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Are There Odd Amicable Numbers Not Divisible by Three?

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Abstract. A conjecture of Bratley and McKay, according to which odd amicable numbers should be divisible by three, is disproved by some counterexamples.

1. Even and Odd Amicable Numbers. Two natural numbers A and B are *amicable* if each of them is the sum of all proper divisors of the other. If $A = B$, they are called *perfect* numbers, otherwise they form an *amicable pair.* The first perfect numbers 6, 28, 496, 8128, and the smallest amicable pair 220, 284, were known to the Greek mathematicians. Two further amicable pairs were discovered by medieval Islamic mathematicians, and rediscovered by Fermat and Descartes. All of these were *even* numbers. In fact, they were found by the famous rules given by *Euclid* for perfect, resp. by *Thabit ibn Kurrah* for amicable numbers (see, e.g., [1], [5] for a survey of this subject), and so were *even by construction.*

L. Euler was the first to study systematically the question whether or not also *odd* numbers with these properties may be found. The existence of *odd perfect numbers* has remained a famous open problem in number theory, while the existence of *odd amicable numbers* was established by Euler. He described several methods to construct numerical examples, one of which is, for example,

$$A = 3^2 * 7 * 13 * 5 * 17 = 69615,$$
$$B = 3^2 * 7 * 13 * 107 = 87633.$$

Since Euler's time, many more even and odd amicable pairs have been found and published: Hundreds of them before, and thousands after the employment of electronic computers in number theory. A superficial glance at the list of hitherto known odd amicable pairs illustrates the fact that the lack of two as a common factor has to be compensated by a sufficient amount of divisibility by the other small prime factors, like three, five, seven. In fact, all odd amicable pairs that we know [2], [6], [7], [8] actually contain some power of three as a common factor. With some familiarity with the various known methods to find odd amicable pairs, it soon becomes clear, that it is actually very hard to avoid three as a common factor. Paul Bratley and John McKay even conjectured that all odd amicable numbers must be divisible by three, see [3], and also R. Guy's book on open problems in number theory [4]. On the other hand, to avoid three is a priori not impossible, but it only leads to very large numbers in all calculations, which are difficult to deal with.

Therefore, we made a systematic attempt to decide the question posed in the title of this note by a *constructive search*, or, in other words, to disprove the conjecture of Bratley and McKay by a counterexample.

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TABLE 1
Prime factor decompositions

1	a * 140453 * 85857199
22	a * 56099 * 214955207
2	a * 40459 * 4075499 * 11247066371
32	a * 6398629999 * 289840477211
3	a * 40063 * 1083014405858114729
22	a * 759883871 * 57100684400759
4	a * 40459 * 4071703 * 347952801041
33	a * 62791913183 * 912890522669
5	a * 40127 * 24316459 * 8637336284693
32	a * 894223226623 * 9425008441529
6	a * 40127 * 24316459 * 131473538420639
32	a * 753760237567 * 170197428069899
7	a * 40459 * 79670932151 * 785235094643
32	a * 4071703 * 621654783217660851119
8	a * 40127 * 456864935591 * 2947460281395319
32	a * 24317567 * 2222097774949919253614239
9	a * 40127 * 24315667 * 4668800173953586602539
32	a * 1410559270197839 * 3229592045605749823
10	a * 40169 * 43850540160157971076585196343983
23	a * 14887001 * 132233805923 * 894802182277066859
11	a * 52937 * 2215291513690331 * 1349652259805170474 68079
33	a * 164729 * 1222569893 * 7859119713501891114234577 8143
12	a * 52937 * 2215291479910079 * 19012939050676327114 72171
33	a * 164729 * 1222569893 * 1107136751268199952431099377023
13	a * 48619 * 1932038626331293043 * 1014575163459101725124 69
33	a * 227597 * 1787122884689 * 2343107037671898940710729 4679
14	a * 40697 * 7837685559226301 * 302073366913892362707570079
33	a * 2565569 * 2695609487 * 1393261045542880864876407 4706747
15	a * 48619 * 2908228327085896 21439 * 1413951571020235225 4726567
33	a * 227597 * 1787121242591 * 4915362264884774321367361013 54399

2. New Results. And here is the answer: Yes, there are odd amicable pairs not divisible by three; see Table 1 and Table 2. The prime factor decompositions of these numbers are of the following form:

$$A = \mathbf{a} * s_1 * \cdots * s_n$$
$$B = \mathbf{a} * p_1 * \cdots * p_m$$

where the common factor is always

$$\mathbf{a} = 5^4 * 7^3 * 11^3 * 13^2 * 17^2 * 19 * 61^2 * 97 * 307.$$

TABLE 2

Decimal representation

1	353804 3844224601 8396504460 7821130625
36D	353808 1696831682 7349549627 3894069375
2	5441 2078286957 7421098242 9810099604 5131589375
44D	5441 3436504035 2070205374 2457772678 1108410625
3	127302 7605743371 3716785701 5197297320 5370406875
46D	127305 9379711520 0172114765 4838692603 4176473125
4	168178 9722864804 0523765509 9240838728 2886525625
46D	168183 1703647644 1457440461 5714272808 6462594375
5	24727315 7305034090 7119626873 2844266174 3800523125
48D	24727932 9737721009 1039519379 3638309215 6354356875
6	376387764 4193359222 5434892386 3490036627 7033469375
49D	376397159 8108343084 9926669250 6866678923 7539330625
7	7426340623 9970095513 8092403777 4524356728 1875956875
50D	7426522352 4179913363 6259325136 5304942166 0541963125
8	15853 7717605693 7222558560 7477952254 8547242544 1456111875
55D	15854 1661985242 0904160160 8995935341 2036729598 8273168125
9	1336559 4670868055 2280720913 7286174768 3767183363 3131101875
57D	1336592 8302882198 6566894534 2143237977 9474160564 5317538125
10	516804264 4293811374 7164549488 7576790952 8935135589 0875156875
59D	516817095 4582968397 1662879237 9226428017 9959385460 9097963125
11	464378 6200632940 6115250682 0758989010 3254259564 1262634632 5840855625
66D	464384 5728895561 5444125973 7497515559 4798338774 4718106164 1268264375
12	6541834 9066755364 3521495944 2672064596 5576933656 8435296080 0595955625
67D	6541918 7658476693 5758002339 9986352493 7155413846 6739033716 5975564375
13	279618675 6494941078 2034012128 5310115476 3148304524 0497186447 3708280625
69D	279623198 2824883038 6487885985 3425146956 3179113713 1957459345 7658119375
14	2826980347 3653977372 3436078177 2108197038 6973416712 8681763928 1454949375
70D	2827048708 4946367121 8320681103 1027577096 3231392433 0289771234 5583770625
15	586 5826024396 0839764629 4216615430 8301026467 4757542512 4583231288 2817194375
73D	586 5920899955 6456714674 1963851001 0211009712 3042155544 4689941699 4584405625

Note. Each double row in the tables gives an amicable pair. In front of each pair, we specify the type of the prime factor decomposition (i.e., the numbers n, m above) in Table 1, resp. the number of decimal digits in Table 2.

3. Open Problems. Although the numbers looked for are necessarily quite large, it is very unlikely that our 36-digit example given here is the smallest one. So one may ask to *find the smallest one.*

Are there amicable number pairs of opposite parity? This question, also considered already by L. Euler, seems to be as hard and unaccessible as the existence of odd perfect numbers.

If, however, we replace in this problem the smallest prime factor two by three, then we obtain the following open problem which seems to us quite tractable by computational methods similar to ours.

Open Problem: Find an odd amicable pair with one, but not both numbers divisible by three.

It may very well be that such number pairs can have smaller size than those with both members prime to six, as those presented above.

4. Comments on Methods. Roughly, our method of construction of the numbers asked for in the title proceeds in three steps:

Step 1. Construction of an appropriate common factor $\mathbf{a}$.

Step 2. Successive computation of a few "complementary" prime factors $s_1, s_2, \ldots, p_1, p_2, \ldots$ to make $(\mathbf{a} * u, \mathbf{a} * v)$ with $u = s_1 s_2 \ldots$, $v = p_1 p_2 \ldots$ a suitable input for the last step, for instance by the method of "breeders" or an appropriate modification thereof; see [2].

Step 3. Computation of the three largest prime factors by the so-called method of *Bilinear Diophantine Equations* (BDE, see [2]), including the necessary *primality tests*.

To be slightly more specific, we have to introduce more notation. Let C denote the largest common divisor of $\mathbf{a}$ with its sum of divisors $\sigma(\mathbf{a})$, put $D = 2\mathbf{a} - \sigma(\mathbf{a})$, and let $\mathbf{a}' = \mathbf{a}/C$, $D' = D/C$.

In Step 1, we proceed by building up $\mathbf{a}$ as a product of powers of different small primes in such a way that $\sigma(\mathbf{a})/\mathbf{a}$ approaches 2 from below. In doing so, we compute $\mathbf{a}, \sigma(\mathbf{a}), \mathbf{a}', D'$ recursively, and we try to get D' as small as possible, without making $\mathbf{a}'$ excessively big. The reason for this will be clear to the reader familiar with the construction of amicable numbers (as in [2]): Essentially, $\mathbf{a}'$ will determine the size of the numbers in all successive computations (e.g., the BDE method), while D' will occur as a denominator in Diophantine problems. So it is best to make D' one, or at least very small, to allow for sufficiently many integer solutions arising in Steps 2 and 3. The crucial point for our present situation is the cancellation by C, that is the replacement of $\mathbf{a}$ and D by $\mathbf{a}'$ and D'. This means that we can obviously work towards all of our goals simultaneously by trying to make C as large as possible.

There are various obvious methods to construct appropriate numbers $\mathbf{a}$ with relatively large C. Using the multiplicativity of σ, we proceed by building up the number $\mathbf{a}$ recursively, introducing one or several prime power factors at each step. In deciding which new prime powers to introduce as a factor, we make extensive use of a table of prime decompositions of $\sigma(p^\nu)$ for all small prime powers p^ν. Let us say that p^ν "*carries*" q^μ, if q^μ divides $\sigma(p^\nu)$. For example, 17^2 carries 307, which carries 7 and 11. Similarly, 13^2 carries 61, and 61^2 carries 13, in addition to 97 (which carries 7^2). The general strategy is then to introduce mainly such new prime factors which are (at least to a large extent) carried by those already previously introduced, because this procedure will increase mainly C, but not $\mathbf{a}'$.

It is convenient to look for "*cycles*" of prime powers carrying each other at least "partially", like 13 and 61 in the example mentioned above. Such "cycles" may be used to get the whole process (of guessing **a**) started, and also to increase its efficiency later on. One may tabulate for this purpose such chains and cycles (or even "trees" and "clusters") of prime powers carrying each other (at least partially). On the basis of such a table, the construction of an appropriate number **a** (with C big and hence $\mathbf{a}'$ and $D' > 0$ relatively small) becomes a nice kind of a number-theoretic puzzle, which can be solved by trial and error without too much computational effort. In the last few steps of guessing appropriate prime factors of **a**, one will change the strategy, and try directly to minimize D' by the last few choices.

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1. W. BORHO, "Befreundete Zahlen," in *Lebendige Zahlen*, Math. Miniaturen, vol. 1, Birkhäuser, Basel, 1981.

2. W. BORHO, H. HOFFMANN, "Breeding amicable numbers in abundance," *Math. Comp.*, v. 46, 1986, pp. 281–293.

3. P. BRATLEY & J. MCKAY, "More amicable numbers," *Math. Comp.*, v. 22, 1968, pp. 677–678.

4. R. K. GUY, *Unsolved Problems in Number Theory*, Vol 1, Springer-Verlag, New York, Heidelberg, Berlin, 1981, pp. 31–32.

5. E. J. LEE & J. S. MADACHY, "The history and discovery of amicable numbers," *J. Recreational Math.*, v. 5, 1972, pp. 77–93, 153–173, 231–249.

6. H. J. J. TE RIELE, "On generating new amicable pairs from given amicable pairs," *Math. Comp.*, v. 42, 1984, pp. 219–223.

7. H. J. J. TE RIELE, W. BORHO, S. BATTIATO, H. HOFFMANN & E. J. LEE, *Table of Amicable Pairs between* 10^{10} *and* 10^{52}, Centrum voor Wiskunde en Informatica, Note NM-N8603, Stichting Math. Centrum, Amsterdam, 1986.

8. H. J. J. TE RIELE, "Computation of all the amicable pairs below 10^{10}," *Math. Comp.*, v. 47, 1986, pp. 361–368.