The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 1980 Mathematics Subject Classification (1985 Revision) can be found in the December index volumes of Mathematical Reviews.

30[65N25, 76W05].—Ralf Gruber & Jacques Rappaz, *Finite Element Methods in Linear Ideal Magnetohydrodynamics*, Springer Series in Computational Physics (H. Cabannes et al., Editors), Springer-Verlag, Berlin, 1985, xi + 180 pp., $23\frac{1}{2}$ cm. Price $34.00.

This book is about finite elements for linearized spectral stability analysis of magnetically confined plasmas. The purpose of such confinement is to reach energy levels at which sustained, controlled thermonuclear fusion can take place in order to generate electric power. This has yet to be accomplished in a practical way, and the Introduction to this book explains why: There are only a few geometries in which confinement has been achieved, and the more efficient the process is energetically, it seems, the more susceptible it is to catastrophic instability which disrupts the confinement.

The magnetohydrodynamic (MHD) equations combine those of compressible, inviscid fluid flow (the plasma) with Maxwell's equations and Ohm's law (the electromagnetic fields). Since the plasma is to be confined, there are regions of magnetic field only (vacuum) and an interface with the confined plasma. Linearization about a stably confined plasma simplifies the problem greatly, but practical geometries are complex (such as helical or toroidal), and the linearized equations are formidable. This is a natural setting for numerical exploration—first to understand the nature of the instabilities, and then to use the understanding to guide the design process. When this reviewer conjures up a picture of the practical applications of his own research, the picture which most often comes to mind is one of some gooey fluid being dolloped out by a dingy industrial process. There is a real flair in being able to model cosmic forces on your computer and watch as the elusive ions escape on the wisps of seconds. This book conveys that romance well, but it is hard for the reader without training in high-energy physics to come away with a more quantitative impression.

So the dutiful reviewer tries to look at the book from the perspective of the finite element analyst, and that approach starts well. It is surprising to find out how easy it is to get into trouble with seemingly simple, symmetric eigenvalue problems in one space dimension! The problems are of the form $|A - \lambda B| = 0$, but the continuum problem can have an infinitely degenerate eigenvalue $\lambda = 0$, and a continuous spectrum. Going at this in a brute-force manner, with linear elements for all fields, leads to what the authors call "spectral pollution". Finite element methods are known to produce inaccurate high-spectrum eigenvalues. In the present problem,

©1987 American Mathematical Society
0025-5718/87 $1.00 + $.25 per page

these eigenvalues are introduced with each mesh refinement, and though each new eigenvalue may actually converge to zero, new inaccurate ones continue to be introduced and cloud the *low* spectrum with nonsense.

A little more investigation shows that spectral pollution results from implied constraints in the multifield Lagrangian. The resolution to pollution is in balancing constrained and unconstrained degrees of freedom: an old finite element story. The problem looks at first like a Mindlin beam, and reduced integration might work. But, no, the derivative is on the wrong term in the constraint equation, and reduced integration may not work. The authors' solution is something that looks like a combination of upwind differencing and lumping—but which is achieved by selection of balanced interpolates.

The point here is that the numerical problems and their resolution seem closely related to, yet distinctly different from, those found in other areas of finite analysis, yet the authors make no connection. Later on in the book, there is another link to constrained media problems. Some solutions (the "Alfvén modes") are characterized by the vanishing of the divergence of one of the fields. The authors develop a rudimentary form of "constraint counting" familiar to Navier-Stokes modelers, but again the authors make no connection. They develop a method—the "finite-hybrid element approach"—argue that it works (but do not present error analysis in this book), and go on to talk about the physics, with its "kink" modes, "ballooning" modes and "Alfvén" modes.

If this book is intended to primarily reach high-energy physicists and tell them new things about the stability of these modes under various confinement conditions, it may well do so admirably. This reviewer is not competent to comment on that. But if that is the case, there seems to be more dwelling on numerical technique than one might expect. Yet, as we have observed, the numerical technique is not comprehensively linked to the numerical world outside of high-energy physics. What, then, *is* the purpose of this book? We do not find out until we reach far into the appendices. These appendices are long and contain a whole code, ERATO 3, with a very thorough write-up. If motivated, we all could compute kink modes. The reviewer is not making light of things here, because in Appendix C we find this revealing and admirable statement:

> "Publishing numerical results is similar to publishing experimental results. The reader of such a publication relies on the integrity of the authors, since it is *often not possible to check the given results*." (Reviewer's emphasis.)

Here, perhaps, we have found this book. It is the authors' consciences speaking, not the other things we thought it might be. This is their integrity on the line: a complete guide to reproducing their results, for those who may doubt them. Along the way, the reader may or may not like their mathematics or physics, but one imagines that the authors' consciences are clear.

DAVID S. MALKUS

Mathematics Research Center
and Engineering Mechanics Department
University of Wisconsin
Madison, Wisconsin 53705