i) \equiv y_i, \\ u_h(x) = u_h(x_i) \equiv u_i \end{cases} \tag{3.1}$$

for all $x \in \Gamma_i$ and $i = 1, 2, \ldots, n(h)$. Let

$V_h =$ the finite-dimensional space spanned by piecewise constant functions on $\tilde{\Gamma}_h$.

Then $(y_h, u_h) \in V_h \times V_h$. V_h is known to satisfy the approximation property

$$\begin{cases} \lim_{h \to 0} \inf_{v_h \in V_h} \|w - v_h\| = 0, \qquad w \in L^2(\Gamma). \\ \text{There exists a } C > 0 \text{ for all } h \text{ such that for any } w \in H^1(\Gamma) \\ \inf_{v_h \in V_h} \|w - v_h\| \leqslant Ch\|w\|_{H^1(\Gamma)}. \end{cases} \tag{3.2}$$

We find the approximate solution (y_h, u_h) by requiring that

$$y_i - \frac{1}{\pi}\sum_{j=1}^{n}\int_{\Gamma_j}\frac{\partial}{\partial s}\theta_{x_i}(\xi(s))\,y_j\,ds = -\frac{1}{\pi}\sum_{j=1}^{n}\int_{\Gamma_j}\ln|x_i - \xi(s)|\,u_j\,ds$$

be satisfied for $i = 1, 2, \ldots, n(h)$. Note here that we have used (2.6). The above gives a system of linear equations

$$\sum_{j=1}^{n}\left(\delta_{ij} - \frac{1}{\pi}\theta_{ij}\right)y_j + \sum_{j=1}^{n} q_{ij}u_j = 0, \qquad i = 1, 2, \ldots, n, \tag{3.3}$$

where

$$\theta_{ij} \equiv \int_{\Gamma_j}\frac{\partial}{\partial s}\theta_{x_i}(\xi(s))\,ds = \text{the angle subtended by the arc } \Gamma_j \text{ centered at } x_i, \text{ cf. Figure 2;} \tag{3.4}$$

$$q_{ij} \equiv \frac{1}{\pi}\int_{\Gamma_j}\ln|x_i - \xi(s)|\,ds. \tag{3.5}$$

Equation (3.3) is now written in matrix form,

$$(I - K_h)y_h = L_h u_h, \tag{3.6}$$

where K_h and L_h are $n \times n$ matrices defined by

$$K_h = \left[\frac{1}{\pi}\theta_{ij}\right]_{n \times n}, \qquad L_h = [-q_{ij}]_{n \times n}.$$

K_h and L_h induce two linear operators on V_h, which by abuse of notations are still denoted as K_h and L_h.

In view of the fact that

$$\sum_{j=1}^{n}\theta_{ij} = \pi, \qquad i = 1, 2, \ldots, n,$$

we see that $\ker(I - K_h)$ contains an eigenvector $(1, 1, \ldots, 1)^T$. For convex domains whose boundary is C^∞ with nonzero curvature everywhere, it is known [4] that $(1, 1, \ldots, 1)^T$ is the only eigenvector of $I - K_h$. Equation (3.6) has a solution if and only if

$$(3.7) \qquad L_h u_h \perp \ker(I - K_h^*).$$

Define

$$AC(\Gamma_h) = \{ f\colon \Gamma \to \mathbf{R} \mid f \text{ is absolutely continuous on the open curve } \Gamma_i, 1 \leqslant i \leqslant n(h) \}.$$

We let $\tilde{\mathbf{P}}_h$ be a projection operator from $AC(\Gamma_h)$ into V_h defined by

$$(\tilde{\mathbf{P}}_h f)(x) = f(x_i) \quad \forall x \in \Gamma_i,\ i = 1, 2, \ldots, n.$$

It is easy to see that

$$K_h w = \tilde{\mathbf{P}}_h K w, \qquad L_h w = \tilde{\mathbf{P}}_h L w \quad \forall\, w \in V_h.$$

We also define the projection operator $\mathbf{P}_h$ from $L^2(\Gamma)$ into V_h by

$$\mathbf{P}_h w = w_h \in V_h, \qquad w \in L^2(\Gamma),$$

where w_h is the unique element in V_h satisfying

$$\langle w, v_h \rangle = \langle w_h, v_h \rangle \quad \forall v_h \in V_h.$$

Then,

$$\mathbf{P}_h = \tilde{\mathbf{P}}_h \quad \text{on } V_h.$$

We now study the discretized mathematical programming problem

$$(3.8) \qquad \begin{cases} \underset{(y_h, u_h)}{\text{Min}} \ \|y_h - z_d\|^2 + \langle N u_h, u_h \rangle \\ \text{for all } (y_h, u_h) \in V_h \times V_h \text{ satisfying constraints (3.6), (3.7).} \end{cases}$$

We repeat the max-min duality argument of Section 2: Let $\lambda_h \in V_h$ be a Lagrange multiplier and solve

$$(3.9) \qquad \underset{(y_h, u_h) \in V_h \times V_h}{\text{Min}} \ \|y_h - z_d\|^2 + \langle N u_h, u_h \rangle + \langle \lambda_h, (I - K_h) y_h - L_h u_h \rangle.$$

The unique minimizing solutions are

$$(3.10) \qquad \hat{y}_h(\lambda_h) = \mathbf{P}_h z_d - \tfrac{1}{2}(I - K_h^*)\lambda_h,$$

$$(3.11) \qquad \hat{u}_h(\lambda_h) = \tfrac{1}{2}(\mathbf{P}_h N \mathbf{P}_h)^{-1} L_h \lambda_h = \tfrac{1}{2}\mathbf{P}_h N^{-1} \mathbf{P}_h L_h \lambda_h = \tfrac{1}{2}\mathbf{P}_h N^{-1} L_h \lambda_h.$$

We obtain the dual problem

$$(3.12) \qquad \underset{\lambda_h \in V_h}{\text{Max}} \ J_\lambda^h \equiv -\tfrac{1}{4} a_h(\lambda_h, \lambda_h) + \tfrac{1}{2}\theta_h(\lambda_h) + \|(I - \mathbf{P}_h) z_d\|^2,$$

where for $w, w_1, w_2 \in V_h$ the forms a_h and θ_h are defined by

$$(3.13) \quad a_h(w_1, w_2) \equiv \langle (I - K_h^*) w_1, (I - K_h^*) w_2 \rangle + \langle \mathbf{P}_h N^{-1} L_h w_1, L_h w_2 \rangle,$$

$$(3.14) \qquad \theta_h(w) = 2\langle w, (I - K_h)\mathbf{P}_h z_d \rangle.$$

Since the last term $\|(I - \mathbf{P}_h) z_d\|^2$ in (3.12) is just a constant, we drop it and redefine

$$(3.15) \qquad J_\lambda^h \equiv -\tfrac{1}{4} a_h(\lambda_h, \lambda_h) + \tfrac{1}{2}\theta_h(\lambda_h).$$

LEMMA 3.1. *There exists a constant $C > 0$ such that*

$$\|\mathbf{P}_h v - \tilde{\mathbf{P}}_h v\| \leqslant Ch\|v\|_{H^1(\lambda)}$$

for all $v \in H^1$ and all $h > 0$.

Proof. Let $v' \in L^2(\Gamma)$ be the weak derivative of $v \in H^1(\Gamma)$. Denote $\hat{v}(x) \equiv (\mathbf{P}_h v)(x)$ and $\tilde{v}(x) \equiv (\tilde{\mathbf{P}}_h v)(x)$. By the definitions of $\tilde{\mathbf{P}}_h$ and $\mathbf{P}_h$, we have, for $x \in \Gamma_i$,

$$\hat{v}(x) = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} v(\sigma)\,d\sigma, \quad \text{except perhaps at the end points of } \Gamma_i,$$

$$\tilde{v}(x) = v(x_i), \quad x_i \text{ is the midpoint of } \Gamma_i,$$

where $|\Gamma_i|$ = length of Γ_i.

Let (α_{i-1}, α_i) be the two end points of Γ_i. Then on Γ_i, we have

$$\begin{aligned}
|\hat{v}(x) - \tilde{v}(x)| &= \left|\frac{1}{|\Gamma_i|}\int_{\Gamma_i} v(\sigma)\,d\sigma - v(x_i)\right| = \frac{1}{|\Gamma_i|}\left|\int_{\Gamma_i}[v(\sigma) - v(x_i)]\,d\sigma\right| \\
&= \frac{1}{|\Gamma_i|}\left|\int_{\Gamma_i}\left[\int_{\alpha_{i-1}}^{\sigma} v'(\tau)\,d\tau - \int_{\alpha_{i-1}}^{x_i} v'(\tau)\,d\tau\right]d\sigma\right| \\
&\leqslant \frac{1}{|\Gamma_i|}\int_{\Gamma_i}\left|\int_{\sigma}^{x_i} v'(\tau)\,d\tau\right|d\sigma \leqslant \left(\frac{h}{2}\right)^{1/2}\frac{1}{|\Gamma_i|}\int_{\Gamma_i}\left[\int_{\sigma}^{x_i}|v'(\tau)|^2\,d\tau\right]^{1/2}d\sigma \\
&\leqslant \left(\frac{h}{2}\right)^{1/2}\frac{1}{|\Gamma_i|}|\Gamma_i|\left[\int_{\Gamma_i}|v'(\tau)|^2\,d\tau\right]^{1/2} = \left(\frac{h}{2}\right)^{1/2}\left[\int_{\Gamma_i}|v'(\tau)|^2\,d\tau\right]^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathbf{P}_h v - \tilde{\mathbf{P}}_h v\|^2 &= \sum_i |\hat{v}(x) - \tilde{v}(x)|^2|\Gamma_i| \leqslant \frac{h}{2}\sum_i \int_{\Gamma_i}|v'(\tau)|^2\,d\tau|\Gamma_i| \\
&\leqslant \frac{h}{2}\max_i|\Gamma_i|\,\|v\|_{H^1}^2 \leqslant Ch^2\|v\|_{H^1}^2. \quad \square
\end{aligned}$$

COROLLARY 3.2. *There exists a constant $C > 0$ such that for all $h > 0$ and any $w \in V_h$,*

$$\|\mathbf{P}_h K^*w - K_h^*w\| = \|\mathbf{P}_h K^*w - \tilde{\mathbf{P}}_h K^*w\| \leqslant Ch\|w\|,$$
$$\|\mathbf{P}_h Lw - L_h w\| = \|\mathbf{P}_h Lw - \tilde{\mathbf{P}}_h Lw\| \leqslant Ch\|w\|.$$

Proof. We just note that

$$K_h^* = \tilde{\mathbf{P}}_h K^*, \qquad L_h = \tilde{\mathbf{P}}_h L$$

and that $K^*, L\colon H^0(\Gamma) \to H^1(\Gamma)$ are continuous. $\square$

LEMMA 3.3. *There exists $\gamma_3 > 1$ such that*

$$\gamma_3^{-1}\|w\|^2 \leqslant a_h(w,w) \leqslant \gamma_3\|w\|^2 \quad \forall w \in V_h, \text{ for all } h > 0 \text{ sufficiently small.} \tag{3.16}$$

Proof. For $w_h \in V_h$, we have

$$(I - K_h^*)w_h = \tilde{\mathbf{P}}_h(I - K^*)w_h = \mathbf{P}_h(I - K^*)w_h + (\tilde{\mathbf{P}}_h - \mathbf{P}_h)(I - K^*)w_h.$$

But

$$(\tilde{\mathbf{P}}_h - \mathbf{P}_h)(I - K^*)w_h = -(\tilde{\mathbf{P}}_h - \mathbf{P}_h)K^*w_h,$$

so

$$\|(\tilde{\mathbf{P}}_h - \mathbf{P}_h)(I - K^*)w_h\| = \|(\tilde{\mathbf{P}}_h - \mathbf{P}_h)K^*w_h\| \leqslant Ch\|w_h\|,$$

by Corollary 3.2.

Similarly,

$$L_h w_h = \tilde{\mathbf{P}}_h L w_h = \mathbf{P}_h L w_h + (\tilde{\mathbf{P}}_h - \mathbf{P}_h) L w_h$$

and

$$\|(\tilde{\mathbf{P}}_h - \mathbf{P}_h) L w_h\| \leqslant Ch\|w_h\|.$$

The rest of the proof follows from the proof of Lemma 2.3 and the fact that $\mathbf{P}_h$ is the identity operator on V_h. □

LEMMA 3.4. *Let $w_h \in V_h$, $w \in L^2(\Gamma)$ and w_h tend to w weakly in $L^2(\Gamma)$. Then*

$$\lim_{h \downarrow 0} \theta_h(w_h) = \theta(w). \tag{3.17}$$

Proof. Routine verification. □

COROLLARY 3.5. *The dual problem* (3.12) *has a unique maximizing solution* λ_h^*. *Furthermore,* $\hat{y}_h(\lambda_h^*)$ *and* $\hat{u}_h(\lambda_h^*)$ *in* (3.10), (3.11) *corresponding to* λ_h^* *are the unique solution of the discretized problem* (3.8).

4. Convergence and Error Estimates. Let a, a_h, θ, θ_h and V_h be defined as before. We wish to establish convergence and error estimates. We argue along the line of perturbations; see [3], for example.

Throughout this section, we let $C > 0$ be a generic constant independent of h.

The optimal multipliers λ^* and λ_h^* for problems (2.16), (3.12), respectively, are solutions to the following variational equations:

$$a(\lambda^*, \mu) = \theta(\mu) \quad \forall \mu \in L^2(\Gamma), \tag{4.1}$$

$$a_h(\lambda_h^*, \mu_h) = \theta_h(\mu_h) \quad \forall \mu_h \in V_h. \tag{4.2}$$

Note that a_h and θ_h are only defined on V_h.

For any $\mu_h \in V_h$, we have

$$\begin{aligned} a(\lambda^* - \lambda_h^*, \mu_h) &= a(\lambda^*, \mu_h) - a(\lambda_h^*, \mu_h) \\ &= \theta(\mu_h) + [a_h(\lambda_h^*, \mu_h) - \theta_h(\mu_h)] - a(\lambda_h^*, \mu_h) \\ &= (\theta - \theta_h)(\mu_h) - (a - a_h)(\lambda_h^*, \mu_h). \end{aligned} \tag{4.3}$$

Thus,

$$\begin{aligned} a(\lambda^* - \lambda_h^*, \lambda^* - \lambda_h^*) &= a(\lambda^* - \lambda_h^*, \lambda^* - \mu_h) + a(\lambda^* - \lambda_h^*, \mu_h - \lambda_h^*) \\ &= a(\lambda^* - \lambda_h^*, \lambda^* - \mu_h) + [(\theta - \theta_h)(\mu_h - \lambda_h^*) - (a - a_h)(\lambda_h^*, \mu_h - \lambda_h^*)]. \end{aligned}$$

By Lemma 2.3,

$$\begin{aligned} \gamma_2^{-1}\|\lambda^* - \lambda_h^*\| &\leqslant a(\lambda^* - \lambda_h^*, \lambda^* - \lambda_h^*) \\ &\leqslant \gamma_2\|\lambda^* - \lambda_h^*\| \, \|\lambda^* - \mu_h\| + |(\theta - \theta_h)(\lambda_h^* - \mu_h)| \\ &\quad + |(a - a_h)(\lambda_h^*, \lambda_h^* - \mu_h)| \quad \forall \mu_h \in V_h. \end{aligned} \tag{4.4}$$

From (4.2), (3.16), (3.14) it is easy to see that there is $C > 0$ such that

$$\|\lambda_h^*\| \leqslant C \quad \forall h > 0. \tag{4.5}$$

In (4.2), we use

$$\mu_h = \mathbf{P}_h \lambda^*. \tag{4.6}$$

Let

$$\varepsilon_{1h} \equiv \left|(\theta - \theta_h)(\lambda_h^* - \mu_h)\right|, \qquad \varepsilon_{2h} \equiv \left|(a - a_h)(\lambda_h^*, \lambda_h^* - \mu_h)\right|. \tag{4.7}$$

By (4.5) and (4.6), λ_h^* has a weak limit in $L^2(\Gamma)$, and μ_h has a strong limit λ^*. Using Lemma 3.4, we get

$$\lim_{h \downarrow 0} \varepsilon_{1h} = 0. \tag{4.8}$$

We note that

$$\begin{aligned}
&(a - a_h)(\lambda_h^*, \lambda_h^* - \mu_h) \\
&\quad = \left\langle (I - K^*)\lambda_h^*, (I - K^*)(\lambda_h^* - \mu_h)\right\rangle + \left\langle N^{-1}L\lambda_h^*, L(\lambda_h^* - \mu_h)\right\rangle \\
&\qquad - \left\langle (I - K_h^*)\lambda_h^*, (I - K_h^*)(\lambda_h^* - \mu_h)\right\rangle - \left\langle \mathbf{P}_h N^{-1} L_h \lambda_h^*, L_h(\lambda_h^* - \mu_h)\right\rangle \\
&\quad = \left\langle \left[(\mathbf{P}_h - I)K^* + (\tilde{\mathbf{P}}_h - \mathbf{P}_h)K^*\right]\lambda_h^*, (I - K^*)(\lambda_h^* - \mu_h)\right\rangle \\
&\qquad + \left\langle (I - K_h^*)\lambda_h^*, \left[(I - \mathbf{P}_h)K^* + (\mathbf{P}_h - \tilde{\mathbf{P}}_h)K^*\right](\lambda_h^* - \mu_h)\right\rangle \\
&\qquad + \left\langle (N^{-1}L - \mathbf{P}_h N^{-1} L_h)\lambda_h^*, L_h(\lambda_h^* - \mu_h)\right\rangle \\
&\qquad + \left\langle \mathbf{P}_h N^{-1} L_h \lambda_h, \left[(I - \mathbf{P}_h)L + (\mathbf{P}_h - \tilde{\mathbf{P}}_h)L\right](\lambda_h^* - \mu_h)\right\rangle.
\end{aligned} \tag{4.9}$$

Now using the property that for $M = K^*$ or L,

$$\begin{aligned}
\|(I - \mathbf{P}_h)Mv_h\| &= \inf_{w_h \in V_h} \|Mv_h - w_h\| \leqslant Ch\|Mv_h\|_{H^1} \\
&\leqslant Ch\|v_h\| \to 0 \quad \text{as } h \downarrow 0, \text{ for any bounded sequence } v_h \in V_h,
\end{aligned} \tag{4.10}$$

and that

$$\begin{aligned}
&\left\|(N^{-1}L - \mathbf{P}_h N^{-1} L_h)\lambda_h^*\right\| \\
&\quad = \left\|(I - \mathbf{P}_h)N^{-1}L\lambda_h^* + \mathbf{P}_h N^{-1}(I - \mathbf{P}_h)L\lambda_h^* + \mathbf{P}_h N^{-1}(\mathbf{P}_h - \tilde{\mathbf{P}}_h)L\lambda_h^*\right\| \\
&\quad \leqslant Ch\|L\lambda_h^*\|_{H^1} \leqslant Ch\|\lambda_h^*\| \to 0 \quad \text{as } h \downarrow 0,
\end{aligned} \tag{4.11}$$

for the bounded sequence λ_h^*, as well as Lemma 3.1, we obtain

$$\lim_{h \downarrow 0} \varepsilon_{2h} = \lim_{h \downarrow 0} \left|(a - a_h)(\lambda_h^*, \lambda_h^* - \mu_h)\right| = 0. \tag{4.12}$$

From (4.4) we thus have

$$\|\lambda^* - \lambda_h^*\|^2 - \gamma_2^2\|\lambda^* - \mu_h\|\,\|\lambda^* - \lambda_h^*\| - \gamma_2(\varepsilon_{1h} + \varepsilon_{2h}) \leqslant 0,$$

yielding

$$\|\lambda^* - \lambda_h^*\| \leqslant \frac{\gamma_2^2\|\lambda^* - \mu_h\| + \left[\gamma_2^4\|\lambda^* - \mu_h\|^2 + 4\gamma_2(\varepsilon_{1h} + \varepsilon_{2h})\right]^{1/2}}{2}, \tag{4.13}$$

by the quadratic solution formula.

By (4.8) and (4.12) it is easy to see that the right-hand side above tends to 0 as $h \downarrow 0$.

We summarize the above in

THEOREM 4.1. *We have*

$$\lim_{h \downarrow 0} \lambda_h^* = \lambda^* \quad \textit{strongly in } L^2(\Gamma). \quad \square$$

COROLLARY 4.2. *We also have the convergence of the optimal state and control,*

$$\lim_{h \downarrow 0} \hat{y}_h(\lambda_h^*) = \hat{y}_{\lambda^*}, \qquad \lim_{h \downarrow 0} \hat{u}_h(\lambda_h^*) = \hat{u}_{\lambda^*}.$$

Proof. This is a consequence of Theorem 4.1, (2.15), (3.10), (3.11) and Corollary 3.2. $\square$

Next, we study error estimates.

Assume that z_d in (1.1) lies in $H^1(\Gamma)$. Then $\lambda^* \in H^1(\Gamma)$ by Theorem 2.5. From (3.2) and (4.6), we have

$$\|\lambda^* - \mu_h\| \leq Ch\|\lambda^*\|_{H^1(\Gamma)}. \tag{4.14}$$

To obtain $\|\lambda^* - \lambda_h^*\|$, it is important to know the orders of magnitude of ε_{1h} and ε_{2h} in (4.7):

$$\begin{aligned} \varepsilon_{1h} &= |(\theta - \theta_h)(\lambda_h^* - \mu_h)| = 2|\langle \lambda_h^* - \mu_h, (I - K)z_d - (I - K_h)\mathbf{P}_h z_d \rangle| \\ &= 2|\langle [(I - K^*) - \mathbf{P}_h(I - K^*)](\lambda_h^* - \mu_h), z_d \rangle| \\ &= 2|\langle (K^* - \mathbf{P}_h K^*)(\lambda_h^* - \mu_h), z_d \rangle| \\ &= 2|\langle (I - \mathbf{P}_h)K^*(\lambda_h^* - \mu_h), z_d \rangle|. \end{aligned} \tag{4.15}$$

From (4.10), we obtain

$$\varepsilon_{1h} \leq 2Ch\|\lambda_h^* - \mu_h\| \, \|z_d\| = O(h). \tag{4.16}$$

Similarly, from (4.9), (4.10) and (4.11) and Corollary 3.2,

$$\varepsilon_{2h} = |(a - a_h)(\lambda_h^*, \lambda_h^* - \mu_h)| \leq Ch\|\lambda_h^*\| \, \|\lambda_h^* - \mu_h\| = O(h). \tag{4.17}$$

Using (4.14), (4.16) and (4.17) on the right of (4.13), we get

$$\|\lambda^* - \lambda_h^*\| \leq C'h^{1/2}\|\lambda_h^* - \mu_h\|^{1/2} + C\gamma_2^2 h\|\lambda^*\|_{H^1(\Gamma)} \tag{4.18}$$

for some $C' > 0$ depending on λ^*, λ_h^* and z_d only, independent of h. Therefore,

$$\|\lambda^* - \lambda_h^*\| \leq C'h^{1/2}\left[\|\lambda^* - \lambda_h^*\|^{1/2} + \|\lambda^* - \mu_h\|^{1/2}\right] + C\gamma_2^2 h\|\lambda^*\|_{H^1(\Gamma)}.$$

By (4.13),

$$\|\lambda^* - \lambda_h^*\| - C'h^{1/2}\|\lambda^* - \lambda_h^*\|^{1/2} - (C'C^{1/2} + C\gamma_2^2)h\|\lambda^*\|_{H^1(\Gamma)} \leq 0.$$

Thus, for h sufficiently small, we obtain

$$\begin{aligned} \|\lambda^* - \lambda_h^*\|^{1/2} &\leq \tfrac{1}{2}\left\{C'h^{1/2} + \left[C'^2 h + 4(C'C^{1/2} + C\gamma_2^2)h\|\lambda^*\|_{H^1(\Gamma)}\right]^{1/2}\right\} \\ &= O(h^{1/2}), \end{aligned}$$

$$\|\lambda^* - \lambda_h^*\| = O(h), \tag{4.19}$$

the optimal rate of convergence with piecewise constant elements V_h.

By (2.15), (3.10), (3.11), (4.12), (4.19) and (3.2), we obtain

$$\|\hat{y}_{\lambda^*} - \hat{y}_h(\lambda_h^*)\| = O(h), \tag{4.20}$$

$$\|\hat{u}_{\lambda^*} - \hat{u}_h(\lambda_h^*)\| = O(h). \tag{4.21}$$

THEOREM 4.3. *Assume that $z_d \in H^1(\Gamma)$. Then the BEM and duality algorithm of Section* 3 *yield the optimal rate of convergence* (4.19), (4.20) *and* (4.21). *Furthermore, we have*

$$|J_{\lambda^*}^h - J_{\lambda^*}| = O(h^2). \quad \square \tag{4.22}$$

5. Numerical Results. We consider the following examples. Let

$$\Omega = \{(r,\theta) \mid r < 1/3\}$$

be a disk centered at the origin with radius 1/3, and let

$$z_d(\theta) = \frac{4}{27}\cos 3\theta + 1 \quad \text{on } \Gamma; \qquad \Gamma = \left\{\left(\frac{1}{3},\theta\right) \mid 0 \leqslant \theta \leqslant 2\pi\right\}.$$

Γ is discretized into n equal arcs, each with arc length $\frac{1}{3} \cdot (2\pi/n)$.

Example 1. Let

$$N = I \quad \text{on } L^2(\Gamma), \tag{5.1}$$

and consider

$$\operatorname*{Min}_u \int_\Gamma \left[\left|y\left(\frac{1}{3},\theta\right) - z_d(\theta)\right|^2 + u^2(\theta)\right] d\theta \tag{5.2}$$

subject to

$$\begin{cases} \dfrac{\partial}{\partial r}\left[r\dfrac{\partial}{\partial r}y(r,\theta)\right] + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}y(r,\theta) = 0 & \text{on } \Omega, \\ \dfrac{\partial}{\partial r}y(r,\theta)|_{r=1/3} = u(\theta). \end{cases}$$

The integral (5.2) is discretized as

$$\sum_{i=1}^{n(h)}\left[\left|y_h\left(\frac{1}{3},\phi_{h,i}\right) - z_d(\phi_{h,i})\right|^2 + u_h^2(\phi_{h,i})\right] \cdot \frac{1}{3}\left[\frac{2\pi}{n(h)}\right], \tag{5.3}$$

and the integral equation (2.6) is discretized to

$$y_h\left(\frac{1}{3},\phi_{h,i}\right) - \frac{1}{\pi}\sum_{j=1}^{n}\theta_{ij}^{(h)}y\left(\frac{1}{3},\phi_{h,i}\right) + \sum_{j=1}^{n}q_{ij}^{(h)}u_h(\phi_{h,i}) = 0, \tag{5.4}$$

$$i = 1,2,\ldots,n(h),$$

where

$$\phi_{h,i} = i \cdot \frac{2\pi}{n}, \qquad i = 1,2,\ldots,n(h),$$

$$\theta_{ij}^{(h)} = \frac{\pi}{n}, \qquad i,j = 1,2,\ldots,n(h),$$

$$q_{ij}^{(h)} \approx \frac{1}{\pi}\int_{\Gamma_j}\ln\left|\frac{1}{3}e^{i\phi_{h,i}} - \frac{1}{3}e^{i\theta}\right| \cdot \frac{1}{3}\,d\theta, \qquad i,j = 1,2,\ldots,n(h), \tag{5.5}$$

$$h = \frac{2\pi}{3n}.$$

The discretized problem is then solved by the max-min duality method as in Section 4.

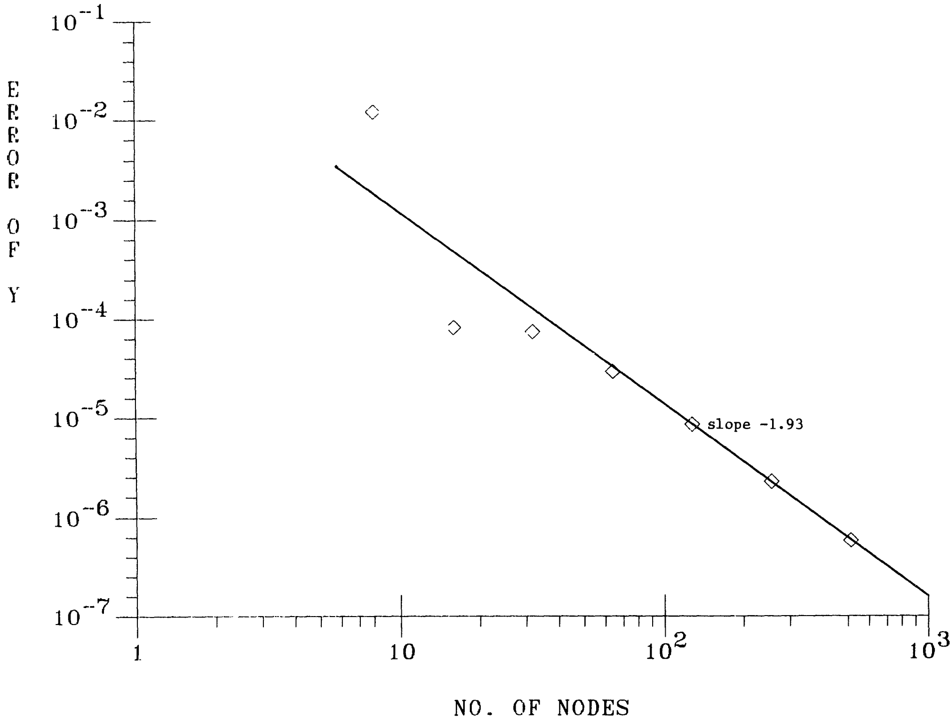

FIGURE 3

Because of the lack of exact solutions to make direct comparisons, we can only estimate the rate of convergence of $\|\hat{u} - \hat{u}_h\|$, $\|\hat{y} - \hat{y}_h\|$ and $\|J_{\lambda^*} - J^h_{\lambda^*}\|$ by comparing two successive solutions.

In Figure 3 we plot the logarithm of errors by calculating the maximum difference of successive solutions at nodes:

$$\max_{1 \leqslant i \leqslant n(2h)} |\hat{y}_h(\phi_{2h,i}) - \hat{y}_{2h}(\phi_{2h,i})|$$

$$\text{for } h = \frac{2\pi}{3} \times \left[\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256} \text{ and } \frac{1}{512}\right],$$

respectively. See the first column of Table 1 for these values.

TABLE 1
Rate of convergence

error of solutions / # of nodes	y	u	J^h_λ
8	1.1952×10^{-2}	2.6843×10^{-2}	1.8542×10^{-3}
16	8.0882×10^{-5}	3.6740×10^{-4}	1.2548×10^{-5}
32	7.4117×10^{-5}	3.3701×10^{-4}	1.1499×10^{-5}
64	2.8951×10^{-5}	1.2964×10^{-4}	4.4915×10^{-6}
128	8.4762×10^{-6}	3.7751×10^{-5}	1.3150×10^{-6}
256	2.2674×10^{-6}	1.0085×10^{-5}	3.5177×10^{-7}
512	5.8421×10^{-7}	2.5969×10^{-6}	9.0635×10^{-8}

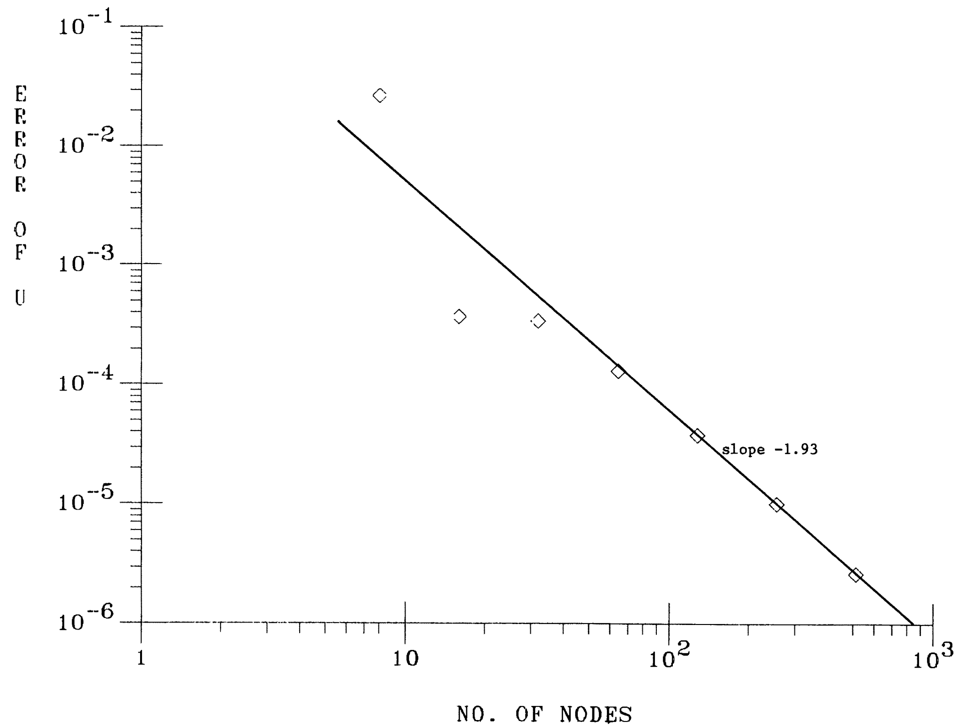

FIGURE 4

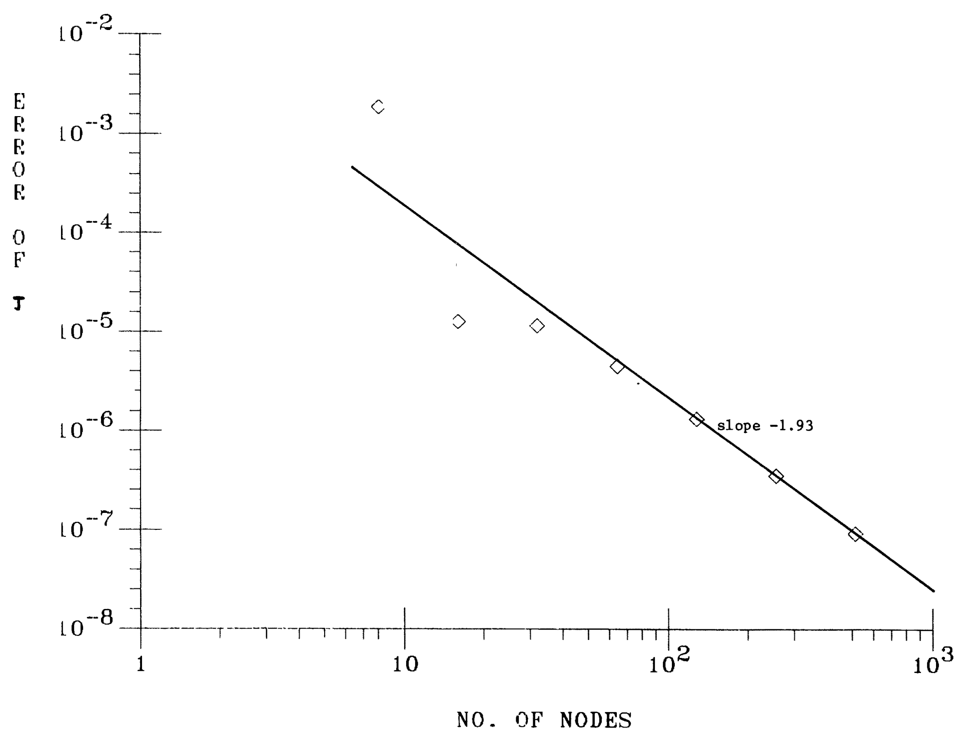

FIGURE 5

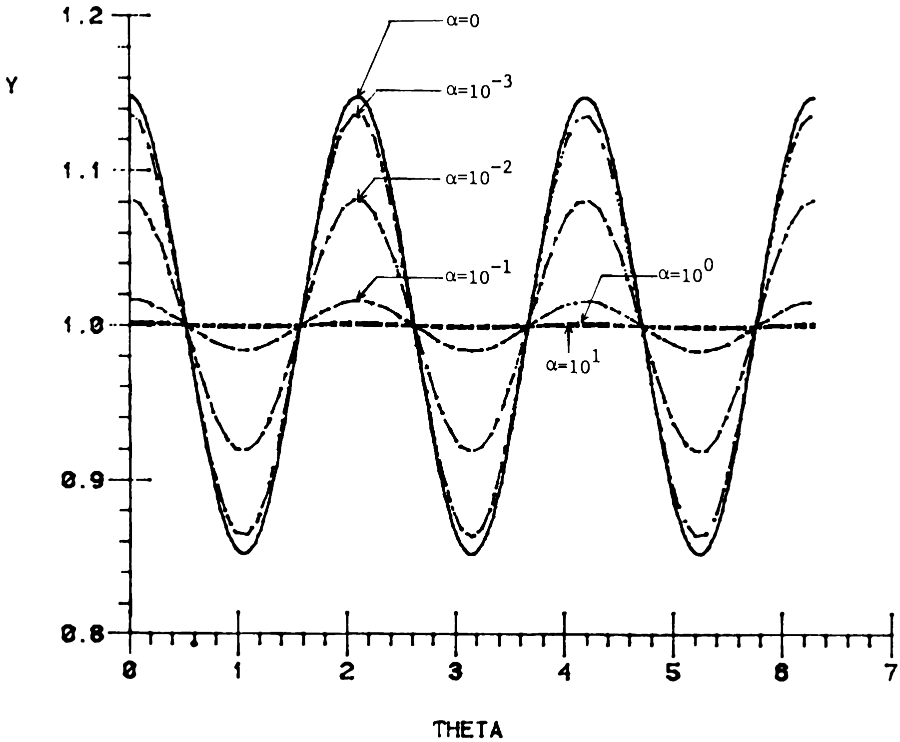

FIGURE 6

The slope of the line passing the last 3 points is measured to be −1.93. This is consistent with the superconvergence rate $O(h^2)$ proved in [4]. As a result (cf. [4]), the L^2-norm convergence rate $O(h)$ follows by linear interpolation. This agrees with our estimate (4.20).

In Figure 4 we do the same for $\hat{u}$, with

$$\max_{1 \leqslant i \leqslant n(2h)} \left| \hat{u}_h(\phi_{2h,i}) - \hat{u}_{2h}(\phi_{2h,i}) \right|.$$

The slope is again measured to be −1.93. See the second column of Table 1.

In Figure 5 we plot the logarithm of errors of J_λ^h by calculating

$$\left| J_{\lambda^*}^h - J_{\lambda^*}^{2h} \right|$$

for the same h values as above. The slope of the line is −1.93; this verifies the theoretical estimate $O(h^2)$ given in Theorem 4.3. See column 3 of Table 1 for values.

We wish to remark that the experimental rate of convergence hinges almost entirely on the order of accuracy of approximating the weakly singular integral (5.5). At first we have tried to evaluate $q_{ij}^{(h)}$ by the Simpson rule. It still yields good accuracy, but J_λ^h converges with a rate of only $O(h^{1.33})$—quite unsatisfactory for the purpose of our paper. Afterwards we decided to evaluate $q_{ii}^{(h)}$ differently (whose integrand contains a logarithmic singularity) by using a series expansion formula, and evaluate $q_{ij}^{(h)}$, $j \neq i$, still by Simpson's rule. This immediately improves the rate to $O(h^{1.93})$ for the convergence of J_λ^h.

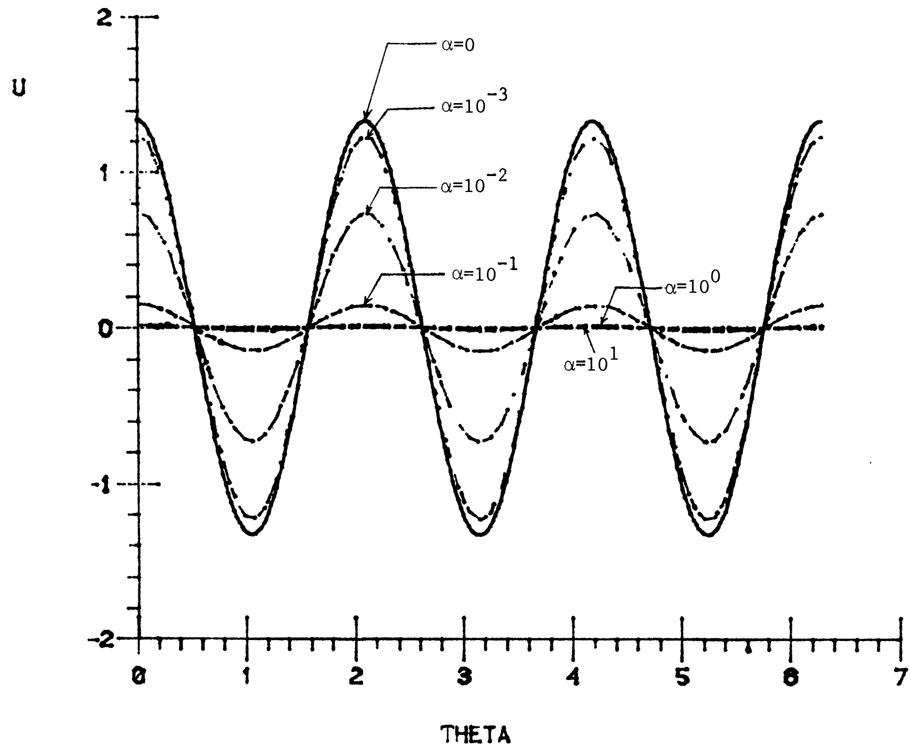

FIGURE 7

Example 2. To test whether our computations are correct, we consider the same example as above, except that now we let

$$N = \alpha I \tag{5.6}$$

with $\alpha = 10, 1, 1/10, 1/10^2$ and $1/10^3$, and $h = 2\pi/64$ throughout.

Note that

$$\tilde{y}(r,\theta) = 4r^3 \cos 3\theta + 1$$

is the exact solution of the problem

$$\begin{cases} \Delta \tilde{y}(r,\theta) = 0, \\ \dfrac{\partial}{\partial r}\tilde{y}(r,\theta)|_{r=1/3} = \dfrac{4}{3}\cos 3\theta = \tilde{u}(\theta), \\ \tilde{y}\left(\dfrac{1}{3},\theta\right) = \dfrac{4}{27}\cos 3\theta + 1 = z_d(\theta). \end{cases}$$

Thus, as $\alpha \to 0$ in (5.6), $(\hat{y}, \hat{u})$ should tend to $(\tilde{y}, \tilde{u})$. □

This is confirmed in Figures 6 and 7. See the data at selected nodes in Tables 2 and 3.

TABLE 2
Values of $\hat{y}(\theta)$ for different α

α \ θ	0	$\pi/4$	$3\pi/4$	$9\pi/8$	$13\pi/8$
10	0.10002×10^1	0.99987×10^0	0.10001×10^1	0.99993×10^0	0.99983×10^0
1	0.10018×10^1	0.99873×10^0	0.10013×10^1	0.99931×10^0	0.99834×10^0
10^{-1}	0.10162×10^1	0.98855×10^0	0.10114×10^1	0.99381×10^0	0.98505×10^0
10^{-2}	0.10816×10^1	0.94229×10^0	0.10577×10^1	0.96877×10^0	0.92460×10^0
10^{-3}	0.11370×10^1	0.90314×10^0	0.10969×10^1	0.94758×10^0	0.87345×10^0
0 ($\bar{y}$)	0.11481×10^1	0.89524×10^0	0.11048×10^1	0.94331×10^0	0.86313×10^0

TABLE 3
Values of $\hat{u}(\theta)$ for different θ

α \ θ	0	$\pi/4$	$3\pi/4$	$9\pi/8$	$13\pi/8$
10	0.16388×10^{-2}	-0.11588×10^{-2}	0.11588×10^{-2}	-0.62713×10^{-3}	-0.15140×10^{-2}
1	0.16209×10^{-1}	-0.11461×10^{-1}	0.11461×10^{-1}	-0.62029×10^{-2}	-0.14975×10^{-1}
10^{-1}	0.14615×10^0	-0.10334×10^0	0.10334×10^0	-0.55929×10^{-1}	-0.13503×10^0
10^{-2}	0.73689×10^0	-0.52106×10^0	0.52106×10^0	-0.28200×10^0	-0.68080×10^0
10^{-3}	0.12368×10^1	-0.87457×10^0	0.87457×10^0	-0.47331×10^0	-0.11427×10^1
0 ($\bar{u}$)	0.13333×10^1	-0.94281×10^0	0.94281×10^0	-0.51024×10^0	-0.12328×10^1

6. Miscellaneous Remarks. (1) For our problem (1.1), (1.2), if we were to treat it numerically by solving a system of equations as in [6], the amount of work would be much larger. In this case, BEM has an advantage of roughly $O(n)$ operations versus $O(n^2)$ operations using FEM. The saving is substantial.

(2) Our method mentioned here can be immediately extended to treat the following problem [6]:

$$(6.1)\qquad \begin{cases} \min\limits_{u \in L^2(\Gamma)} J(u) = \int_\Omega |y(x) - z_d(x)|^2\, dx + \langle Nu, u\rangle_{L^2(\Gamma)} \\ \text{subject to} \\ \Delta y = f \quad \text{on } \Omega,\ f \text{ given on } \Omega, \\ \dfrac{\partial y}{\partial \nu} = u \quad \text{on } \Gamma, \\ \int_\Gamma u(x)\, d\sigma = \int_\Omega f\, dx. \end{cases}$$

However, Ω must now also be discretized in order to evaluate $\int_\Omega f(\xi) v(\xi, x)\, d\xi$. The efficiency of the BEM is lost to some extent.

(3) For (6.1), one can also use the boundary integral equation formulation to obtain regularity of $(\hat{y}, \hat{u})$, as we have done in Theorem 2.5.

(4) Although for convenience we have only used piecewise constant elements (in Section 3), it is understood that any (r, s)-system of finite element spaces V_h on Γ can be used to approximate the variational equation (4.1) and give error estimates. But generally the nice and simple geometric relation (3.3) is lost.

(5) Many restrictions such as (diameter Ω) < 1, convexity of Ω and C^∞-smoothness of Γ can be removed or relaxed without causing any computational and theoretical difficulty.

For elliptic linear quadratic problems with Dirichlet control (as opposed to Neumann control here), how does one use BEM to study them? This will be discussed in Part II.

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