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On the 3-Sylow Subgroup of the Class Group of Quadratic Fields

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Abstract. In this paper we give a large number of quadratic fields $\mathbf{Q}(\sqrt{d})$ whose class group $H(d)$ has 3-Sylow subgroup $H_3(d)$ with rank $r_3(d) > 1$. They include 20 discriminants $d < 0$ with $r_3(d) = 5$. The associate real fields have $r_3(d') = 4$. The distribution of the $H_3(d)$ is studied and its relative frequency is compared with the heuristic conjectures of Cohen and Lenstra. Some of the $H(d)$ are recorded since they are of special interest. Some related topics are also considered.

1. Introduction. For every prime p let $H_p(d)$ be the p-Sylow subgroup of the ideal class group $H(d)$ of the quadratic field k with discriminant d. Let $r_p(d)$ be the rank of $H_p(d)$, and let $h(d)$ be the order of $H(d)$, that is, the *class number* of k. In this paper we are interested in the study of $H_3(d)$ when $r_3(d) > 1$, and especially when $r_3(d) > 2$, since in this case k has an infinite class field tower (cf. [8]).

Following Scholz [9], we denote $r_3(d)$ by r or s according as $d < 0$ or $d > 0$. If $d < 0$, let k' be the *associate real field* of k, that is, the quadratic field of discriminant $d' = -3d$ if $3 \nmid d$ and $d' = -d/3$ if $3 \mid d$. By Scholz's theorem [9], the ranks of $H_3(d)$ and $H_3(d')$ are related by

$$r = s \quad \text{or} \quad r = s + 1.$$

We call the condition on the right the *escalatory case*.

In [11] Shanks defines polynomials $D_3(y)$ and $D_6(z)$ whose values produce d with $r \geqslant 2$. Using these polynomials, the first d with $r = 4$ were obtained: two in [12] and one more in [6]. The distribution of $H_3(d)$ for 250 d with $2 \leqslant r \leqslant 4$ is given in [6]. All fields k obtained by this method are in the escalatory case [11]. In particular, $s = 3$ in the three cases with $r = 4$.

In [3] Diaz y Diaz gives a method to find fields k with $r \geqslant 3$, and he obtains 13 fields with $r = 4$ and 119 fields with $r \geqslant 4$. In [4] it is proved that all these 132 fields have $r = 4$ and that the associates k' have $s = 3$.

Craig [2] constructs infinitely many fields k with $r \geqslant 4$. The smallest of these has discriminant $d \approx -428 \times 10^{100}$. Diaz y Diaz showed that $r = s$ for these fields, so that their associates have $s \geqslant 4$ (cf. [4]). Up to now, these are the only real fields with $s \geqslant 4$ ever discovered, and no imaginary field was known with $r \geqslant 5$.

In this paper we give a large number of fields k with $H_3(d)$ of interest. In particular, 20 imaginary fields with $r = 5$ and associate real fields with $s = 4$ are

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given. In our search we have used the methods given in [11], [3] and [5]. For our computations of $H(d)$ with $d < 0$ we have used a version of the method given in [10].

In Section 2 the distribution of $H_3(d)$ is given for the d obtained by each method (Tables 1–3). In Section 3 these results are compared with Cohen and Lenstra's heuristic conjectures [1]. Section 4 gives numerical examples of special interest (Tables 6–9). In Section 5 we present the 20 fields with $r = 5$ that we know for $-35102371403731 \leqslant d \leqslant -1798827299449207043$ (Table 10). We also consider the 121 equations $4A^3 = B^2 - C^2 d$ associated with each such field (cf. [3]), and we give the solutions in two cases (Tables 11 and 12). In the last section we deal with real quadratic fields. We verify that all our fields with $r = 5$ are in the escalatory case and we give $H(d')$ (Table 13).

Some of the results presented here are contained in [7].

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2. Distribution of the $H_3(d)$ for $d < 0$. In [11] Shanks defines the following polynomials:

$$D_3(y) = 27y^4 - 74y^3 + 84y^2 - 48y + 12,$$
$$D_6(z) = 108z^4 - 148z^3 + 84z^2 - 24z + 3,$$

and he proves that

(1) $d = -4D_3(y)$ with $y \equiv 5 \pmod 6$, $y \neq -1$, and $D_3(y)$ square-free, and

(2) $d = -D_6(z)$ with $z \equiv 1 \pmod 3$, $z \neq 1$, and $D_6(z)$ square-free

are discriminants of imaginary quadratic fields with $r = s + 1 \geqslant 2$.

We have computed $H(d)$ for the 3249 d in (1) for $-12005 \leqslant y \leqslant 11999$ and for the 5680 d in (2) with $-10498 \leqslant z \leqslant 10501$. In Table 1 we give the distribution of $H_3(d)$ for these 8929 fields. As usual, $C(n)$ denotes the cyclic group of order n.

We have also computed $H(d)$ for some other values of y and z. In particular, we have found a new example of $r = 5$ for $d = -D_6(-11360)$.

In [3] Diaz y Diaz gives a method to obtain imaginary quadratic fields with $r \geqslant 3$ with discriminants $d = DD(m)$ depending on a parameter m. The main idea in this method lies in finding solutions of the equation

$$4A^3 = B^2 - C^2 d. \tag{3}$$

In [3], $d = DD(m)$ are considered for all values of m in the interval $11 \leqslant m \leqslant 2000$ and certain values of m in the interval $2000 < m < 10000$.

We have obtained 39519 such $d = DD(m)$ for m prime in several subintervals contained in the interval $30000 < m < 100000$. Among them, 1824 have $r = 4$ and 16 have $r = 5$. Table 2 gives the structures of $H_3(d)$ for those 1840 fields with $r > 3$.

In [5] Mestre gives a method to obtain discriminants d of quadratic fields with $r_p(d) \geqslant 2$ for $p = 3, 5$ and 7. This method is based on the theory of elliptic curves. In [5] the method is developed for $p = 5$ and 7. In [7] it is developed for $p = 3$. Elliptic curves with a rational point of order three are parametrized by integers u, v:

$$C(u, v)\colon\ y^2 + uxy + vy = x^3$$

and, for each such curve, a polynomial of eighth degree $M(t)$ is constructed. For t satisfying certain congruences, this polynomial attains values enabling us to obtain discriminants d of quadratic fields with $r_3(d) \geqslant 2$ (cf. [7]).

Using this method, we have computed $H(d)$ for 3884 fields with $d < 0$. We observe that the frequency of $r > 2$ depends on the curve $C(u, v)$ chosen, being larger for the curve $C(2,1)$. In Table 3 are listed the structures of $H_3(d)$ for the 680 d obtained from $C(2,1)$ and the 3204 d obtained from other curves.

TABLE 1

Distribution of the $H_3(d)$ for $d = -4D_3(y)$ and $d = -D_6(z)$.

STRUCTURE	$-4D_3(y)$	$-D_6(z)$	Total
C(3)×C(3)	1550	2595	4145
C(3)×C(9)	660	1185	1845
C(3)×C(27)	237	414	651
C(3)×C(81)	72	134	206
C(3)×C(243)	22	48	70
C(3)×C(729)	5	13	18
C(3)×C(2187)	5	1	6
C(3)×C(6561)	2	3	5
C(3)×C(19683)	1	0	1
C(3)×C(59049)	1	0	1
C(3)×C(531441)	0	1	1
C(9)×C(9)	17	34	51
C(9)×C(27)	7	10	17
C(9)×C(81)	2	2	4
C(9)×C(243)	1	2	3
C(9)×C(729)	1	0	1
C(9)×C(6561)	0	1	1
C(27)×C(27)	1	0	1
C(27)×C(81)	0	1	1
TOTAL $r=2$	2584	4444	7028
C(3)×C(3)×C(3)	353	711	1064
C(3)×C(3)×C(9)	171	321	492
C(3)×C(3)×C(27)	74	96	170
C(3)×C(3)×C(81)	20	30	50
C(3)×C(3)×C(243)	5	11	16
C(3)×C(3)×C(729)	4	9	13
C(3)×C(3)×C(531441)	0	1	1
C(3)×C(9)×C(9)	5	7	12
C(3)×C(9)×C(27)	3	9	12
C(3)×C(9)×C(81)	2	2	4
C(3)×C(9)×C(243)	1	0	1
TOTAL $r=3$	638	1197	1835
C(3)×C(3)×C(3)×C(3)	17	23	40
C(3)×C(3)×C(3)×C(9)	5	10	15
C(3)×C(3)×C(3)×C(27)	1	5	6
C(3)×C(3)×C(3)×C(81)	1	0	1
C(3)×C(3)×C(9)×C(81)	1	0	1
TOTAL $r=4$	25	38	63
C(3)×C(3)×C(3)×C(3)×C(9)	1	1	2
C(3)×C(3)×C(3)×C(3)×C(27)	1	0	1
TOTAL $r=5$	2	1	3

TABLE 2
Distribution of the $H_3(d)$ for $d = DD(m)$ with $r \geqslant 4$.

STRUCTURE	Number
C(3)×C(3)×C(3)×C(3)*	1043
C(3)×C(3)×C(3)×C(9)	512
C(3)×C(3)×C(3)×C(27)	152
C(3)×C(3)×C(3)×C(81)	53
C(3)×C(3)×C(3)×C(243)	21
C(3)×C(3)×C(3)×C(729)	8
C(3)×C(3)×C(3)×C(2187)	2
C(3)×C(3)×C(3)×C(6561)	1
C(3)×C(3)×C(9)×C(9)	15
C(3)×C(3)×C(9)×C(27)	12
C(3)×C(3)×C(9)×C(81)	3
C(3)×C(3)×C(9)×C(243)	2
TOTAL $r=4$	1824
C(3)×C(3)×C(3)×C(3)×C(3)	9
C(3)×C(3)×C(3)×C(3)×C(9)	6
C(3)×C(3)×C(3)×C(3)×C(27)	1
TOTAL $r=5$	16

TABLE 3
Distribution of the $H_3(d)$ for d obtained from $C(u, v)$.

STRUCTURE	$C(2,1)$	Other C	Total
C(3)×C(3)	213	1532	1745
C(3)×C(9)	132	685	817
C(3)×C(27)	46	206	252
C(3)×C(81)	15	87	102
C(3)×C(243)	4	21	25
C(3)×C(729)	1	5	6
C(3)×C(2187)	1	5	6
C(9)×C(9)	2	21	23
C(9)×C(27)	1	8	9
C(9)×C(81)	0	8	8
C(9)×C(243)	0	3	3
C(9)×C(729)	0	1	1
C(27)×C(27)	0	1	1
TOTAL $r=2$	415	2583	2998
C(3)×C(3)×C(3)	140	358	498
C(3)×C(3)×C(9)	78	164	242
C(3)×C(3)×C(27)	27	54	81
C(3)×C(3)×C(81)	7	15	22
C(3)×C(3)×C(243)	2	3	5
C(3)×C(3)×C(729)	1	1	2
C(3)×C(3)×C(2187)	1	0	1
C(3)×C(9)×C(9)	2	12	14
C(3)×C(9)×C(27)	1	4	5
C(3)×C(9)×C(81)	0	1	1
C(3)×C(9)×C(243)	0	1	1
TOTAL $r=3$	259	613	872
C(3)×C(3)×C(3)×C(3)	3	6	9
C(3)×C(3)×C(3)×C(9)	2	1	3
C(3)×C(3)×C(3)×C(27)	1	1	2
TOTAL $r=4$	6	8	14

TABLE 4

Frequency of different r.

Method	$r=2$	$r=3$	$r=4$	$r=5$
Cohen - Lenstra	99 5579	0.4418	0.0003	10^{-6}
$D_3(y)$ and $D_6(z)$	78.7098	20.5510	0.7056	0 0336
$C(2,1)$	61.0294	38.0882	0.8824	0
$C(u,v)$, $(u,v)\neq(2,1)$	80.6180	19.1323	0.2497	0.

TABLE 5

Frequency of different r.

Method	$r=3$	$r=4$	$r=5$
Cohen - Lenstra	99 9531446	0.0468530	0.0000024
$DD(m)$	95.3440	4.6155	0.0405

TABLE 6

Example of $H_3(d) = \mathrm{C}(3) \times \mathrm{C}(3) \times \mathrm{C}(3^{12})$.

$d=-$ 126 690 112 721 206 499
$h(d) = 162620946$
$H(d) = \mathrm{C}(2)\times \mathrm{C}(3)\times \mathrm{C}(3)\times \mathrm{C}(531441)\times \mathrm{C}(17)$
$F_1 = (140910333, 72182527, 2340148291)$
$F_2 = (59700143, 57614491, 5444272651)$
$F_3 = (26050129, 15485097, 12181312131)$

3. On Cohen and Lenstra's Heuristic Conjectures. Let G be a 3-group. Cohen and Lenstra conjectured that the probability of $H_3(d)$ being isomorphic to G is proportional to $(\#\mathrm{Aut}(G))^{-1}$ (cf. [1]). Analyzing our statistics in the preceding section, we observe that this conjecture holds with enough accuracy among the G having the same rank. However, it is not accurate for groups G with different rank. In Tables 1–3 we observe a frequency of the larger r much greater than that theoretically conjectured. Indeed, let $P(r_1, r_2)$ be the probability of $r = r_2$ provided $r \geqslant r_1$. In Table 4 we give the values of $P(2, r) \times 100$ computed following Cohen and Lenstra's conjecture and those obtained from Tables 1 and 3.

In Table 5 theoretical values of $P(3, r) \times 100$ and those obtained following our computations using Diaz y Diaz's method are compared.

Differences observed in Tables 4 and 5 could relate to the particular methods used to obtain these fields. Actually, the larger r is, the easier it will be to find two elements of order three in $H(d)$ satisfying certain conditions (Shanks's method) or three independent solutions of (3) with the same C (Diaz y Diaz's method). Something similar happens with the method of elliptic curves, even though the different behavior of different curves requires special study.

TABLE 7
Examples of $H_3(d) = C(9) \times C(9) \times C(9)$.

$d = -880\,370\,610\,667$ $h(d) = 93312$ $H(d) = C(2) \times C(2) \times C(32) \times C(9) \times C(9) \times C(9)$	$d = -95\,580\,523\,719\,067$ $h(d) = 1251693$ $H(d) = C(9) \times C(9) \times C(9) \times C(17) \times C(101)$
$F_1 = (245507, 226703, 948817)$ $F_2 = (105307, 19003, 2090867)$ $F_3 = (96689, 52359, 2283383)$	$F_1 = (2132057, 94137, 11208587)$ $F_2 = (2423933, 154227, 9860453)$ $F_3 = (2136181, 1279425, 11377483)$

TABLE 8
Distribution of d modulo 9.

d (mod 9)	1	2	3	4	5	6	7	8
$r=4$	221	41	56	456	670	52	19	309
$r=4$, d prime	0	3	0	66	102	0	0	0
$r=5$	1	0	0	7	7	0	0	1

4. Imaginary Quadratic Fields of Interest. Among the numerous imaginary quadratic fields considered, some have sufficient mathematical significance to be recorded. Here we only include those with $r \leqslant 4$. For fields with $r = 5$ see Section 5.

(A) On the basis of the two last sections it appears that for each 3-group G there will be discriminants $d < 0$ with $H_3(d) \approx G$, and, in particular, for $G = G_k = C(3) \times C(3) \times C(3^k)$. In Table 1 such G_k appear for $1 \leqslant k \leqslant 6$ and one case with $k = 12$. In Table 3, G_k appear for $1 \leqslant k \leqslant 7$. Among the 37679 $d = DD(m)$ with $r = 3$ which we have considered, there are also G_k for $1 \leqslant k \leqslant 9$. We do not know any d with $H_3(d) \approx G_k$ for $k = 10$ and $k = 11$. The extreme case with $k = 12$ in Table 1 is recorded in Table 6. We give its discriminant d, class number $h(d)$, the structure of $H(d)$ and three generators $F_i = (a_i, b_i, c_i)$ $(1 \leqslant i \leqslant 3)$ of the 3-Sylow subgroup of the class group of quadratic forms of discriminant d. The orders of F_i are respectively 3, 3 and 3^{12}.

(B) During our computations of class groups for discriminants $d = DD(m)$, we have detected two d with $H_3(d) \approx C(9) \times C(9) \times C(9)$. Since $3^6 \mid h(d)$ for these d, while no generator of $H_3(d)$ has order $> 3^2$, we could expect $r \geqslant 4$. According to [1], the probability that $H_3(d) \approx C(9) \times C(9) \times C(9)$ for $d < 0$ is close to 1.82×10^{-9}. Since this $H_3(d)$ is so rare, we give in Table 7 these two d with their $H(d)$ and three generators F_i of the 3-Sylow subgroup of the class group of quadratic forms of discriminant d. Every d in Table 7 has an additional interest. The first one has $r_2(d) = r_3(d)$, and $r_p(d) = 0$ for $p > 3$, $H(d)$ has exponent 288 and a subgroup isomorphic to $C(18) \times C(18) \times C(18)$. The second d is prime, hence this field has an infinite class field tower while having only one ramified prime.

TABLE 9

d verifying certain congruences (mod 9) *with* $-d$ *prime and* $r = 4$.

d	$H(d)$
-2 720 548 377 763	C(3)×C(3)×C(3)×C(3)×C(3001)
-15 166 956 005 107	C(3)×C(3)×C(3)×C(3)×C(29)×C(167)
-368 190 940 567 291	C(3)×C(3)×C(3)×C(3)×C(13)×C(37)×C(73)
-908 301 095 627	C(3)×C(3)×C(3)×C(3)×C(25)×C(109)
-1 461 910 779 779	C(3)×C(3)×C(3)×C(3)×C(17)×C(97)
-5 647 091 265 923	C(3)×C(3)×C(3)×C(3)×C(7)×C(17)×C(79)

(C) In [4], 13 fields were listed with $r = 4$ and d prime. All of them have $d \equiv 5 \pmod 9$. We have found 171 d prime with $r = 4$. They all also have an infinite class field tower with only one ramified prime. In Table 8 we give the distribution (mod 9) of the 1824 discriminants $d = DD(m)$ with $r = 4$ that we know, of the 171 such d that are prime, and of the $16 d = DD(m)$ with $r = 5$.

We always have $d = -4D_3(y) \equiv 1 \pmod 9$ and $d = -D_6(z) \equiv 4 \pmod 9$. We think that the particular distribution of the $d = DD(m) \pmod 9$ is associated with the method itself. In Table 8 we observe that no prime $d \equiv 1$, 7 or 8 (mod 9) with $r = 4$ has been found. The $d \equiv 7 \pmod 9$ are rare in general, but we think that prime discriminants satisfying this congruence probably exist. However, we wonder if there exist prime discriminants $d \equiv \pm 1 \pmod 9$ with $r = 4$. In Table 9 we give six fields with $r = 4$ and d prime: the three $d \equiv 2 \pmod 9$ that we have found, and three examples of $d \equiv 4 \pmod 9$.

5. Imaginary Quadratic Fields with $r = 5$. In Table 10 we give the twenty imaginary quadratic fields with $r = 5$ that we have found. Sixteen of them correspond to discriminants $d = DD(m)$, m being the parameter for which we have found them. Two correspond to discriminants $d = -4D_3(y)$, and the other two correspond to discriminants $d = -D_6(z)$. For each of them we give its discriminant d, structure of $H_3 = H_3(d)$ and $H = H(d)$, and five quadratic forms $F_i = (a_i, b_i, c_i)$ $(1 \leqslant i \leqslant 5)$ which generate the corresponding 3-Sylow subgroup of the class group of binary quadratic forms of discriminant d. For $1 \leqslant i \leqslant 4$, F_i is of order three. F_5 is of order 3, 9 or 27 according to the structure of $H_3(d)$.

For these 20 fields we have computed the 121 corresponding equations (3). They are listed in Table 11 for $d = DD(40879)$ and in Table 12 for $d = DD(46649)$.

Table 12 confirms the observation in [6] of the frequent occurrence of $\mathrm{C} = 2n^3$. Here we have 21 cases of $\mathrm{C} = 2$ and 19 cases of $\mathrm{C} = 2n^3$ with $2 \leqslant n \leqslant 9$. In Table 11 the occurrence of $\mathrm{C} = n^3$ is frequent. There are 21 cases of $\mathrm{C} = 1$, 20 cases of $\mathrm{C} = n^3$ with $2 \leqslant n \leqslant 8$, and 3 cases of $\mathrm{C} = 11^3$.

Analyzing other fields, we observe the frequent appearance of $\mathrm{C} = en^3$, where e is a small divisor of the discriminant.

Table 10
Imaginary quadratic fields with $r = 5$.

$DD(40879) = -35102371403731$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 8 \times 149$

2749399	1190223	3320635
681001	617409	13026253
2371793	1044623	3815005
2736385	219623	3211409
1170173	1107959	7761661

$DD(46649) = -56736325657288$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 2 \times 8 \times 7 \times 31$

1281241	860740	11215142
314642	131680	45093841
2513254	286328	5651867
39398	25744	360024547
265318	81536	53466947

$DD(46457) = -119810819605171$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 41 \times 73$

2094367	1912683	14738245
374099	345177	80145875
987503	861721	30519751
152713	20831	196137941
1204237	97195	24874727

$DD(67607) = -126543066437083$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 64 \times 5 \times 7$

5150903	1720863	6285521
3234781	1810753	10033283
5437169	579497	5833867
4025993	1696553	8036611
1969403	1663453	16414891

$DD(43543) = -130276398889003$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2 \times 2 \times 25 \times 29$

4812661	123653	6768173
45961	21539	708627221
1334713	300055	24418439
5869339	4792933	6527507
4090003	2661925	8396219

$DD(61927) = -211714885631083$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2731$

6210269	2276229	8731349
1879609	324171	28173409
6304201	4679061	9264001
1957471	1350105	27272137
3799297	1618503	14103559

$DD(47837) = -261174328819763$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2 \times 2203$

508973	325041	128336857
1232969	401299	52989039
557073	237125	117233539
3485557	313791	18739673
7314479	6199659	10240309

$DD(48779) = -348377316181547$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 24443$

4914673	4682581	18836649
2993829	2263697	29519191
813847	137599	107021421
2252517	1690933	38982677
1114131	455897	78219069

$DD(96233) = -467771196253976$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 49 \times 11 \times 83$

115025	85232	1016688678
49167	13022	2378482345
510543	119494	229062721
3102191	724644	37739158
4426495	1730528	26587962

$DD(62327) = -555319071324131$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 17 \times 1877$

2176787	1671409	64098219
3496883	566315	39723933
6243185	1421203	22317891
1359041	13725	102152779
2122347	790417	65486915

TABLE 10 (*continued*)

$DD(61441) = -699569567169259$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 5 \times 2401$

12855059	8353655	14962069
10702001	4005785	16716871
12909449	1033793	13568323
4487377	1359427	39077261
1200605	138759	145674227

$DD(62299) = -799388793124267$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2 \times 31 \times 73$

4847833	3422067	41827933
4325851	1690555	46363523
5757361	471207	34721239
4280423	3608831	47449309
10063571	9345317	22028059

$DD(73517) = -1240132748242027$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 4 \times 32 \times 229$

286777	263215	1081155419
7643323	1810641	40669849
18597031	13626755	19167323
5995759	2471901	51963523
6518917	2952159	47893231

$DD(95603) = -1635609136827227$

$H_3 = 3 \times 3 \times 3 \times 3 \times 27$

$H = H_3 \times 2 \times 2 \times 25 \times 41$

2274751	506771	179785167
18459429	17687027	26388141
18718827	5488051	22246691
17026351	6002979	24544967
3524179	1673663	116226381

$DD(84521) = -2332020861166019$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 32141$

5641677	413557	103346571
12663615	1814759	46102835
19161437	10296195	31809103
11638959	10485119	52452255
21541271	20080867	31744437

$DD(96329) = -3519139760972131$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 227 \times 241$

27732253	2539581	31782391
7112825	3164037	124041815
4236811	243359	207656123
20670661	14671425	45165349
13353143	659415	65894123

$-D_6(-2915) = -7801572518313863$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2 \times 8 \times 6079$

42747042	18804871	47694503
46065282	39207853	50682574
43734618	17447929	46336307
12924369	8535035	152317288
43737123	32856775	50764314

$-4 \times D_3(-10237) = -11863991835388346 12$

$H_3 = 3 \times 3 \times 3 \times 3 \times 9$

$H = H_3 \times 2 \times 2 \times 8 \times 5 \times 3889$

9799822	6926178	30267058817
114161109	52265288	2604062921
11722661	4968366	25301931622
262065382	248025666	1190462381
13974249	13705010	21228099822

$-4 \times D_3(10559) = -13421534902336291 40$

$H_3 = 3 \times 3 \times 3 \times 3 \times 27$

$H = H_3 \times 2 \times 2 \times 2 \times 2 \times 4 \times 5 \times 613$

52983174	48453118	6344000809
398205469	48840030	844123790
442749165	254794090	794509414
74613589	31562758	4500352134
350504709	169836232	977874049

$-D_6(-11360) = -17988272994492070 43$

$H_3 = 3 \times 3 \times 3 \times 3 \times 3$

$H = H_3 \times 2 \times 2 \times 11 \times 33713$

555880903	394255233	878904311
389271879	137911457	1167466087
248906069	52179843	1809467767
318337001	260423653	1465936863
666906913	550453193	787900821

TABLE 11

Solutions of $4A^3 = B^2 - C^2 d$ *corresponding to* $d = -35\ 102\ 371\ 403\ 731$.

A	B	C	A	B	C
20627	50651	1	1394285	1699640038	476
21787	2502891	1	1426085	2170763941	443
23447	4056931	1	1441865	1669888394	512
24085	4558863	1	1462931	2145622217	475
29597	8282719	1	1468165	2852111433	359
36325	12514863	1	1474355	473908813	599
40879	15432045	1	1479727	3198060331	279
45215	18293363	1	1525577	867503021	619
46567	19204611	1	1527829	3448548834	260
83425	8712654	8	1577659	1315609245	631
83641	9695190	8	1589989	2669531799	505
100657	63594471	1	1598041	4040005770	8
115865	63049654	8	1623277	2311226669	579
118879	81761805	1	1669873	1434702377	687
170837	141097841	1	1687025	2305973827	629
190345	44677699	27	1701625	4439162346	8
203597	4941601	31	1741771	1010507415	757
237605	231564113	1	1776371	4610702840	182
238255	174268188	26	1778867	1101998041	779
248155	247166613	1	1857935	3493412447	619
262685	232240151	23	1897765	4352652007	489
271445	109748222	44	1940693	3525277762	692
279179	293530759	5	1972079	5531196445	49
282235	125103552	46	2007323	5176141088	398
295385	321024887	1	2008465	5680163982	64
321485	47846893	61	2086081	1938924155	963
366397	358876374	44	2113855	5656493628	406
385925	478909298	4	2195737	3563536239	919
426355	455686648	54	2197435	1476258523	1071
517691	360879872	110	2297275	2238636269	1113
534197	780853681	1	2315483	3451012897	1037
582451	491842977	125	2334613	3190957067	1077
598933	913131493	27	2349671	6231220412	610
601793	932481838	8	2371793	5900553823	727
629575	986192449	27	2454649	7655816661	125
641737	411963401	159	2458985	5102586538	976
665299	404286264	170	2497331	4580027233	1085
681001	1032395123	75	2509753	7741186533	307
684781	857886483	125	2522885	4738190467	1091
691135	912011916	118	2645107	8603888229	1
691475	1079308321	67	2674865	7530143726	752
699245	138103898	196	2678315	6061070047	1069
711719	1176036005	41	2680309	8766689285	69
730349	1004901139	125	2686037	7377618581	811
759097	334538966	216	2696675	3303427481	1387
763129	1197348235	99	2704369	8855334475	141
768857	1326274769	41	2718185	8729478991	343
834641	1524297410	8	2730025	4381998597	1331
887617	1672498311	1	2736385	329891551	1527
940955	1129927904	242	2749399	1034025195	1529
946549	1807178114	60	2796359	5217922996	1310
1034125	2103235137	1	2797147	5035262751	1331
1046401	2008624827	125	2814607	6254916	1594
1074835	2228646363	1	2823679	9147916420	426
1117229	2361793535	1	2841439	7798337345	939
1195807	685244396	426	2847265	9491222661	253
1203635	342437696	442	2864215	9639823588	174
1221821	1492374838	380	2865659	5652007775	1331
1323265	3020683518	64	2988467	6169021139	1399
1369595	1716157859	457	3084947	8324823889	1171
1381199	1817842660	454			

TABLE 12

Solutions of $4A^3 = B^2 - C^2d$ *corresponding to* $d = -56\ 736\ 325\ 657\ 288$.

A	B	C	A	B	C
39398	4203496	2	1695259	2890258302	443
41234	7313408	2	1733558	1164426280	586
46649	13383262	2	1792921	4753344638	90
55817	21648430	2	1798819	2273864394	565
56338	22097856	2	1809401	2084504474	584
71618	35247824	2	1877707	2581595138	591
80089	42753918	2	1939547	5358661058	91
94342	55962360	2	1945046	2574590504	634
135593	98715826	2	1966211	3212366218	595
153857	6625538	16	1981622	5171182360	278
173074	143214912	2	1992086	216693704	746
213442	196643280	2	2066086	5622961704	254
234923	177140270	19	2164657	5920231250	312
255497	228450358	16	2164838	6026833400	274
265318	272909976	2	2190721	3623726486	714
274307	69915350	37	2205977	5330435842	506
297641	301575466	16	2216402	1824208240	842
300059	174332162	37	2262889	6764592818	102
314642	352662400	2	2382742	7344810488	54
322054	339454888	18	2426747	3784226782	869
322054	365219448	2	2451779	7230336262	343
323761	336234318	20	2461441	400125186	1024
381634	471279792	2	2466409	6683776746	520
387538	482269344	2	2513254	5961496552	702
391379	9533366	65	2554619	5486572738	803
405907	346321478	51	2565593	7044845566	562
436259	278261078	67	2570857	5527889610	812
450001	591589326	16	2602883	7946332090	361
456802	464582432	54	2641963	2233258450	1101
507883	661610050	39	2829473	9172149386	338
519766	629467976	54	2886209	535191646	1300
663697	489744270	128	2895682	7195495952	894
699211	750947934	119	2956747	6362857330	1053
715403	257067086	157	2963969	8135819818	818
806278	1447883784	2	2971222	9839460440	378
810553	1095697446	128	2976242	9122543392	626
832169	547573642	188	3005809	2822345998	1332
848731	612957006	191	3017522	6364081568	1106
888998	1676345896	2	3020566	5417660104	1194
904049	1719100538	2	3053129	3034643218	1358
966326	251686648	250	3119083	2206990146	1433
973667	933172730	223	3142802	11025882928	214
1025609	877054546	250	3145538	10285898320	574
1027667	2050621942	49	3172642	10051816752	686
1089814	1277280024	250	3185491	4234983914	1401
1100923	2201522366	93	3209473	3410491606	1458
1112089	1398373374	250	3238537	5933104810	1332
1133014	1895680168	198	3249574	11715728088	2
1134338	1772562176	218	3267011	9188740282	985
1281241	2857601666	66	3286658	11359936816	478
1315123	1206862506	367	3290339	11695632362	317
1387369	3013347818	168	3325681	12110764358	90
1390619	992683406	415	3386066	11380555136	674
1427347	3215307498	151	3433267	8548900058	1251
1456379	3157816834	205	3505702	7192462840	1458
1464289	1403643742	432	3524809	13100641746	250
1543091	3815884714	49	3725062	10430552392	1314
1555907	969044278	499	3734393	12709748486	908
1599442	4045570080	2	3751099	12675578478	943
1632569	3924249658	188	3951161	15672425018	140
1671449	2390615998	478			

6. Real Quadratic Fields with $s = 4$. For each of the twenty imaginary quadratic fields k with $r = 5$ in the preceding section we consider the corresponding real field k'. We compute the class number $h = h(d')$ for each k' from

$$2hR = L(1)(d')^{1/2} \tag{4}$$

by computing its regulator R and estimating its Dirichlet function $L(1)$ with sufficient accuracy from its Euler product. Since we know that $81 \mid h$, and we are able to calculate $r_2(d')$ from the factorization of d' (obtained from the elements of order two of $H(d)$), the computation of h from (4) is therefore simplified.

For fields with $d = -4D_3(y)$ or $d = -D_6(z)$ we know that $s = r - 1 = 4$. For the fields with $d = DD(m)$ and $243 \nmid h(d')$ (13 cases) we have also $s = 4$. Only the three cases of $d = DD(m)$ and $243 \mid h(d')$ require some additional computations. We found that $s = 4$ in all cases.

In Table 13 these 20 fields k' with $s = 4$ are listed. We give their discriminant d', their regulator R, and the structure of their class group $H(d')$.

TABLE 13
Real quadratic fields with $s = 4$.

d'	R	$H(d')$
105 307 114 211 193	43729.634928	$3 \times 3 \times 3 \times 3 \times 2$
170 208 976 971 864	4078.362734	$3 \times 3 \times 3 \times 9 \times 2 \times 2 \times 2$
359 432 458 815 513	62553.717921	$3 \times 3 \times 3 \times 3 \times 2$
379 629 199 311 249	125422.880210	$3 \times 3 \times 3 \times 3 \times 2$
390 829 196 667 009	87929.228902	$3 \times 3 \times 3 \times 3 \times 2 \times 2$
635 144 656 893 249	137624.222149	$3 \times 3 \times 3 \times 9$
783 522 986 459 289	148666.779943	$3 \times 3 \times 3 \times 3 \times 2$
1 045 131 948 544 641	503036.085084	$3 \times 3 \times 3 \times 3$
1 403 313 588 761 928	101923.181791	$3 \times 3 \times 3 \times 3 \times 2$
1 665 957 213 972 393	380575.492973	$3 \times 3 \times 3 \times 3$
2 098 708 701 507 777	281503.774940	$3 \times 3 \times 3 \times 3 \times 2$
2 398 166 379 372 801	490167.135347	$3 \times 3 \times 3 \times 3 \times 2$
3 720 398 244 726 081	33610.381950	$3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 7$
4 906 827 410 481 681	86036.266612	$3 \times 3 \times 3 \times 9 \times 2 \times 2$
6 996 062 583 498 057	438660.121764	$3 \times 3 \times 3 \times 3 \times 2$
10 557 419 282 916 393	887253.978149	$3 \times 3 \times 3 \times 3$
23 404 717 554 941 589	19560.013055	$3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 11$
3 559 197 550 616 503 836	899807.527959	$3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 4$
4 026 460 470 700 887 420	367154.920008	$3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2 \times 2$
5 396 481 898 347 621 129	4991959.583518	$3 \times 3 \times 3 \times 3 \times 2 \times 4$

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