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Bi-Cyclide and Flat-Ring Cyclide Coordinate Surfaces: Correction of Two Expressions

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Abstract. Bi-cyclide and flat-ring cyclide coordinates are three-dimensional rotational coordinate systems based on conformal transformations using the Jacobian elliptic function sn. We have checked the previously published formulae of these systems (P. Moon and D. E. Spencer, *Field Theory Handbook*, Springer-Verlag, Berlin, 1971). In both cases the expression for the rotation-cyclide surfaces was incorrect: thus we present rederivations. The expressions were verified with the symbolic-algebraic computation package MACSYMA.

1. Introduction. Novel orthogonal coordinate systems in two dimensions can be generated by conformal transformations using analytic functions of complex variables; three-dimensional systems follow by rotation about either the real or the imaginary axes [8], [9]. Our interest in these systems is related to the calculation of the magnetic potential in and around nonspherical objects introduced into a uniform magnetic field; of particular interest are the biconcave-disc shapes of some red blood cells [3]. Among the analytic functions that yield coordinate curves that are similar to the cross section of biconcave discs is the Jacobian elliptic function $z = x + iy = a\,\mathrm{sn}(w, k)$ [6], [7], where a is real and the complex numbers $w = \mu + i\nu$ and k are the *argument* and *modulus*, respectively [1]. Separation of the real and imaginary parts of the elliptic function yields two coordinate-transformation equations, in x and y [6]–[9]:

$$x = \frac{a}{\Lambda}\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu', \tag{1.1}$$

$$y = \frac{a}{\Lambda}\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu'\,\mathrm{cn}\,\nu', \tag{1.2}$$

$$\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu', \tag{1.3}$$

$$0 \leqslant \mu \leqslant K, \qquad 0 \leqslant \nu \leqslant K', \tag{1.4}$$

where K and K' are the definite elliptic integrals of the first kind with respect to k and its complement k', respectively [2], [5]; and the prime on ν' specifies that k' is used in the elliptic function. The series of coordinate curves shown in Figure 1.1 were drawn for three different values of k in order to emphasize the effects of changes in k on the concavity of the ν = constant curves. Moon and Spencer have already presented similar curves, but only for $k^2 = 0.5$ [6]–[9].

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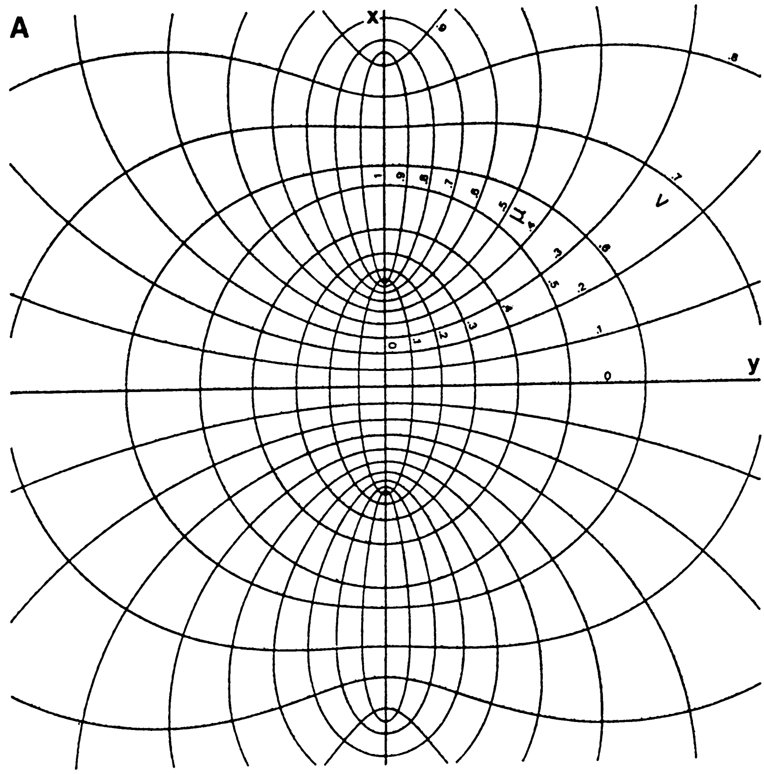
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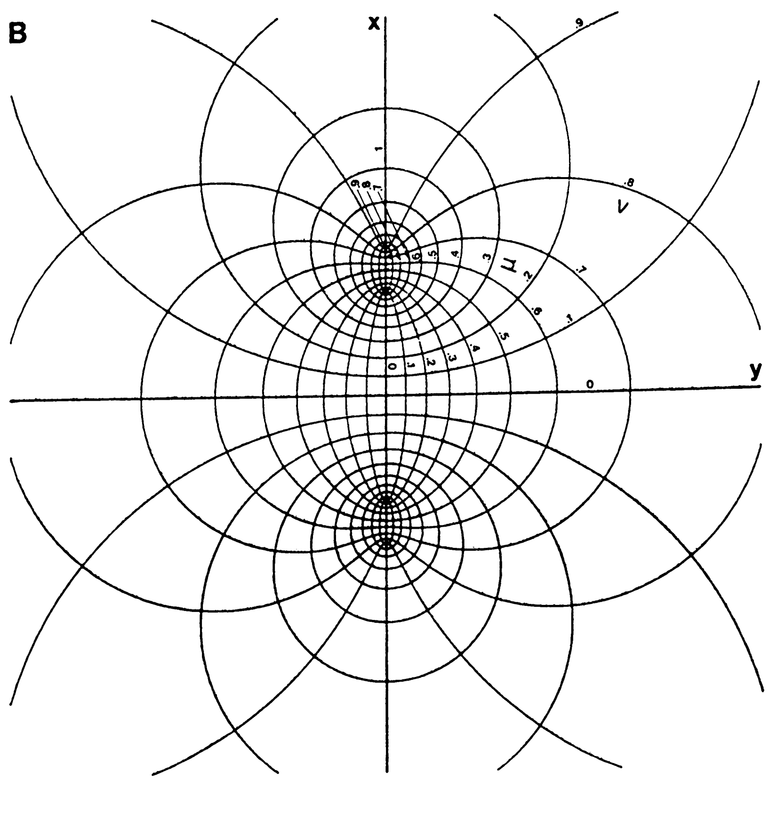
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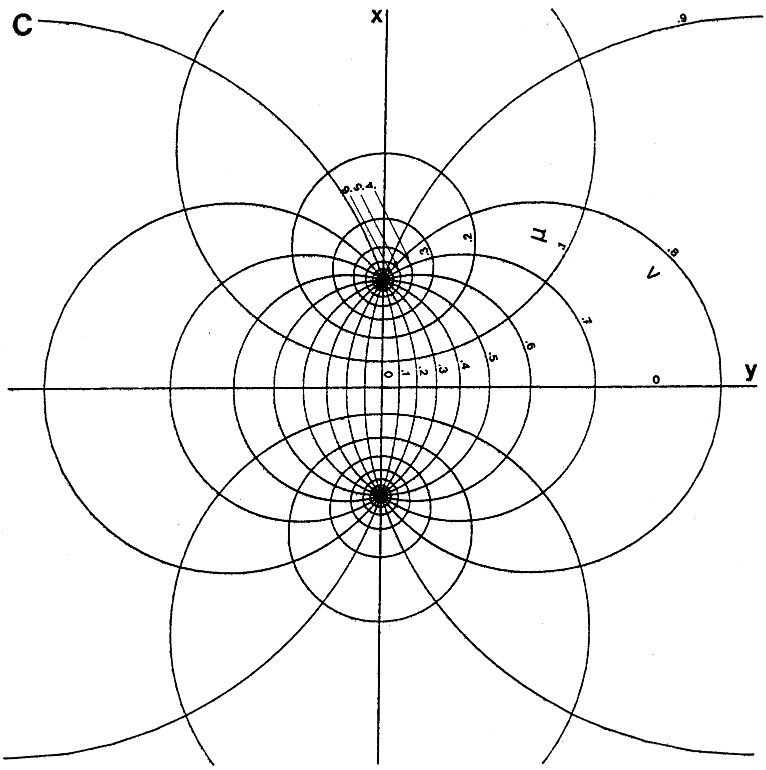

FIGURE 1.1

Orthogonal closed-curve coordinate system described by (1.1)–(1.4). The following values of real a and k^2 were used: A, 30, 0.1; B, 30, 0.5; C, 30, 0.9. K and K' were calculated by computer using the hypergeometric series expression [2, p. 298] and sn was evaluated using the series expression [5, p. 13] programmed in BASIC. The scale-values of μ and ν are fractions of K and K', respectively. The curves were plotted using a Hewlett-Packard Series 9000 model 220 computer and a 7475A plotter.

When the maps of Figure 1.1 are rotated about the x-axis, we obtain an orthogonal family of surfaces [6], [7]. The Cartesian transformations are given in terms of the bi-cyclide coordinates (μ, ν, ψ);

$$x = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu' \operatorname{cn}\nu' \cos\psi, \tag{1.5}$$

$$y = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu' \operatorname{cn}\nu' \sin\psi, \tag{1.6}$$

$$z = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu', \tag{1.7}$$

$$0 \leqslant \mu \leqslant K, \quad 0 \leqslant \nu \leqslant K', \quad 0 \leqslant \psi < 2\pi, \quad \Lambda \text{ as in } (1.3). \tag{1.8}$$

Expressions for the three families of coordinate surfaces (bi-cyclides, μ = constant; rotation cyclides, ν = constant; meridional half-planes, ψ = constant) are obtained by elimination of two of the three bi-cyclide variables from (1.5) to (1.7).

2. Derivation of Expressions for Coordinate Surfaces. The process of variable-elimination from (1.5) to (1.7) was as follows. Let,

$$s = \mathrm{sn}(\mu, k), \qquad c = \mathrm{cn}(\mu, k), \quad d = \mathrm{dn}(\mu, k),$$
$$S = \mathrm{sn}(\nu, k'), \qquad C = \mathrm{cn}(\nu, k'), \quad D = \mathrm{dn}(\nu, k'), \quad k'^2 = 1 - k^2,$$
$$r^2 = x^2 + y^2 + z^2.$$

2.1. $\nu =$ *constant*. The Cartesian coordinate surface for this condition was derived by eliminating ψ and μ from (1.5) and (1.6); ψ was eliminated by squaring these equations followed by addition and using $\cos^2\psi + \sin^2\psi = 1$. Thus, from (1.3),

$$\Lambda = 1 - d^2S^2 = C^2 + k^2S^2s^2, \tag{2.1}$$

from (1.7),

$$\Lambda z/a = sD, \tag{2.2}$$

and from (1.5) to (1.7),

$$\Lambda^2(r^2 - z^2)/a^2 = c^2d^2S^2C^2. \tag{2.3}$$

Squaring both sides of (2.2) and using $D^2 = 1 - k'^2S^2 = C^2 + k^2S^2$ gives

$$\Lambda^2z^2/a^2 = s^2(C^2 + k^2S^2) = (1 - c^2)C^2 + (1 - d^2)S^2 = 1 - c^2C^2 - d^2S^2.$$

Adding this to (2.3) gives

$$\begin{aligned} \Lambda^2r^2/a^2 &= (1 - c^2C^2)(1 - d^2S^2) = (1 - c^2C^2)\Lambda, \\ \Lambda r^2/a^2 &= 1 - c^2C^2 = S^2 + s^2C^2 = s^2 + c^2S^2. \end{aligned} \tag{2.4}$$

We now have

$$\Lambda^2 = (C^2 + k^2S^2s^2)^2, \tag{2.5}$$

$$\Lambda^2z^2/a^2 = D^2s^2, \tag{2.6}$$

$$\Lambda^2r^2/a^2 = S^2C^2 + (C^4 + k^2S^4)s^2 + k^2S^2C^2s^4, \tag{2.7}$$

$$\Lambda^2r^4/a^4 = (S^2 + C^2s^2)^2. \tag{2.8}$$

The right-hand sides are four linear combinations of the three quantities s^4, s^2, and s^0, which can be *eliminated* to yield a linear homogeneous relation between the four left sides. Using the identity $C^4 - k^2S^4 = C^2 - S^2D^2$ and cancellation of Λ^2 gives

$$\frac{k^2r^4}{a^4} - \left(\frac{C^4 + k^2S^4}{S^2C^2}\right)\frac{r^2}{a^2} + \frac{(C^2 - S^2D^2)^2}{S^2C^2D^2}\frac{z^2}{a^2} + 1 = 0. \tag{2.9}$$

To obtain the basic equation-form given by Moon and Spencer [6], [7], [9], we expand the coefficients of (2.9) in sn ν' only:

$$(x^2 + y^2 + z^2)^2 - P(x^2 + y^2) - Qz^2 - R = 0, \tag{2.10}$$

where

$$P = \frac{a^2}{k^2}\left[\frac{(1+k^2)\,\mathrm{sn}^4 \nu' - 2\,\mathrm{sn}^2 \nu' + 1}{(1-\mathrm{sn}^2 \nu')\,\mathrm{sn}^2 \nu'}\right], \tag{2.11}$$

$$\begin{aligned} Q &= P - \frac{(C^2 - S^2D^2)^2}{S^2C^2D^2}\frac{a^2}{k^2} = \frac{a^2}{k^2}\left[\frac{S^2C^2D^4 + k^2S^2C^2}{S^2C^2D^2}\right] \\ &= \frac{a^2}{k^2}\left[\frac{(k^2-1)^2\,\mathrm{sn}^4 \nu' + 2(k^2-1)\,\mathrm{sn}^2 \nu' + (k^2+1)}{(k^2-1)\,\mathrm{sn}^2 \nu' + 1}\right], \end{aligned} \tag{2.12}$$

$$R = -\frac{a^4}{k^2}. \tag{2.13}$$

Expression (2.10) and its coefficients differs from that given by Moon and Spencer [9, p. 124]. The expressions were, in fact, first derived here using the symbolic-algebraic computation package MACSYMA [4]. Batch-mode procedure files used for computations relating to this and other sections are available from the authors.

2.2. μ = *constant*. The right-hand sides of (2.5) to (2.8) can be re-expressed as linear combinations of S^4, S^2, and S^0 with coefficients depending on the lower-case letters. Elimination of these capitals yielded the coordinate surfaces defined by Eq.

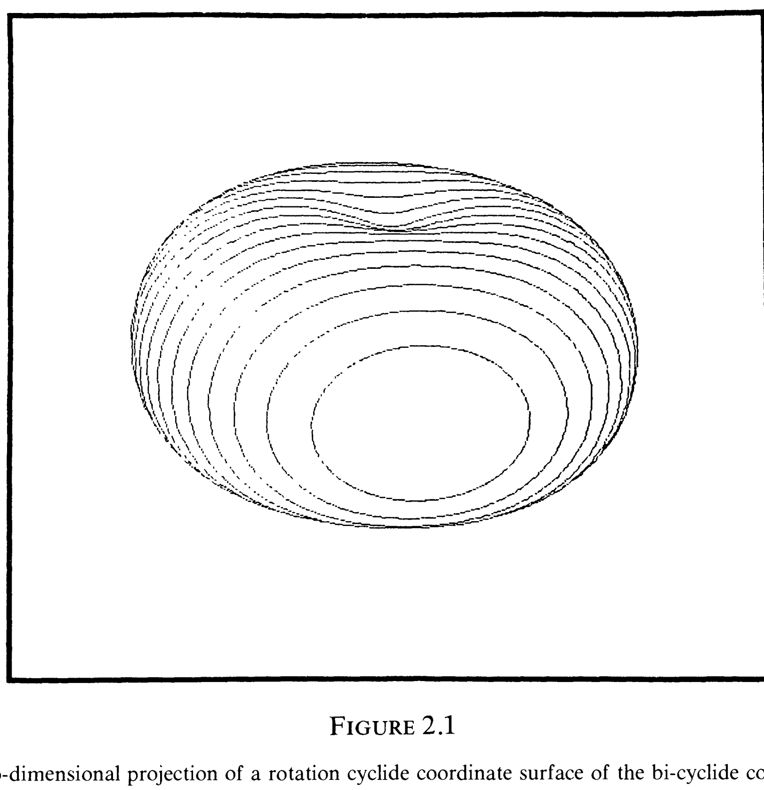

FIGURE 2.1

Two-dimensional projection of a rotation cyclide coordinate surface of the bi-cyclide coordinate system. The parameter values used in Eqs. (2.10) to (2.13) for this computer-based drawing were $a = 3.0$, $k^2 = 0.1$, $\nu = 0.8K'$. The curves were plotted using a Hewlett-Packard Think Jet printer from a screen dump from the computer mentioned in the caption of Figure 1.1.

(2.10) with the following coefficients:

$$(2.14)\qquad P = -\frac{a^2}{k^2}\left[\frac{k^2 \operatorname{cn}^4 \mu + \operatorname{dn}^4 \mu}{\operatorname{cn}^2 \mu \operatorname{dn}^2 \mu}\right],$$

$$(2.15)\qquad Q = \frac{a^2}{k^2}\left[k^2 \operatorname{sn}^2 \mu + \frac{1}{\operatorname{sn}^2 \mu}\right],$$

$$(2.16)\qquad R = -\frac{a^4}{k^2}.$$

These expressions, after some rearrangement, are the same as Moon and Spencer's [9, p. 124].

2.3. ψ = *constant*. This coordinate surface is simply the half-plane given by $\tan \psi = y/x$ [9].

2.4. *Graphical Representation of Surfaces*. Figure 2.1 is a two-dimensional projection of a rotation cyclide obtained using computer graphics which relied on the expressions (2.10) to (2.13).

3. Flat-Ring Cyclide Coordinates (μ, ν, ψ). The Cartesian transformations for this coordinate system are [7], [8],

$$(3.1)\qquad x = \frac{a}{\Lambda} \operatorname{sn} \mu \operatorname{dn} \nu' \cos \psi,$$

$$(3.2)\qquad y = \frac{a}{\Lambda} \operatorname{sn} \mu \operatorname{dn} \nu' \sin \psi,$$

$$(3.3)\qquad z = \frac{a}{\Lambda} \operatorname{cn} \mu \operatorname{dn} \mu \operatorname{sn} \nu' \operatorname{cn} \nu',$$

where Λ and the ranges of the variables are as in (1.3) and (1.8).

The equations of the coordinate surfaces were derived in the same way as the bi-cyclide cases, after noting that (2.8) still holds, although the roles of z^2 and $r^2 - z^2$ are interchanged in the derivation. We confirmed the correctness of Moon and Spencer's expression [8, p. 127] for the flat ring-cyclides (μ = constant). However, the formula for the rotation cyclides (ν = constant) was shown to be wrong. The correct expressions for the coefficients in (2.10) are,

$$(3.4)\qquad P = \frac{a^2}{k^2}\left[\frac{(k^2-1)^2 \operatorname{sn}^4 \nu' + 2(k^2-1)\operatorname{sn}^2 \nu' + (k^2+1)}{(k^2-1)\operatorname{sn}^2 \nu' + 1}\right] \equiv (2.12),$$

$$(3.5)\qquad Q = \frac{a^2}{k^2}\left[\frac{(1+k^2)\operatorname{sn}^4 \nu' - 2\operatorname{sn}^2 \nu' + 1}{(1-\operatorname{sn}^2 \nu')\operatorname{sn}^2 \nu'}\right] \equiv (2.11),$$

$$(3.6)\qquad R = -\frac{a^4}{k^2}.$$

4. Discussion. That the earlier versions of the expressions for the coordinate surfaces, ν = constant, are incorrect can be demonstrated readily by choosing values of ν and substituting the corresponding values of the relevant elliptic functions into them. Fortuitously, if k^2 has the value 0.5 (as was used by Moon and Spencer [7], [8]), then, for a wide range of ν and a values the previous 'equality' is satisfied to within $< 0.01a$. However, if $k^2 \neq 0.5$, a much larger error can appear with the previously published equations; this is not the case with our expressions. We are uncertain whether a *systematic* error arose in the earlier derivations of the formulae; but we have excluded, by use of MACSYMA [4], the suggestion that k instead of k' was used in the expressions containing ν.

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