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Incompressible Finite Element Methods for Navier-Stokes Equations with Nonstandard Boundary Conditions in $\mathbf{R}^3$

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Abstract. This paper is devoted to the steady state, incompressible Navier-Stokes equations with nonstandard boundary conditions of the form $\mathbf{u}\cdot\mathbf{n} = 0$, $\mathbf{curl}\ \mathbf{u}\times\mathbf{n} = \mathbf{0}$, either on the entire boundary or mixed with the standard boundary condition $\mathbf{u} = \mathbf{0}$ on part of the boundary. The problem is expressed in terms of vector potential, vorticity and pressure. The vorticity and vector potential are approximated with **curl**-conforming finite elements and the pressure with standard continuous finite elements. The error estimates yield nearly optimal results for the purely nonstandard problem.

1. Introduction. In this paper we propose to solve a Navier-Stokes problem of the following type:

$$-\nu\Delta\mathbf{u} + \sum_{j\leq 3} u_j \partial\mathbf{u}/\partial x_j + \nabla p = \mathbf{f}, \qquad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1}$$

with boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \mathbf{curl}\ \mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \tag{1.2a}$$

or

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \mathbf{curl}\ \mathbf{u}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_0, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma\setminus\Gamma_0, \tag{1.2b}$$

where Ω is a bounded, convex domain of $\mathbf{R}^3$ with a polyhedral boundary Γ, Γ_0 is a connected portion of Γ, either empty or with strictly positive measure and $\mathbf{n}$ is the exterior unit normal to Γ. The case where Γ_0 is empty corresponds, of course, to the standard Navier-Stokes equations. We shall use a mixed incompressible finite element method that approximates the vector potential and vorticity of $\mathbf{u}$, using the **curl**-conforming elements of Nédélec.

The convexity assumption on Ω is a well-known theoretical consequence of the fact that Γ is not smooth. There is no practical evidence that it is necessary, and this assumption is disregarded in practice: instead we can assume that Ω is simply connected and Γ is connected. The case where Γ is not connected or Ω not simply connected is more intricate and is not studied here. It might be done through the approach of Bendali, Dominguez and Gallic [4].

With easy modifications, our analysis extends to the case where Γ_0 is not connected. We can also handle boundary conditions of the form

$$\mathbf{u}\times\mathbf{n} = \mathbf{0}, \qquad \tilde{p} = 0 \quad \text{on } \Gamma, \tag{1.2c}$$

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where $\tilde{p}$ stands for the dynamical pressure:

$$\tilde{p} = p + (1/2)\mathbf{u} \cdot \mathbf{u}$$

(alone or combined with the previous conditions), but this case presents a (yet unsolved) theoretical difficulty arising from the roughness of Γ.

Navier-Stokes equations with nonstandard boundary conditions are of growing interest. Bègue, Conca, Murat and Pironneau present in [3] a thorough theoretical and practical study of the subject; they consider more general domains as well as nonhomogeneous boundary conditions. For the numerical solution, they propose a "velocity-pressure" Hood-Taylor scheme in [2] and a "P_1-P_1" scheme with a finer mesh for the pressure in [3]. In [22], Verfürth studies a related Stokes problem with a nonhomogeneous boundary condition of type (1.2c) on a curved domain.

Sections 2 to 5 are dedicated to the theoretical and numerical analysis of system (1.1) with the boundary conditions (1.2a). They are simpler to handle than the conditions (1.2b), studied in Section 6, and many results relative to the former carry over with straightforward modifications to the latter.

It turns out that the **curl**-conforming finite elements of Nédélec are particularly well adapted to express the nonstandard boundary conditions (1.2a), (1.2b) and (1.2c). We shall derive nearly optimal error estimates for (1.1) with (1.2a), but not with (1.2b), although we believe that this can be improved. The difficulty arises not from the nonlinearity, but from the mixed formulation itself and occurs also in two dimensions.

2. A "vector potential-vorticity" formulation for (1.1), (1.2a). Let us first recall the classical Sobolev space $W^{m,p}(\Omega)$ or $H^m(\Omega)$ when $p = 2$:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega); \partial^\alpha v \in L^p(\Omega)\ \forall\ |\alpha| \le m\},$$

equipped with the following seminorm and norm:

$$|v|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v(x)|^p \, dx \right\}^{1/p}$$

$$\|v\|_{m,p,\Omega} = \left\{ \sum_{k\le m} |v|^p_{k,p,\Omega} \right\}^{1/p}$$

We make the usual modification when $p = \infty$ and we agree to omit p when $p = 2$. As usual, $(\cdot,\cdot)$ denotes the scalar product of $L^2(\Omega)$. Also, recall the space

$$H_0^1(\Omega) = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma\}.$$

Apart from these, we require the following Hilbert spaces relative to the divergence and rotation operators:

$$\begin{aligned} H(\mathrm{div};\Omega) &= \{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div} \mathbf{v} \in L^2(\Omega)\}, \\ H_0(\mathrm{div};\Omega) &= \{\mathbf{v} \in H(\mathrm{div};\Omega); \mathbf{v} \cdot \mathbf{n}_{|\Gamma} = 0\}, \\ H(\mathbf{curl};\Omega) &= \{\mathbf{v} \in L^2(\Omega)^3; \mathbf{curl}\ \mathbf{v} \in L^2(\Omega)^3\}, \\ H_0(\mathbf{curl};\Omega) &= \{\mathbf{v} \in H(\mathbf{curl};\Omega); \mathbf{v} \times \mathbf{n}_{|\Gamma} = \mathbf{0}\}, \end{aligned}$$

equipped with the norms

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} = \{\|\mathbf{v}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2\}^{1/2},$$
$$\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)} = \{\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\ \mathbf{v}\|_{0,\Omega}^2\}^{1/2}.$$

We refer to Duvaut and Lions [10] and Girault and Raviart [12] for an extensive study of these spaces. In order to handle the Navier-Stokes equations, we also introduce the Banach spaces

$$H^p(\mathbf{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^3; \mathbf{curl}\ \mathbf{v} \in L^p(\Omega)^3\},$$
$$H_0^p(\mathbf{curl};\Omega) = \{\mathbf{v} \in H^p(\mathbf{curl};\Omega); \mathbf{v} \times \mathbf{n}_{|\Gamma} = \mathbf{0}\},$$

which we shall use with the exponents $p = 4$ and $p = 4/3$. It can be shown, in particular, that for this range of p the trace operator $\mathbf{v} \times \mathbf{n}_{|\Gamma}$ can be defined in a weak sense.

In $\mathbf{R}^3$, it is not altogether trivial to formulate the Navier-Stokes (or even the Stokes) equations in terms of vector potential and vorticity, because the vector potential of $\mathbf{u}$ is not easily characterized. Our formulation derives from the three fundamental theorems below, due to Bernardi [5], Girault and Raviart [12] and Nédélec [16]. The assumptions on the domain are: Ω *is bounded, simply connected, with a polyhedral, connected boundary* Γ.

THEOREM 2.1. *Each function* $\mathbf{u} \in L^2(\Omega)^3$ *that satisfies* $\operatorname{div}\mathbf{u} = 0$ *in* Ω *has a unique vector potential* $\boldsymbol{\psi} \in L^2(\Omega)^3$ *characterized by*

$$\mathbf{curl}\ \boldsymbol{\psi} = \mathbf{u}, \qquad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\psi}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma.$$

If, in addition, Ω *is convex, then* $\boldsymbol{\psi} \in H^1(\Omega)^3$.

THEOREM 2.2. *Each function* $\mathbf{u} \in L^2(\Omega)^3$ *that satisfies* $\operatorname{div}\mathbf{u} = 0$ *in* Ω, $\mathbf{u}\cdot\mathbf{n} = 0$ *on* Γ, *has a unique vector potential* $\boldsymbol{\psi} \in L^2(\Omega)^3$ *characterized by*

$$\mathbf{curl}\ \boldsymbol{\psi} = \mathbf{u}, \qquad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\psi}\times\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

If, in addition, Ω *is convex, then* $\boldsymbol{\psi} \in H^1(\Omega)^3$. *Moreover, there exists a real* $s > 2$ *depending on the angles of* Γ *such that*

$$\boldsymbol{\psi} \in W^{1,t}(\Omega)^3 \text{ whenever } \mathbf{u} \in L^t(\Omega)^3 \quad \forall t \in [2,s]. \tag{2.1}$$

Remark 2.1. The extra regularity (2.1) stems from a powerful result of Grisvard [14] concerning the solution of $-\Delta u = f$, $u_{|\Gamma} = 0$ on a convex polyhedron. The same regularity for the vector potential of Theorem 2.1 would require an analogous result for a nonhomogeneous Neumann problem. Although such a result is now well known in polygons of $\mathbf{R}^2$, to the author's knowledge it is not yet proved in $\mathbf{R}^3$. But there is a strong conjecture by Dauge [9] that it does hold. □

THEOREM 2.3. *Let* Ω *be convex. All functions* $\boldsymbol{\psi} \in L^2(\Omega)^3$ *that satisfy*

$$\operatorname{div}\boldsymbol{\psi} = 0, \qquad \mathbf{curl}\ \boldsymbol{\psi} \in L^2(\Omega)^3, \qquad \boldsymbol{\psi}\cdot\mathbf{n} = 0 \quad (\text{or } \boldsymbol{\psi}\times\mathbf{n} = \mathbf{0}) \quad \text{on } \Gamma,$$

belong to $H^1(\Omega)^3$ *and*

$$\|\boldsymbol{\psi}\|_{1,\Omega} \leq C\|\mathbf{curl}\ \boldsymbol{\psi}\|_{0,\Omega}. \tag{2.2}$$

In addition, when $\boldsymbol{\psi} \times \mathbf{n} = \mathbf{0}$ *on* Γ *and* $\mathbf{curl}\ \boldsymbol{\psi} \in L^s(\Omega)^3$ *with the real* $s > 2$ *of Theorem* 2.2, *then for each* $t \in [2, s]$ *we have* $\boldsymbol{\psi} \in W^{1,t}(\Omega)^3$ *and*

$$\|\boldsymbol{\psi}\|_{1,t,\Omega} \leq C(t)\|\mathbf{curl}\ \boldsymbol{\psi}\|_{0,t,\Omega}. \tag{2.3}$$

Now assume that the right-hand side $\mathbf{f}$ of (1.1) belongs to $L^{4/3}(\Omega)^3$. As for the classical Navier-Stokes equations (cf., for instance, Témam [21]), it is easy to prove that the system (1.1), (1.2a) has at least one solution:

$$(\mathbf{u}, p) \in H^1(\Omega)^3 \times L^2(\Omega).$$

Let $(\mathbf{u}, p)$ be one of these solutions, and suppose that $\mathbf{curl}\ \mathbf{u} \in H^{4/3}(\mathbf{curl}; \Omega)$ and $p \in W^{1,4/3}(\Omega)$. Setting

$$\boldsymbol{\omega} = \mathbf{curl}\ \mathbf{u}, \qquad \mathbf{u} = \mathbf{curl}\ \boldsymbol{\psi} \text{ with } \boldsymbol{\psi} \text{ characterized by Theorem 2.2,}$$

and using the identities

$$-\Delta \mathbf{u} = \mathbf{curl}\ \mathbf{curl}\ \mathbf{u}, \qquad \sum_{j\leq 3} u_j \partial \mathbf{u}/\partial x_j = \mathbf{curl}\ \mathbf{u} \times \mathbf{u} + (1/2)\nabla(\mathbf{u}\cdot\mathbf{u}),$$

we derive from (1.1) that

$$\nu(\mathbf{curl}\ \boldsymbol{\omega}, \mathbf{curl}\ \boldsymbol{\varphi}) + (\boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\varphi}) = (\mathbf{f}, \mathbf{curl}\ \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in H_0^4(\mathbf{curl}; \Omega).$$

The relationship between $\boldsymbol{\omega}$ and $\boldsymbol{\psi}$ can be expressed by

$$(\mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\omega}, \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in H_0^{4/3}(\mathbf{curl}; \Omega).$$

As far as the pressure is concerned, setting

$$\tilde{p} = p + (1/2)\mathbf{u}\cdot\mathbf{u},$$

we also derive from (1.1), (1.2a) that

$$(\nabla \tilde{p}, \nabla q) = (\mathbf{f} - \boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \nabla q) \quad \forall q \in W^{1,4}(\Omega).$$

Thus, we propose for (1.1), (1.2a) the following formulation:

Find a pair $(\boldsymbol{\psi}, \boldsymbol{\omega}) \in H_0^4(\mathbf{curl}; \Omega) \times H_0^{4/3}(\mathbf{curl}; \Omega)$ *and* $\tilde{p} \in W^{1,4/3}(\Omega)/\mathbf{R}$ *such that*

$$\nu(\mathbf{curl}\ \boldsymbol{\omega}, \mathbf{curl}\ \boldsymbol{\varphi}) + (\boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\varphi}) = (\mathbf{f}, \mathbf{curl}\ \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in H_0^4(\mathbf{curl}; \Omega), \tag{2.4}$$

$$(\mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\omega}, \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in H_0^{4/3}(\mathbf{curl}; \Omega), \tag{2.5}$$

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad \textit{in } \Omega, \tag{2.6}$$

$$(\nabla \tilde{p}, \nabla q) = (\mathbf{f} - \boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \nabla q) \quad \forall q \in W^{1,4}(\Omega). \tag{2.7}$$

THEOREM 2.4. *Let* Ω *be convex and assume that the right-hand side* $\mathbf{f}$ *and the solutions* $(\mathbf{u}, p)$ *of the system* (1.1), (1.2a) *have the regularity*

$$\mathbf{f} \in L^{4/3}(\Omega)^3, \quad \boldsymbol{\omega} = \mathbf{curl}\ \mathbf{u} \in H^{4/3}(\mathbf{curl}; \Omega), \quad p \in W^{1,4/3}(\Omega). \tag{2.8}$$

Then the mixed formulation (2.4)–(2.7) *is equivalent to* (1.1), (1.2a).

Proof. We have just seen that, under the assumption (2.8), each solution of (1.1), (1.2a) is also a solution of (2.4)–(2.7). The converse follows from the fact that all functions $\mathbf{v} \in H^1(\Omega)^3$ with $\mathbf{v}\cdot\mathbf{n} = 0$ have the decomposition

$$\mathbf{v} = \nabla q + \mathbf{curl}\ \boldsymbol{\varphi} \quad \text{with } q \in W^{1,4}(\Omega) \text{ and } \boldsymbol{\varphi} \in H_0^4(\mathbf{curl}; \Omega). \quad \square$$

The constraint (2.5) is conveniently expressed by means of the space

$$\mathbf{V} = \{\mathbf{v} = (\boldsymbol{\varphi}, \boldsymbol{\theta}) \in H_0^4(\mathbf{curl}; \Omega) \times L^2(\Omega)^3; \operatorname{div} \boldsymbol{\varphi} = 0, \\ (\mathbf{curl}\ \boldsymbol{\varphi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\theta}, \boldsymbol{\mu})\ \forall \boldsymbol{\mu} \in H_0^{4/3}(\mathbf{curl}; \Omega)\}, \tag{2.9}$$

normed by

$$\|\mathbf{v}\| = \|\boldsymbol{\varphi}\|_{0,\Omega} + \|\mathbf{curl}\ \boldsymbol{\varphi}\|_{0,4,\Omega} + \|\boldsymbol{\theta}\|_{0,\Omega}.$$

It is a matter of routine to prove that the pairs $\mathbf{v} = (\boldsymbol{\varphi}, \boldsymbol{\theta})$ of $\mathbf{V}$ satisfy

$$-\Delta \boldsymbol{\varphi} = \boldsymbol{\theta}.$$

In addition, when Ω is convex, then $\boldsymbol{\varphi} \in W^{1,s}(\Omega)^3$ with the exponent s of Theorem 2.2, $\mathbf{curl}\ \boldsymbol{\varphi} \in H^1(\Omega)^3$ and

$$\|\mathbf{curl}\ \boldsymbol{\varphi}\|_{1,\Omega} \leq C\|\boldsymbol{\theta}\|_{0,\Omega}. \tag{2.10}$$

As a consequence, the seminorm

$$|\mathbf{v}| = |(\boldsymbol{\varphi}, \boldsymbol{\theta})| = \|\boldsymbol{\theta}\|_{0,\Omega} \tag{2.11}$$

is a *norm on* $\mathbf{V}$ *equivalent to the above norm.*

Remark 2.2. Note that formula (2.5) implies that $\operatorname{div} \boldsymbol{\omega} = 0$. □

Remark 2.3. When $\mathbf{f} \in L^2(\Omega)^3$, the Stokes problem

$$-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \qquad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.12}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \mathbf{curl}\ \mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.13a}$$

has the equivalent formulation (because $\boldsymbol{\omega} = \mathbf{curl}\ \mathbf{u} \in H(\mathbf{curl}; \Omega)$):

Find a pair $(\boldsymbol{\psi}, \boldsymbol{\omega}) \in H_0(\mathbf{curl}; \Omega) \times H_0(\mathbf{curl}; \Omega)$ *and* $p \in H^1(\Omega)/\mathbf{R}$ *such that*

$$\nu(\mathbf{curl}\ \boldsymbol{\omega}, \mathbf{curl}\ \boldsymbol{\varphi}) = (\mathbf{f}, \mathbf{curl}\ \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in H_0(\mathbf{curl}; \Omega), \tag{2.14}$$

$$(\mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\omega}, \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in H_0(\mathbf{curl}; \Omega), \tag{2.15}$$

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad \textit{in } \Omega, \tag{2.16}$$

$$(\nabla p, \nabla q) = (\mathbf{f}, \nabla q) \quad \forall q \in H^1(\Omega). \tag{2.17}$$

It is easy to prove that this problem has a unique solution that satisfies the following bounds:

$$\|\mathbf{curl}\ \boldsymbol{\omega}\|_{0,\Omega} \leq (1/\nu)\|\mathbf{f}\|_{0,\Omega}, \qquad \|\boldsymbol{\omega}\|_{1,\Omega} \leq (C/\nu)\|\mathbf{f}\|_{0,\Omega},$$
$$\|\boldsymbol{\psi}\|_{1,\Omega} \leq C\|\boldsymbol{\omega}\|_{0,\Omega}, \qquad \|\mathbf{curl}\ \boldsymbol{\psi}\|_{1,\Omega} \leq C\|\boldsymbol{\omega}\|_{0,\Omega};$$

if $\mathbf{f} \in L^s(\Omega)^3$ (with the exponent s of Theorem 2.2), then

$$\|\boldsymbol{\omega}\|_{1,s,\Omega} \leq (C(s)/\nu)\|\mathbf{f}\|_{0,s,\Omega};$$

if $\mathbf{f} \in H(\mathbf{curl}; \Omega)$, then

$$\|\mathbf{curl}\ \boldsymbol{\omega}\|_{1,\Omega} \leq C\|\mathbf{curl}\ \mathbf{f}\|_{0,\Omega}. \quad \square$$

Remark 2.4. It follows readily from (2.17) that the pressure p satisfies (in a weak sense) the boundary condition

$$\partial p/\partial n = \mathbf{f} \cdot \mathbf{n} \quad \text{on } \Gamma. \tag{2.18}$$

Owing to the nonlinear term in the Navier-Stokes equation, the normal derivative of the dynamical pressure $\tilde{p}$ is not directly related to $\mathbf{f}$ but depends also upon the velocity. □

3. Families of curl-Conforming and div-Conforming Finite Elements. There are two families of **curl**-conforming finite elements that can be used to approximate Problem (2.4)–(2.7). Both were developed by Nédélec, the first one in [15] and [16] and the second one in [17]. For the sake of simplicity, we shall work here with the second family. It is more costly as far as the number of degrees of freedom is concerned, since the velocity involves complete polynomials of degree k versus incomplete polynomials for the first family, but it is easier to describe and more accurate in some situations.

As usual, we denote by P_k the space of polynomials of three variables of degree at most k, and by $\tilde{P}_k$ the subspace of homogeneous polynomials of degree exactly k. Let us *fix* an integer $k \geq 1$ and define the following subspace of P_k^3:

$$\mathbf{D}_k = (P_{k-1})^3 \oplus \{p(x)\mathbf{x}; \forall p \in \tilde{P}_{k-1}\}. \tag{3.1}$$

Definition 3.1. Let κ be a tetrahedron with faces denoted by f and edges denoted by e, τ being the direction vector of e, and let $\mathbf{u} \in W^{1,t}(\kappa)^3$ for some $t > 2$. We define the three sets of moments of $\mathbf{u}$ on κ:

$$M_e(\mathbf{u}) = \left\{ \int_e (\mathbf{u} \cdot \tau) q de; \forall q \in P_k(e), \text{ for the six edges } e \text{ of } \kappa \right\}, \tag{3.2}$$

$$\begin{aligned} M_f(\mathbf{u}) = \Big\{ \int_f \mathbf{u} \cdot \mathbf{q}\, ds; \forall \mathbf{q} \in \mathbf{D}_{k-1}(f) & \text{ tangent to the face } f, \\ & \text{for the four faces } f \text{ of } \kappa \Big\}, \end{aligned} \tag{3.3}$$

$$M_\kappa(\mathbf{u}) = \left\{ \int_\kappa \mathbf{u} \cdot \mathbf{q}\, dx; \forall \mathbf{q} \in \mathbf{D}_{k-2}(\kappa) \right\}. \quad \square \tag{3.4}$$

Nédélec proves in [17] that this set of moments is *unisolvent and **curl**-conforming on* $(P_k)^3$. Hence it determines the following interpolation operator:

(3.5) $r_\kappa(\mathbf{u})$ is the unique polynomial of $(P_k)^3$ that has the same moments on κ as $\mathbf{u}$.

Parallel to these elements, we introduce the following family of div-conforming finite elements, developed by Nédélec in [15], that generalize to $\mathbf{R}^3$ the elements of Raviart and Thomas [19].

Definition 3.2. Let κ be a tetrahedron with faces denoted by f and let $\mathbf{u} \in H^1(\kappa)^3$. We define the two sets of moments of $\mathbf{u}$ on κ:

$$N_f(\mathbf{u}) = \left\{ \int_f (\mathbf{u} \cdot \mathbf{n}) q\, ds; \forall q \in P_{k-1}(f), \text{ for the four faces } f \text{ of } \kappa \right\}, \tag{3.6}$$

$$N_\kappa(\mathbf{u}) = \left\{ \int_\kappa \mathbf{u} \cdot \mathbf{q}\, dx; \forall \mathbf{q} \in (P_{k-2}(\kappa))^3 \right\}. \quad \square \tag{3.7}$$

Again, Nédélec proves in [15] that this set of moments is *unisolvent and* div-*conforming on* $\mathbf{D}_k$; the associated interpolation operator is:

(3.8) $\omega_\kappa(\mathbf{u})$ is the unique polynomial of $\mathbf{D}_k$ that has the same moments on κ as $\mathbf{u}$.

Since the divergence-free vectors of $\mathbf{D}_k$ belong in fact to $(P_{k-1})^3$, these two interpolation operators are linked by a valuable relation:

$$\omega_\kappa(\mathbf{curl}\ \mathbf{u}) = \mathbf{curl}\ r_\kappa(\mathbf{u}) \quad \forall \mathbf{u} \in H^2(\kappa)^3. \tag{3.9}$$

Now we turn to the finite element spaces. Since Ω is a polyhedron, we can triangulate it entirely with tetrahedra. Thus, let $\mathscr{T}_h$ be a triangulation of $\overline{\Omega}$ made of tetrahedra κ with diameters bounded by h. For each integer $k \geq 1$, we define the following finite element spaces:

$$\mathbf{M}_h = \{\boldsymbol{\mu}_h \in H(\mathbf{curl};\Omega); \boldsymbol{\mu}_{h|\kappa} \in (P_k)^3\ \forall \kappa \in \mathscr{T}_h\}, \tag{3.10a}$$

$$\mathbf{F}_h = \mathbf{M}_h \cap H_0(\mathbf{curl};\Omega), \tag{3.10b}$$

$$\mathbf{Q}_h^k = \{q_h \in C^0(\overline{\Omega}); q_{h|\kappa} \in P_k\ \forall \kappa \in \mathscr{T}_h\}, \tag{3.11a}$$

$$\Theta_h = \mathbf{Q}_h^{k+1} \cap H_0^1(\Omega). \tag{3.11b}$$

Next we define the interpolation operator r_h from $W^{1,t}(\Omega)^3$ for some $t > 2$ onto $\mathbf{M}_h$:

$$r_h\mathbf{u} = r_\kappa(\mathbf{u}) \quad \text{on } \kappa\ \forall \kappa \in \mathscr{T}_h. \tag{3.12}$$

Nédélec establishes in [17] that r_h has the following crucial properties:

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \Rightarrow r_h\mathbf{u} \times \mathbf{n} = \mathbf{0}, \qquad \mathbf{curl}\ \mathbf{u} = \mathbf{0} \Rightarrow \mathbf{curl}\ r_h\mathbf{u} = \mathbf{0},$$
$$\mathbf{u} = \nabla p \text{ with } p_{|\Gamma} = 0 \Rightarrow r_h\mathbf{u} = \nabla p_h \text{ with } p_h \in \Theta_h.$$

Remark 3.1. In general, the moments (3.2) are not defined when $\mathbf{u}$ has no more than H^1-regularity. This is why a $W^{1,t}$ (or an $H^{1+\varepsilon}$)-regularity is required to define r_h. This is one of the drawbacks of these finite elements. Unfortunately, there seems to be no way of bypassing the moments (3.2), because they are necessary to preserve vanishing **curls** and vanishing tangential components. □

As far as the div-conforming finite element spaces are concerned, we set

$$\mathbf{D}_h = \{\mathbf{v}_h \in H(\text{div};\Omega); \mathbf{v}_{h|\kappa} \in \mathbf{D}_k\ \forall \kappa \in \mathscr{T}_h\}, \tag{3.13a}$$

$$\mathbf{D}_{0h} = \mathbf{D}_h \cap H_0(\text{div};\Omega), \tag{3.13b}$$

together with the interpolation operator ω_h from $H^1(\Omega)^3$ onto $\mathbf{D}_h$:

$$\omega_h\mathbf{u} = \omega_\kappa(\mathbf{u}) \quad \text{on } \kappa\ \forall \kappa \in \mathscr{T}_h. \tag{3.14}$$

As for r_h, ω_h has the following important properties:

$$\mathbf{u} \cdot \mathbf{n} = 0 \Rightarrow \omega_h\mathbf{u} \cdot \mathbf{n} = 0, \qquad \text{div}\,\mathbf{u} = 0 \Rightarrow \text{div}\,\omega_h\mathbf{u} = 0,$$
$$\{\mathbf{u}_h \in \mathbf{D}_{0h}; \text{div}\,\mathbf{u}_h = 0\} = \{\mathbf{curl}\ \mathbf{f}_h; \mathbf{f}_h \in \mathbf{F}_h\}.$$

The following theorem, proved by Nédélec in [16] and [17] (cf. also Girault and Raviart [12]), collects the main approximation properties of these two interpolation operators. First, let us recall the notion of a *regular* (resp. *uniformly regular*) *triangulation*:

> there exists a constant $\sigma > 0$ (and a constant $\tau > 0$, resp.) independent of h and κ such that $h_\kappa/\rho_\kappa \leq \sigma$ (resp. $\tau h \leq h_\kappa \leq \sigma\rho_\kappa$) $\forall \kappa \in \mathscr{T}_h$,

where h_κ denotes the diameter of κ and ρ_κ the maximum diameter of the balls inscribed in κ.

THEOREM 3.1. *Assume that the triangulation $\mathcal{T}_h$ is regular. Then the interpolation operators r_h and ω_h satisfy the following stability estimates:*

$$\|\mathbf{u} - r_h\mathbf{u}\|_{0,\Omega} + h\|\mathbf{curl}(\mathbf{u} - r_h\mathbf{u})\|_{0,\Omega} \leq C(t)h|\mathbf{u}|_{1,t,\Omega}, \quad \forall \mathbf{u} \in W^{1,t}(\Omega)^3 \text{ for some } t > 2, \tag{3.15}$$

$$\|\mathbf{u} - \omega_h\mathbf{u}\|_{0,\Omega} + h\|\operatorname{div}(\mathbf{u} - \omega_h\mathbf{u})\|_{0,\Omega} \leq Ch|\mathbf{u}|_{1,\Omega} \quad \forall \mathbf{u} \in H^1(\Omega)^3. \tag{3.16}$$

Moreover, when $\mathbf{u} \in H^k(\Omega)^3$ with the integer k of (3.1), *then*

$$\|\mathbf{u} - \omega_h\mathbf{u}\|_{0,\Omega} \leq Ch^k|\mathbf{u}|_{k,\Omega}, \tag{3.17}$$

and when $\mathbf{u} \in H^{k+1}(\Omega)^3$, we have

$$\|\mathbf{u} - r_h\mathbf{u}\|_{0,\Omega} + h\|\mathbf{curl}(\mathbf{u} - r_h\mathbf{u})\|_{0,\Omega} \leq Ch^{k+1}|\mathbf{u}|_{k+1,\Omega}, \tag{3.18}$$

$$\|\operatorname{div}(\mathbf{u} - \omega_h\mathbf{u})\|_{0,\Omega} \leq Ch^k|\mathbf{u}|_{k+1,\Omega}. \tag{3.19}$$

All the above constants are independent of h.

It remains to impose a divergence-free condition on the functions of $\mathbf{M}_h$. Since $\mathbf{M}_h \not\subset H(\operatorname{div};\Omega)$, the best we can do is to approximate this condition by Green's formula. Thus, like Nédélec, we define the space

$$\mathbf{F}_{0h} = \{\mathbf{u}_h \in \mathbf{F}_h; (\mathbf{u}_h, \nabla q_h) = 0 \ \forall q_h \in \Theta_h\}. \tag{3.20}$$

In turn, this yields an approximation of the space $\mathbf{V}$ defined by (2.9):

$$\begin{aligned} \mathbf{V}_h = \{\mathbf{v}_h = (\boldsymbol{\varphi}_h, \boldsymbol{\theta}_h) \in \mathbf{F}_{0h} \times \mathbf{F}_h;& \\ (\mathbf{curl}\ \boldsymbol{\varphi}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\boldsymbol{\theta}_h, \boldsymbol{\mu}_h)\ \forall \boldsymbol{\mu}_h \in \mathbf{F}_h\}.& \end{aligned} \tag{3.21}$$

Remark 3.2. Note that formula (3.21) implies that $\boldsymbol{\theta}_h \in \mathbf{F}_{0h}$. □

With these spaces, we discretize the Navier-Stokes system (2.4)–(2.7) by:

Find a pair $(\boldsymbol{\psi}_h, \boldsymbol{\omega}_h) \in \mathbf{F}_{0h} \times \mathbf{F}_h$ and $\tilde{p}_h \in \mathbf{Q}_h^{k'}/\mathbf{R}$ such that

$$\nu(\mathbf{curl}\,\boldsymbol{\omega}_h, \mathbf{curl}\ \boldsymbol{\varphi}_h) + (\boldsymbol{\omega}_h \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \boldsymbol{\varphi}_h) = (\mathbf{f}, \mathbf{curl}\ \boldsymbol{\varphi}_h) \quad \forall \boldsymbol{\varphi}_h \in \mathbf{F}_h, \tag{3.22}$$

$$(\mathbf{curl}\ \boldsymbol{\varphi}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\boldsymbol{\omega}_h, \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h \in \mathbf{F}_h, \tag{3.23}$$

$$(\nabla \tilde{p}_h, \nabla q_h) = (\mathbf{f} - \boldsymbol{\omega}_h \times \mathbf{curl}\ \boldsymbol{\psi}_h, \nabla q_h) \quad \forall q_h \in \mathbf{Q}_h^{k'}, \tag{3.24}$$

where $k' = \max(k-1, 1)$.

Remark 3.3. The reason for choosing polynomials of lesser degree for the pressure arises from the fact that the error on the pressure is measured in the L^2 norm. The error analysis of Section 5 will show that (theoretically) one does not gain accuracy by using pressure elements of higher degree. □

Remark 3.4. A similar discretization of the Stokes system (2.14)–(2.17) can be obtained by deleting the nonlinear convection term from (3.22)–(3.24). But in this linear case, the pressure is entirely dissociated from the other variables and here it is worthwhile to compute the pressure in $\mathbf{Q}_h^k$. □

Remark 3.5. The other family of **curl**-conforming finite elements defined by Nédélec in [15] is cheaper, considering that it involves half as many degrees of freedom as is required by Definition 3.1. It is less accurate as far as the interpolation error on $\mathbf{u}$ is concerned, but it yields the same interpolation error for **curl u**. □

4. Discrete Sobolev's Inequality and Compactness in $\mathbf{V}_h$. In order to analyze the nonlinear problem (3.22)–(3.24), we require a discrete (uniform with respect to h) Sobolev's inequality for the above finite element spaces. This result is established in a previous paper for a slightly different space $\mathbf{V}_h$ (cf. Girault [11]), but the proof extends easily here. We give the proof for the reader's convenience. First, we recall an important property of the space $\mathbf{F}_{0h}$ proved by Nédélec in [16].

THEOREM 4.1. *Assume that Ω is a convex polyhedron and $\mathscr{T}_h$ a uniformly regular triangulation of $\overline{\Omega}$. There exists a constant C, independent of h, such that*

$$\|\varphi_h\|_{0,\Omega} \leq C\|\mathbf{curl}\ \varphi_h\|_{0,\Omega} \quad \forall \varphi_h \in \mathbf{F}_{0h}. \tag{4.1}$$

Besides that, we shall use the following theoretical result.

LEMMA 4.1. *Let Ω be a convex polyhedron. For each function $\mathbf{g}$ in $L^2(\Omega)^3$, the problem:*

Find $\mathbf{w} \in H^1(\Omega)^3$ and $p \in H^1(\Omega)$ such that

$$\mathbf{curl\ curl\ w} + \nabla p = \mathbf{g}, \qquad \operatorname{div} \mathbf{w} = 0 \quad in\ \Omega, \tag{4.2}$$

$$\mathbf{w} \times \mathbf{n} = \mathbf{0}, \qquad p = 0 \quad on\ \Gamma, \tag{4.3}$$

has the equivalent variational formulation:

Find $\mathbf{w} \in H_0(\mathbf{curl};\Omega)$ and $p \in H_0^1(\Omega)$ such that

$$\begin{gathered}(\mathbf{curl\ w}, \mathbf{curl\ v}) + (\nabla p, \mathbf{v}) = (\mathbf{g}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega),\\ \operatorname{div} \mathbf{w} = 0 \quad in\ \Omega.\end{gathered} \tag{4.4}$$

This problem has a unique solution that satisfies the following bounds:

$$\|\mathbf{curl\ curl\ w}\|_{0,\Omega} \leq \|\mathbf{g}\|_{0,\Omega}, \qquad |p|_{1,\Omega} \leq \|\mathbf{g}\|_{0,\Omega}, \tag{4.5}$$

$$\|\mathbf{curl\ w}\|_{1,\Omega} \leq C\|\mathbf{g}\|_{0,\Omega}. \tag{4.6}$$

The theorem below states a discrete Sobolev's inequality.

THEOREM 4.2. *With the hypotheses of Theorem 4.1, there exists a constant C, independent of h, such that*

$$\|\mathbf{curl}\ \varphi_h\|_{0,4,\Omega} \leq C\|\boldsymbol{\theta}_h\|_{0,\Omega} \quad \forall \mathbf{v}_h = (\varphi_h, \boldsymbol{\theta}_h) \in \mathbf{V}_h. \tag{4.7}$$

Proof. Let us apply Lemma 4.1 with $\boldsymbol{\theta}_h$ for the right-hand side: there exists a unique pair $(\varphi(h), p(h))$ in $H^1(\Omega)^3 \times H^1(\Omega)$, solution of

$$\begin{gathered}(\mathbf{curl}\ \varphi(h), \mathbf{curl}\ \boldsymbol{\mu}) + (\nabla p(h), \boldsymbol{\mu}) = (\boldsymbol{\theta}_h, \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in H_0(\mathbf{curl};\Omega),\\ \operatorname{div} \varphi(h) = 0 \quad \text{in } \Omega,\\ \varphi(h) \times \mathbf{n} = \mathbf{0}, \qquad p(h) = 0 \quad \text{on } \Gamma.\end{gathered} \tag{4.8}$$

Moreover, we have the bounds

$$\|\mathbf{curl}\ \varphi(h)\|_{1,\Omega} \leq C_1\|\boldsymbol{\theta}_h\|_{0,\Omega}, \qquad |p(h)|_{1,\Omega} \leq \|\boldsymbol{\theta}_h\|_{0,\Omega}.$$

Now let $\mathbf{v}_h = (\varphi_h, \boldsymbol{\theta}_h) \in \mathbf{V}_h$; we derive from (4.8) that

$$(\mathbf{curl}(\varphi(h) - \varphi_h), \mathbf{curl}\ \boldsymbol{\mu}_h) + (\nabla(p(h) - q_h), \boldsymbol{\mu}_h) = 0\ \forall \boldsymbol{\mu}_h \in \mathbf{F}_{0h}, \forall q_h \in \Theta_h.$$

Hence,

$$(4.9)\quad \begin{aligned}(\mathbf{curl}(\boldsymbol{\lambda}_h - \boldsymbol{\varphi}_h), \mathbf{curl}\ \boldsymbol{\mu}_h) &= (\mathbf{curl}(\boldsymbol{\lambda}_h - \boldsymbol{\varphi}(h)), \mathbf{curl}\ \boldsymbol{\mu}_h) \\ &\quad + (\nabla(q_h - p(h)), \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h, \boldsymbol{\lambda}_h \in \mathbf{F}_{0h}, \forall q_h \in \Theta_h.\end{aligned}$$

On the one hand, considering that $\mathbf{curl}\ \boldsymbol{\varphi}(h)$ is divergence-free and belongs to $H^1(\Omega)^3$, we can apply the relation (3.9) and choose $\boldsymbol{\lambda}_h \in \mathbf{F}_{0h}$ such that

$$\mathbf{curl}\ \boldsymbol{\lambda}_h = \omega_h\, \mathbf{curl}\ \boldsymbol{\varphi}(h).$$

With this choice, Theorem 3.1 yields

$$(4.10)\qquad \|\mathbf{curl}(\boldsymbol{\lambda}_h - \boldsymbol{\varphi}(h))\|_{0,\Omega} \le C_2 h \|\mathbf{curl}\ \boldsymbol{\varphi}(h)\|_{1,\Omega} \le C_1 C_2 h \|\boldsymbol{\theta}_h\|_{0,\Omega}.$$

On the other hand, nothing can be gained from $\nabla(q_h - p(h))$ because $p(h)$ is not sufficiently smooth, but we can take advantage of the structure of $\boldsymbol{\mu}_h$. Thus, we split $\boldsymbol{\mu}_h$ as follows:

$$\boldsymbol{\mu}_h = \mathbf{w} + \nabla t,$$

where $t \in H_0^1(\Omega)$ is the solution of

$$(\nabla t, \nabla v) = (\boldsymbol{\mu}_h, \nabla v) \quad \forall v \in H_0^1(\Omega),$$

and $\mathbf{w}$ satisfies

$$\operatorname{div} \mathbf{w} = 0, \quad \mathbf{curl}\ \mathbf{w} = \mathbf{curl}\ \boldsymbol{\mu}_h, \quad \mathbf{w} \times \mathbf{n} = \mathbf{0}.$$

Since $\mathbf{curl}\ \boldsymbol{\mu}_h \in L^p(\Omega)^3\ \forall p$, Theorem 2.3 implies that $\mathbf{w} \in W^{1,s}(\Omega)^3$ for some $s > 2$. Hence $\boldsymbol{\mu}_h$ can also be split into

$$\boldsymbol{\mu}_h = r_h \mathbf{w} + \nabla t_h \quad \text{with } t_h \in \Theta_h.$$

Therefore, if we choose for q_h the $H_0^1(\Omega)$-projection of $p(h)$ onto Θ_h, we obtain

$$(4.11)\qquad (\nabla(q_h - p(h)), \boldsymbol{\mu}_h) = (\nabla(q_h - p(h)), r_h\mathbf{w} - \mathbf{w})$$

and

$$|q_h - p(h)|_{1,\Omega} \le |p(h)|_{1,\Omega} \le \|\boldsymbol{\theta}_h\|_{0,\Omega}.$$

As far as $\mathbf{w}$ is concerned, Nédélec proves in [16] that

$$(4.12)\qquad \|\mathbf{w} - r_h\mathbf{w}\|_{0,\Omega} \le C_3(s) h^{1+3/s-3/2} \|\mathbf{curl}\ \boldsymbol{\mu}_h\|_{0,\Omega}.$$

Thus, if we take $\boldsymbol{\mu}_h = \boldsymbol{\lambda}_h - \boldsymbol{\varphi}_h$ and substitute (4.10)–(4.12) into (4.9), we find

$$(4.13)\qquad \|\mathbf{curl}(\boldsymbol{\lambda}_h - \boldsymbol{\varphi}_h)\|_{0,\Omega} \le (C_1 C_2 h + C_3(s) h^{1+3/s-3/2}) \|\boldsymbol{\theta}_h\|_{0,\Omega}.$$

Finally, let us write

$$\begin{aligned}\|\mathbf{curl}\ \boldsymbol{\varphi}_h\|_{0,4,\Omega} \le \|\mathbf{curl}(\boldsymbol{\varphi}_h - \boldsymbol{\lambda}_h)\|_{0,4,\Omega} &+ \|\mathbf{curl}(\boldsymbol{\lambda}_h - \boldsymbol{\varphi}(h))\|_{0,4,\Omega} \\ &+ \|\mathbf{curl}\ \boldsymbol{\varphi}(h)\|_{0,4,\Omega}.\end{aligned}$$

On the one hand, we have the inverse inequality

$$(4.14)\qquad \|\mathbf{curl}(\boldsymbol{\varphi}_h - \boldsymbol{\lambda}_h)\|_{0,4,\Omega} \le C_4 h^{-3/4} \|\mathbf{curl}(\boldsymbol{\varphi}_h - \boldsymbol{\lambda}_h)\|_{0,\Omega}.$$

On the other hand, we can readily prove the following variant of (3.16):

$$\|\omega_h\, \mathbf{curl}\ \boldsymbol{\varphi}(h) - \mathbf{curl}\ \boldsymbol{\varphi}(h)\|_{0,4,\Omega} \le C_5 h^{1/4} \|\mathbf{curl}\ \boldsymbol{\varphi}(h)\|_{1,\Omega}.$$

And, of course, Sobolev's inequality holds for **curl** $\varphi(h)$:

$$\|\mathbf{curl}\ \varphi(h)\|_{0,4,\Omega} \le C_6 \|\mathbf{curl}\ \varphi(h)\|_{1,\Omega}.$$

Collecting all the above inequalities, we obtain

$$\|\mathbf{curl}\ \varphi_h\|_{0,4,\Omega} \le h^{1/4}[C_7 + C_8(s)h^{3/s-3/2}]\|\boldsymbol{\theta}_h\|_{0,\Omega} + C_1C_6\|\boldsymbol{\theta}_h\|_{0,\Omega}.$$

Now it suffices to choose s so that the power of h be nonnegative. This is the case if $2 < s \le 12/5$. □

Remark 4.1. Observe that the proof of Theorem 4.2 is valid as long as $\boldsymbol{\theta}_h$ belongs to $L^2(\Omega)^3$; it need not belong to a finite-dimensional space. □

The following theorem states a discrete compactness result. Its proof is identical to that of a similar result given in Girault [11].

THEOREM 4.3. *We retain the assumptions of Theorem* 4.1. *Let* $(\varphi_h, \boldsymbol{\theta}_h)$ *be a family of pairs of* $\mathbf{V}_h$ *that satisfy*

$$\operatorname*{weak-lim}_{h\to 0} \boldsymbol{\theta}_h = \boldsymbol{\theta} \quad in\ L^2(\Omega)^3.$$

Then there exists φ *in* $H_0^4(\mathbf{curl};\Omega)$ *such that* $(\varphi, \boldsymbol{\theta}) \in \mathbf{V}$ *and*

$$\lim_{h\to 0} \varphi_h = \varphi \quad in\ H^4(\mathbf{curl};\Omega).$$

We end this section with an error estimate concerning the projection operator P_h on $\mathbf{F}_{0h}$ defined by

$$(4.15) \qquad P_h \in \mathscr{L}(H_0(\mathbf{curl};\Omega); \mathbf{F}_{0h}), (\mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi}), \mathbf{curl}\ \boldsymbol{\mu}_h) = 0 \quad \forall \boldsymbol{\mu}_h \in \mathbf{F}_h.$$

LEMMA 4.2. *Let* $\boldsymbol{\psi} \in H_0(\mathbf{curl};\Omega)$ *with* $\operatorname{div}\boldsymbol{\psi} = 0$. *Under the assumptions of Theorem* 4.1, $P_h\boldsymbol{\psi}$ *satisfies the bound*

$$(4.16) \qquad \begin{aligned} \|P_h\boldsymbol{\psi} - \boldsymbol{\psi}\|_{0,\Omega} &\le (Ch + C(s)h^{1+3/s-3/2})\|\mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi})\|_{0,\Omega} \\ &\quad + \inf_{\varphi_h \in \mathbf{F}_h} (\|\varphi_h - \boldsymbol{\psi}\|_{0,\Omega} + C(s)h^{1+3/s-3/2}\|\mathbf{curl}(\varphi_h - \boldsymbol{\psi})\|_{0,\Omega}), \end{aligned}$$

where $C(s)$ *is the constant of Theorem* 2.3.

Proof. For $\mathbf{g} \in L^2(\Omega)^3$, let $\mathbf{w} \in H_0(\mathbf{curl};\Omega)$ and $p \in H_0^1(\Omega)$ be the solution of the Stokes problem of Lemma 4.1:

$$(\mathbf{curl}\ \mathbf{w}, \mathbf{curl}\ \mathbf{v}) + (\nabla p, \mathbf{v}) = (\mathbf{g}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\mathbf{curl};\Omega),$$
$$\operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega.$$

Then,

$$\begin{aligned} (\mathbf{g}, P_h\boldsymbol{\psi} - \boldsymbol{\psi}) &= (\mathbf{curl}(\mathbf{w} - \mathbf{w}_h), \mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi})) + (\nabla(p - p_h), P_h\boldsymbol{\psi} - \boldsymbol{\psi}) \\ &\qquad \forall \mathbf{w}_h \in \mathbf{F}_h, \forall p_h \in \Theta_h, \\ &= (\mathbf{curl}(\mathbf{w} - \mathbf{w}_h), \mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi})) + (\nabla(p - p_h), P_h\boldsymbol{\psi} - \varphi_h) \\ &\quad + (\nabla(p - p_h), \varphi_h - \boldsymbol{\psi}) \quad \forall \mathbf{w}_h, \varphi_h \in \mathbf{F}_h, \forall p_h \in \Theta_h. \end{aligned}$$

Let us choose for p_h the $H_0^1(\Omega)$-projection of p on Θ_h. Then the technique used in the proof of Theorem 4.2 gives here

$$|(\nabla(p - p_h), P_h\boldsymbol{\psi} - \varphi_h)| \le C(s)h^{1+3/s-3/2}\|\mathbf{curl}(P_h\boldsymbol{\psi} - \varphi_h)\|_{0,\Omega}\|\mathbf{g}\|_{0,\Omega}.$$

Likewise, if we take $\mathbf{w}_h \in \mathbf{F}_h$ such that $\mathbf{curl}\ \mathbf{w}_h = \omega_h\, \mathbf{curl}\ \mathbf{w}$, we obtain

$$|(\mathbf{curl}(\mathbf{w} - \mathbf{w}_h), \mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi}))| \le Ch\|\mathbf{curl}(P_h\boldsymbol{\psi} - \boldsymbol{\psi})\|_{0,\Omega}\|\mathbf{g}\|_{0,\Omega}.$$

The desired result follows from these two inequalities. □

5. Error Analysis of Scheme (3.22)–(3.24). Let us make the following assumptions which guarantee that Problem (2.4)–(2.7) has a unique solution. First, we retain the hypotheses of Theorem 2.4, so that the Navier-Stokes system (1.1), (1.2a) is equivalent to its mixed formulation (2.4)–(2.7). Next, we introduce the two quantities

$$N = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}} (\omega \times \mathbf{curl}\ \varphi, \mathbf{curl}\ \boldsymbol{\xi})/(|\mathbf{u}||\mathbf{v}||\mathbf{w}|), \tag{5.1}$$

$$B = \sup_{\mathbf{v}\in\mathbf{V}} \|\mathbf{curl}\ \varphi\|_{0,4,\Omega}/|\mathbf{v}|, \tag{5.2}$$

where $\mathbf{u} = (\boldsymbol{\psi}, \omega)$, $\mathbf{v} = (\varphi, \boldsymbol{\theta})$, $\mathbf{w} = (\boldsymbol{\xi}, \boldsymbol{\eta})$. Then, a classical argument establishes that if

$$[NB\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 < 1, \tag{5.3}$$

then Problem (2.4)–(2.7) has a *unique solution.*

Likewise, we define analogous quantities for the space $\mathbf{V}_h$:

$$N_h = \sup(\omega_h \times \mathbf{curl}\ \varphi_h, \mathbf{curl}\ \boldsymbol{\xi}_h)/(|\mathbf{u}_h||\mathbf{v}_h||\mathbf{w}_h|), \tag{5.4}$$

$$B_h = \sup \|\mathbf{curl}\ \varphi_h\|_{0,4,\Omega}/|\mathbf{v}_h|, \tag{5.5}$$

where the sup is taken over all pairs $\mathbf{u}_h = (\boldsymbol{\psi}_h, \omega_h), \mathbf{v}_h = (\varphi_h, \boldsymbol{\theta}_h)$ and $\mathbf{w}_h = (\boldsymbol{\xi}_h, \boldsymbol{\eta}_h)$ of $\mathbf{V}_h$. Owing to Theorem 4.2, the families N_h and B_h are bounded independently of h. Moreover, using Theorem 4.3 and a standard argument of Girault and Raviart [13], we can show that

$$\limsup_{h\to 0} N_h \le N \quad \text{and} \quad \limsup_{h\to 0} B_h \le B.$$

Therefore, if the condition (5.3) holds, say

$$[NB\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 \le 1 - \delta \quad \text{for some } \delta > 0,$$

then for all sufficiently small h, say $h \le h_0$, we shall have

$$[N_h B_h\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 \le 1 - \delta/2. \tag{5.6}$$

Now, let us study the nonlinear scheme (3.22)–(3.24). A familiar finite-dimensional application of Brouwer's fixed point theorem (cf. Girault [11]) permits us to prove that the scheme always has a solution. Similarly, a classical argument shows that under the condition

$$[N_h B_h\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 < 1$$

the solution is unique.

THEOREM 5.1. *Let Ω be a convex polyhedron and assume that the Navier-Stokes system* (1.1), (1.2a) *has the following regularity*:

$$\mathbf{f} \in L^{4/3}(\Omega)^3, \quad \omega = \mathbf{curl}\ \mathbf{u} \in H^{4/3}(\mathbf{curl}; \Omega), \quad p \in W^{1,4/3}(\Omega) \tag{2.8}$$

and satisfies the condition

$$[NB\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 \le 1 - \delta \quad \textit{for some } \delta > 0. \tag{5.7}$$

If the triangulation $\mathscr{T}_h$ is uniformly regular, then the mixed approximation (3.22)–(3.24) *has a unique solution* $\{\mathbf{u}_h = (\boldsymbol{\psi}_h, \omega_h), \tilde{p}_h\}$*, which bears the following relations with the solution* $\{\mathbf{u} = (\boldsymbol{\psi}, \omega), \tilde{p}\}$ *of the mixed formulation* (2.4)–(2.7):

$$\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,\Omega} \le \inf_{\boldsymbol{\varphi}_h \in \mathbf{F}_h} \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\varphi}_h)\|_{0,\Omega} + C_1 \|\omega - \omega_h\|_{0,\Omega}, \tag{5.8}$$

$$\begin{aligned} &\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,4,\Omega} \\ &\quad \le \inf_{\boldsymbol{\varphi}_h \in \mathbf{F}_h} \{\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\varphi}_h)\|_{0,4,\Omega} + C_2 h^{-3/4} \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\varphi}_h)\|_{0,\Omega}\} \\ &\quad + C_3 \|\omega - \omega_h\|_{0,\Omega}. \end{aligned} \tag{5.9}$$

Moreover, when $\omega \in H(\mathbf{curl}; \Omega)$*, we have the error estimate*

$$\begin{aligned} &\|\omega - \omega_h\|_{0,\Omega} \\ &\quad \le K_1(\nu, \omega, \mathbf{f}) \left\{ \inf_{\boldsymbol{\varphi}_h \in \mathbf{F}_h} \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\varphi}_h)\|_{0,\Omega} \right. \\ &\qquad + \inf_{\boldsymbol{\mu}_h \in \mathbf{F}_h} [\|\omega - \boldsymbol{\mu}_h\|_{0,\Omega} \\ &\qquad\qquad \left. + C(s) h^{1+3/s-3/2} \|\mathbf{curl}(\omega - \boldsymbol{\mu}_h)\|_{0,\Omega}] \right\}. \end{aligned} \tag{5.10}$$

If, in addition, $\tilde{p} \in H^1(\Omega)$*, we have*

$$\begin{aligned} \|\tilde{p} - \tilde{p}_h\|_{0,\Omega} \le K_2(\nu, \omega, \mathbf{f}) \{\|\omega - \omega_h\|_{0,\Omega} + \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,\Omega}\} \\ + C_4 h \inf_{q_h \in \mathbf{Q}_h^{k'}} |\tilde{p} - q_h|_{1,\Omega}. \end{aligned} \tag{5.11}$$

All constants involved are independent of h.

Proof. We have already checked the existence and uniqueness of the solution. The estimates (5.8) and (5.9) are easy to prove. Let us establish the estimate (5.10).

From the continuous and discrete formulations, we derive

$$\begin{aligned} &\nu(\mathbf{curl}(\omega_h - \boldsymbol{\theta}_h), \mathbf{curl}\ \boldsymbol{\varphi}_h) + ((\omega_h - \boldsymbol{\theta}_h) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \boldsymbol{\varphi}_h) \\ &\quad + (\omega \times \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h), \mathbf{curl}\ \boldsymbol{\varphi}_h) \\ &\quad = \nu(\mathbf{curl}(\omega - \boldsymbol{\theta}_h), \mathbf{curl}\ \boldsymbol{\varphi}_h) + ((\omega - \boldsymbol{\theta}_h) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \boldsymbol{\varphi}_h) \\ &\quad + (\omega \times \mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h), \mathbf{curl}\ \boldsymbol{\varphi}_h) \quad \forall \boldsymbol{\varphi}_h, \boldsymbol{\theta}_h, \boldsymbol{\lambda}_h \in \mathbf{F}_h. \end{aligned}$$

Let us choose $\boldsymbol{\theta}_h = P_h \omega$, the projection of ω on $\mathbf{F}_{0h}$ defined by (4.15), and let $\boldsymbol{\lambda}_h \in \mathbf{F}_{0h}$ be defined by

$$(\mathbf{curl}\ \boldsymbol{\lambda}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\boldsymbol{\theta}_h, \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h \in \mathbf{F}_h,$$

so that the pair $(\boldsymbol{\lambda}_h, \boldsymbol{\theta}_h)$ belongs to $\mathbf{V}_h$. Now let us take $\boldsymbol{\varphi}_h = \boldsymbol{\psi}_h - \boldsymbol{\lambda}_h$; in view of (3.23), we are left with

$$\begin{aligned} &\nu\|\omega_h - P_h\omega\|_{0,\Omega}^2 + ((\omega_h - P_h\omega) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h)) \\ &\quad = ((\omega - P_h\omega) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h)) \\ &\quad + (\omega \times \mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h), \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h)). \end{aligned}$$

Hence, applying (5.4), (5.5), (5.6), we obtain

$$\begin{aligned} (\nu\delta/2)\|\omega_h - P_h\omega\|_{0,\Omega} \le B_h\{&\|\omega - P_h\omega\|_{0,\Omega}\|\mathbf{curl}\ \boldsymbol{\psi}_h\|_{0,4,\Omega} \\ &+ \|\omega\|_{0,4,\Omega}\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h)\|_{0,\Omega}\}. \end{aligned}$$

Therefore, since

$$\|\mathbf{curl}\ \psi_h\|_{0,4,\Omega} \le (B_h)^2 \|\mathbf{f}\|_{0,4/3,\Omega}/\nu$$

and the constant B_h is bounded with respect to h, we can write

$$\|\omega_h - P_h\omega\|_{0,\Omega} \le C_1(\nu,\omega,\mathbf{f})\{\|\omega - P_h\omega\|_{0,\Omega} + \|\mathbf{curl}(\psi - \lambda_h)\|_{0,\Omega}\}. \tag{5.12}$$

The first term on the right-hand side is evaluated by Lemma 4.2. For the second term, we write

$$\|\mathbf{curl}(P_h\psi - \lambda_h)\|_{0,\Omega}^2 \le \|\omega - P_h\omega\|_{0,\Omega}\|P_h\psi - \lambda_h\|_{0,\Omega}.$$

As $P_h\psi - \lambda_h \in \mathbf{F}_{0h}$, this implies

$$\|\mathbf{curl}(P_h\psi - \lambda_h)\|_{0,\Omega} \le C_2\|\omega - P_h\omega\|_{0,\Omega},$$

where C_2 is the constant of Theorem 4.1. Hence,

$$\|\mathbf{curl}(\psi - \lambda_h)\|_{0,\Omega} \le \inf_{\varphi_h \in \mathbf{F}_h} \|\mathbf{curl}(\psi - \varphi_h)\|_{0,\Omega} + C_2\|\omega - P_h\omega\|_{0,\Omega}, \tag{5.13}$$

and the estimate (5.10) follows from (5.12), (5.13) and (4.16).

Finally, let us prove (5.11). On the one hand, we have

$$(\nabla(\tilde{p}_h - \tilde{p}), \nabla q_h) = ((\omega - \omega_h) \times \mathbf{curl}\ \psi_h, \nabla q_h) + (\omega \times \mathbf{curl}(\psi - \psi_h), \nabla q_h) \qquad \forall q_h \in \mathbf{Q}_h^{k'}.$$

Therefore,

$$\begin{aligned} |(\nabla(\tilde{p}_h - \tilde{p}), \nabla q_h)| \le \{&\|\omega - \omega_h\|_{0,\Omega}\|\mathbf{curl}\ \psi_h\|_{0,4,\Omega} \\ &+ \|\omega\|_{0,4,\Omega}\|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega}\}|q_h|_{1,4,\Omega} \quad \forall q_h \in \mathbf{Q}_h^{k'} \end{aligned} \tag{5.14}$$

and

$$|\tilde{p}_h - \pi_h\tilde{p}|_{1,\Omega} \le C_3(\nu,\omega,\mathbf{f})h^{-3/4}\{\|\omega - \omega_h\|_{0,\Omega} + \|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega}\}, \tag{5.15}$$

where π_h denotes the $H^1(\Omega)$-projection onto $\mathbf{Q}_h^{k'}$. On the other hand, let us choose for $\tilde{p}_h$ and $\tilde{p}$ their representatives in $L^2(\Omega)$ with zero mean value (denoted by $L_0^2(\Omega)$), i.e., $(\tilde{p}_h, 1) = (\tilde{p}, 1) = 0$. A classical duality argument (cf. Aubin [1] and Nitsche [18]) yields

$$\|\tilde{p}_h - \tilde{p}\|_{0,\Omega} = \sup_{g \in L_0^2(\Omega)} \{(\tilde{p}_h - \tilde{p}, g)/\|g\|_{0,\Omega}\},$$

where $g = -\Delta q$, $\partial q/\partial n_{|\Gamma} = 0$, $q \in H^2(\Omega)$, $\|q\|_{2,\Omega} \le C_4\|g\|_{0,\Omega}$. Thus,

$$(\tilde{p}_h - \tilde{p}, g) = (\nabla(\tilde{p}_h - \tilde{p}), \nabla q) = (\nabla(\tilde{p}_h - \tilde{p}), \nabla(q - q_h)) + (\nabla(\tilde{p}_h - \tilde{p}), \nabla q_h) \qquad \forall q_h \in \mathbf{Q}_h^{k'}.$$

As $k' \ge 1$, let us take $q_h = s_h q$, the standard interpolant of q in $\mathbf{Q}_h^{k'}$. The familiar theory of finite element interpolation (cf. Ciarlet [7]) gives

$$|s_h q|_{1,4,\Omega} \le C_5\|q\|_{2,\Omega}, \qquad |q - s_h q|_{1,\Omega} \le C_6 h\|q\|_{2,\Omega}.$$

Now we easily derive (5.11) from (5.14) and (5.15). □

COROLLARY 5.1. *With the notations and assumptions of Theorem* 5.1, $\{\mathbf{u}_h = (\psi_h, \omega_h), \tilde{p}_h\}$ *converges to* $\{\mathbf{u} = (\psi, \omega), \tilde{p}\}$. *In addition, when the solution is sufficiently smooth, we have the following orders of convergence:*

$$\|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega} + \|\omega - \omega_h\|_{0,\Omega} \leq Ch^k\{|\mathbf{curl}\ \psi|_{k,\Omega} + C(s)h^{3/s-3/2}|\omega|_{k,\Omega}\},$$
$$\|\tilde{p} - \tilde{p}_h\|_{0,\Omega} \leq Ch^k\{|\mathbf{curl}\ \psi|_{k,\Omega} + C(s)h^{3/s-3/2}|\omega|_{k,\Omega} + |\tilde{p}|_{k,\Omega}\}.$$

If $\omega \in H^{k+1}(\Omega)^3$, *the factor of* $C(s)$ *becomes* $h^{1+3/s-3/2}$.

Proof. When $k > 1$, we substitute the estimates of Theorem 3.1 into those of Theorem 5.1. When $k = 1$, ω is not sufficiently smooth for the interpolation operator r_h. Instead, we can use a local regularization operator like the ones defined by Clément in [8] and Bernardi in [6], that preserves the constraint $\omega \times \mathbf{n} = \mathbf{0}$ and is of order one. □

The error estimates of Corollary 5.1 are nearly optimal in the sense that polynomials of degree k yield an error of the order of $h^{k-\varepsilon}$ for ω, **curl** ψ and $\tilde{p}$ in the L^2 norm. On the other hand, the error for **curl** ψ in the L^4 norm is only of the order of $h^{k-3/4}$, but we believe that this bound can be refined.

We finish this section with a similar, but more accurate analysis for the Stokes problem (2.14)–(2.17). The corresponding scheme is:

Find a pair $(\psi_h, \omega_h) \in \mathbf{F}_{0h} \times \mathbf{F}_h$ *and* $p_h \in \mathbf{Q}_h^k/\mathbf{R}$ *such that*

$$\nu(\mathbf{curl}\ \omega_h, \mathbf{curl}\ \varphi_h) = (\mathbf{f}, \mathbf{curl}\ \varphi_h) \quad \forall \varphi_h \in \mathbf{F}_h, \tag{5.16}$$
$$(\mathbf{curl}\ \psi_h, \mathbf{curl}\ \mu_h) = (\omega_h, \mu_h) \quad \forall \mu_h \in \mathbf{F}_h, \tag{5.17}$$
$$(\nabla p_h, \nabla q_h) = (\mathbf{f}, \nabla q_h) \quad \forall q_h \in \mathbf{Q}_h^k. \tag{5.18}$$

It is easy to derive the following expressions for the error:

$$\|\mathbf{curl}(\omega - \omega_h)\|_{0,\Omega} = \inf_{\mu_h \in \mathbf{F}_h} \|\mathbf{curl}(\omega - \mu_h)\|_{0,\Omega}, \tag{5.19}$$
$$\|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega} \leq \inf_{\varphi_h \in \mathbf{F}_h} \|\mathbf{curl}(\psi - \varphi_h)\|_{0,\Omega} + C_1\|\omega - \omega_h\|_{0,\Omega}, \tag{5.8}$$
$$|p - p_h|_{1,\Omega} = \inf_{q_h \in \mathbf{Q}_h^k} |p - q_h|_{1,\Omega}. \tag{5.20}$$

Then Lemma 4.2 and the duality argument of Theorem 5.1 yield

$$\begin{aligned} &\|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega} \\ &\quad \leq \inf_{\varphi_h \in \mathbf{F}_h} \|\mathbf{curl}(\psi - \varphi_h)\|_{0,\Omega} \\ &\qquad + C_1 \inf_{\mu_h \in \mathbf{F}_h} [\|\omega - \mu_h\|_{0,\Omega} + C(s)h^{1+3/s-3/2}\|\mathbf{curl}(\omega - \mu_h)\|_{0,\Omega}], \end{aligned} \tag{5.21}$$

$$\|p - p_h\|_{0,\Omega} \leq C_2 h \inf_{q_h \in \mathbf{Q}_h^k} |p - q_h|_{1,\Omega}. \tag{5.22}$$

Hence, when the solution is sufficiently smooth, this scheme has the following orders of convergence:

$$\|\mathbf{curl}(\omega - \omega_h)\|_{0,\Omega} = O(h^k), \qquad \|\omega - \omega_h\|_{0,\Omega} = O(h^{k+1-\varepsilon}),$$
$$\|\mathbf{curl}(\psi - \psi_h)\|_{0,\Omega} = O(h^k), \quad |p - p_h|_{1,\Omega} = O(h^k), \quad \|p - p_h\|_{0,\Omega} = O(h^{k+1}).$$

They are nearly optimal for ω and optimal for the other variables.

6. The System (1.1) With Boundary Conditions (1.2b). Because of its boundary conditions, Problem (1.1), (1.2b) cannot be decoupled as neatly as above; nevertheless, its analysis is quite similar to that of the preceding sections. Since $\mathbf{u} \cdot \mathbf{n} = 0$ on the whole of Γ, we can still write the velocity $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{curl}\ \boldsymbol{\psi}, \quad \text{with } \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega \text{ and } \boldsymbol{\psi} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma.$$

Hence, if we set

$$H^{4/3}_{\Gamma_0}(\mathbf{curl};\Omega) = \{\mathbf{v} \in H^{4/3}(\mathbf{curl};\Omega); \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_0\},$$

we have the following variational formulation for Problem (1.1), (1.2b):

Find a pair $(\boldsymbol{\psi}, \boldsymbol{\omega}) \in H^4_0(\mathbf{curl};\Omega) \times H^{4/3}_{\Gamma_0}(\mathbf{curl};\Omega)$ *and* $\tilde{p} \in W^{1,4/3}(\Omega)/\mathbf{R}$, *such that*

$$\nu(\mathbf{curl}\ \boldsymbol{\omega}, \mathbf{curl}\ \boldsymbol{\varphi}) + (\boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\varphi}) = (\mathbf{f}, \mathbf{curl}\ \boldsymbol{\varphi}) \qquad \forall \boldsymbol{\varphi} \in H^4_0(\mathbf{curl};\Omega), \tag{6.1}$$

$$(\mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\omega}, \boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in H^{4/3}_{\Gamma_0}(\mathbf{curl};\Omega), \tag{6.2}$$

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad in\ \Omega, \tag{6.3}$$

$$(\nabla \tilde{p}, \nabla q) = (\mathbf{f} - \nu\, \mathbf{curl}\, \boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{curl}\ \boldsymbol{\psi}, \nabla q) \quad \forall q \in W^{1,4}(\Omega). \tag{6.4}$$

Like in Theorem 2.4, it is easy to prove that if the right-hand side $\mathbf{f}$ and the solutions $(\mathbf{u}, p)$ of the system (1.1), (1.2b) have the regularity

$$\mathbf{f} \in L^{4/3}(\Omega)^3, \quad \boldsymbol{\omega} = \mathbf{curl}\ \mathbf{u} \in H^{4/3}(\mathbf{curl};\Omega), \quad p \in W^{1,4/3}(\Omega), \tag{2.8}$$

then the mixed formulation (6.1)–(6.4) is equivalent to (1.1), (1.2b).

The corresponding space $\mathbf{V}$ is

$$\mathbf{V} = \{\mathbf{v} = (\boldsymbol{\varphi}, \boldsymbol{\theta}) \in H^4_0(\mathbf{curl};\Omega) \times L^2(\Omega)^3; \operatorname{div} \boldsymbol{\varphi} = 0, \quad (\mathbf{curl}\ \boldsymbol{\varphi}, \mathbf{curl}\ \boldsymbol{\mu}) = (\boldsymbol{\theta}, \boldsymbol{\mu})\ \forall \boldsymbol{\mu} \in H^{4/3}_{\Gamma_0}(\mathbf{curl};\Omega)\},$$

and the seminorm

$$|\mathbf{v}| = |(\boldsymbol{\varphi}, \boldsymbol{\theta})| = \|\boldsymbol{\theta}\|_{0,\Omega} \tag{2.11}$$

is again an equivalent norm on $\mathbf{V}$. With this space $\mathbf{V}$, we use the expressions (5.1) and (5.2) to define the constants N and B and a standard argument proves that under the condition

$$[NB\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 < 1, \tag{6.5}$$

the solution of Problem (6.1)–(6.4) is unique.

As far as the approximation is concerned, we retain all the finite element spaces of Section 3, and we introduce the spaces

$$\mathbf{M}_{h,\Gamma_0} = \{\boldsymbol{\mu}_h \in \mathbf{M}_h; \boldsymbol{\mu}_h \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_0\}, \tag{6.6}$$

$$\mathbf{V}_h = \{\mathbf{v}_h = (\boldsymbol{\varphi}_h, \boldsymbol{\theta}_h) \in \mathbf{F}_{0h} \times \mathbf{M}_{h,\Gamma_0}; \quad (\mathbf{curl}\ \boldsymbol{\varphi}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\boldsymbol{\theta}_h, \boldsymbol{\mu}_h)\ \forall \boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}\}. \tag{6.7}$$

It is easy to verify that the discrete "Sobolev inequality" Theorem 4.2 and the discrete "compactness" Theorem 4.3 are still valid for this space $\mathbf{V}_h$.

The corresponding discretization of Problem (6.1)–(6.4) is:
Find a pair $(\boldsymbol{\psi}_h, \omega_h) \in \mathbf{F}_{0h} \times \mathbf{M}_{h,\Gamma_0}$ *and* $\tilde{p}_h \in \mathbf{Q}_h^{k'}/\mathbf{R}$ *such that*

$$\nu(\mathbf{curl}\ \omega_h, \mathbf{curl}\ \varphi_h) + (\omega_h \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \varphi_h) = (\mathbf{f}, \mathbf{curl}\ \varphi_h) \quad \forall \varphi_h \in \mathbf{F}_h, \tag{6.8}$$

$$(\mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\omega_h, \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}, \tag{6.9}$$

$$(\nabla \tilde{p}_h, \nabla q_h) = (\mathbf{f} - \nu\, \mathbf{curl}\ \omega_h - \omega_h \times \mathbf{curl}\ \boldsymbol{\psi}_h, \nabla q_h) \quad \forall q_h \in \mathbf{Q}_h^{k'}, \tag{6.10}$$

where $k' = \max(k-1, 1)$.

The following theorem establishes the convergence properties of this scheme.

THEOREM 6.1. *Let* Ω *be a convex polyhedron and assume that the Navier-Stokes system* (1.1), (1.2b) *has the following regularity:*

$$\mathbf{f} \in L^{4/3}(\Omega)^3, \quad \omega = \mathbf{curl}\ \mathbf{u} \in H^{4/3}(\mathbf{curl}; \Omega), \quad p \in W^{1,4/3}(\Omega) \tag{2.8}$$

and satisfies the condition

$$[NB\|\mathbf{f}\|_{0,4/3,\Omega}]/\nu^2 \le 1 - \delta \quad \textit{for some } \delta > 0. \tag{6.11}$$

If the triangulation $\mathcal{T}_h$ *is uniformly regular, then the mixed approximation* (6.8)–(6.10) *has a unique solution* $\{\mathbf{u}_h = (\boldsymbol{\psi}_h, \omega_h), \tilde{p}_h\}$, *which bears the following relations with the solution* $\{\mathbf{u} = (\boldsymbol{\psi}, \omega), \tilde{p}\}$ *of the mixed formulation* (6.1)–(6.4):

$$\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,\Omega} \le \inf_{\varphi_h \in \mathbf{F}_h} \|\mathbf{curl}(\boldsymbol{\psi} - \varphi_h)\|_{0,\Omega} + C_1\|\omega - \omega_h\|_{0,\Omega}, \tag{6.12}$$

$$\begin{aligned} &\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,4,\Omega} \\ &\quad \le \inf_{\varphi_h \in \mathbf{F}_h} \{\|\mathbf{curl}(\boldsymbol{\psi} - \varphi_h)\|_{0,4,\Omega} + C_2 h^{-3/4}\|\mathbf{curl}(\boldsymbol{\psi} - \varphi_h)\|_{0,\Omega}\} \\ &\quad\quad + C_3\|\omega - \omega_h\|_{0,\Omega}. \end{aligned} \tag{6.13}$$

Moreover, when $\omega \in H(\mathbf{curl}; \Omega)$, *we have the error estimate*

$$\begin{aligned} &\|\omega - \omega_h\|_{0,\Omega} \\ &\quad \le K_1(\nu, \omega, \mathbf{f}) \Big\{ C_4 h^{-1} \inf_{\varphi_h \in \mathbf{F}_h} \|\mathbf{curl}(\boldsymbol{\psi} - \varphi_h)\|_{0,\Omega} \\ &\quad\quad + \inf_{\boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}} [\|\mathbf{curl}(\omega - \boldsymbol{\mu}_h)\|_{0,\Omega} + \|\omega - \boldsymbol{\mu}_h\|_{0,\Omega}] \Big\}. \end{aligned} \tag{6.14}$$

If, in addition, $\tilde{p} \in H^1(\Omega)$, *we have*

$$\begin{aligned} \|\tilde{p} - \tilde{p}_h\|_{0,\Omega} &\le K_2(\nu, \omega, \mathbf{f})\{\|\omega - \omega_h\|_{0,\Omega} + \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,\Omega}\} \\ &\quad + C_5 h \inf_{q_h \in \mathbf{Q}_h^{k'}} |\tilde{p} - q_h|_{1,\Omega}. \end{aligned} \tag{6.15}$$

Again, all constants involved are independent of h.

Proof. We shall just sketch the proof, because it is very similar to that of Theorem 5.1. Here also, we write

$$\begin{aligned} &\nu(\mathbf{curl}(\omega_h - \boldsymbol{\theta}_h), \mathbf{curl}\ \varphi_h) + ((\omega_h - \boldsymbol{\theta}_h) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \varphi_h) \\ &\quad\quad + (\omega \times \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h), \mathbf{curl}\ \varphi_h) \\ &\quad = \nu(\mathbf{curl}(\omega - \boldsymbol{\theta}_h), \mathbf{curl}\ \varphi_h) + ((\omega - \boldsymbol{\theta}_h) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ \varphi_h) \\ &\quad\quad + (\omega \times \mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h), \mathbf{curl}\ \varphi_h) \quad \forall \varphi_h, \boldsymbol{\lambda}_h \in \mathbf{F}_h, \forall \boldsymbol{\theta}_h \in M_{h,\Gamma_0}. \end{aligned}$$

Let us fix $\boldsymbol{\lambda}_h$ in $\mathbf{F}_{0h}$ and associate $\boldsymbol{\theta}_h$ in $\mathbf{M}_{h,\Gamma_0}$ with

$$(\mathbf{curl}\ \boldsymbol{\lambda}_h, \mathbf{curl}\ \boldsymbol{\mu}_h) = (\boldsymbol{\theta}_h, \boldsymbol{\mu}_h) \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0},$$

so that the pair belongs to $\mathbf{V}_h$. Let us choose $\varphi_h = \boldsymbol{\psi}_h - \boldsymbol{\lambda}_h$ and observe that

$$\begin{aligned}(\mathbf{curl}(\omega - \boldsymbol{\theta}_h), \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h)) &= (\mathbf{curl}(\omega - \boldsymbol{\mu}_h), \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\lambda}_h)) \\ &\quad + (\omega_h - \boldsymbol{\theta}_h, \boldsymbol{\mu}_h - \boldsymbol{\theta}_h) \quad \forall \boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}.\end{aligned}$$

Thus, if h is sufficiently small, we obtain

$$\begin{aligned}\|\omega_h - \boldsymbol{\theta}_h\|_{0,\Omega} \le C_1(\nu, \omega, \mathbf{f}) \Big\{ &\inf_{\boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}} [\|\mathbf{curl}(\omega - \boldsymbol{\mu}_h)\|_{0,\Omega} + \|\boldsymbol{\mu}_h - \boldsymbol{\theta}_h\|_{0,\Omega}] \\ &+ \|\omega - \boldsymbol{\theta}_h\|_{0,\Omega} + \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h)\|_{0,\Omega} \Big\}.\end{aligned}$$

It remains to evaluate $\|\omega - \boldsymbol{\theta}_h\|_{0,\Omega}$. Unfortunately, this computation is not optimal. At the present stage all we can say is that

$$\begin{aligned}\|\omega - \boldsymbol{\theta}_h\|_{0,\Omega} \le 2 &\inf_{\boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}} \|\omega - \boldsymbol{\mu}_h\|_{0,\Omega} \\ &+ \sup_{\mathbf{g}_h \in \mathbf{M}_{h,\Gamma_0}} \{\|\mathbf{curl}\ \mathbf{g}_h\|_{0,\Omega} / \|\mathbf{g}_h\|_{0,\Omega}\} \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h)\|_{0,\Omega},\end{aligned}$$

i.e.,

$$\|\omega - \boldsymbol{\theta}_h\|_{0,\Omega} \le 2 \inf_{\boldsymbol{\mu}_h \in \mathbf{M}_{h,\Gamma_0}} \|\omega - \boldsymbol{\mu}_h\|_{0,\Omega} + C_2 h^{-1} \|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\lambda}_h)\|_{0,\Omega}. \tag{6.16}$$

This proves (6.14).

As far as the pressure is concerned, we proceed much like in Theorem 2.7, Chapter III of Girault and Raviart [12]. First we choose for $\tilde{p}_h$ and $\tilde{p}$ their representatives in $L_0^2(\Omega)$; then we associate with $\tilde{p}_h - \tilde{p}$ a function $\mathbf{v}$ in $H_0^1(\Omega)^3$ such that

$$\tilde{p}_h - \tilde{p} = \operatorname{div} \mathbf{v} \quad \text{and} \quad |\mathbf{v}|_{1,\Omega} \le C_3 \|\tilde{p}_h - \tilde{p}\|_{0,\Omega}.$$

Let us split $\mathbf{v}$ into a gradient and a rotation:

$$\mathbf{v} = \nabla q + \mathbf{curl}\ \varphi,$$

where $q \in H^2(\Omega)$ is the solution of

$$\Delta q = \operatorname{div} \mathbf{v} \quad \text{in } \Omega, \qquad \partial q / \partial n = 0 \quad \text{on } \Gamma \quad \text{and } \|q\|_{2,\Omega} \le C_4 \|\tilde{p}_h - \tilde{p}\|_{0,\Omega};$$

and according to Theorem 2.2,

$$\operatorname{div} \varphi = 0 \quad \text{in } \Omega, \qquad \varphi \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma, \qquad \varphi \in W^{1,s}(\Omega)^3, \qquad \mathbf{curl}\ \varphi \in H^1(\Omega)^3$$

and $\|\mathbf{curl}\ \varphi\|_{1,\Omega} \le C_5 \|\tilde{p}_h - \tilde{p}\|_{0,\Omega}$. In addition, let us set

$$\mathbf{v}_h = \nabla \pi_h q + \mathbf{curl}\ r_h \varphi,$$

where π_h denotes the $H^1(\Omega)$-projection onto $\mathbf{Q}_h^{k'}$. Thus, we can write

$$\begin{aligned}\|\tilde{p}_h - \tilde{p}\|_{0,\Omega}^2 &= (\tilde{p}_h - \tilde{p}, \operatorname{div} \mathbf{v}) = -(\nabla(\tilde{p}_h - \tilde{p}), \nabla q) \\ &= -(\nabla(t_h - \tilde{p}), \nabla(q - \pi_h q)) - (\nabla(\tilde{p}_h - \tilde{p}), \nabla \pi_h q) \quad \forall t_h \in \mathbf{Q}_h^{k'}.\end{aligned}$$

The first term in the right-hand side is easily estimated, and it remains to evaluate the second term.

We have

$$(\nabla(\tilde{p}_h - \tilde{p}), \nabla\pi_h q) = \nu(\mathbf{curl}(\omega - \omega_h), \nabla\pi_h q) + ((\omega - \omega_h) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \nabla\pi_h q) + (\omega \times \mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h), \nabla\pi_h q).$$

The last two terms can be bounded as in Theorem 5.1. As far as the first term is concerned, we write

$$\begin{aligned}\nu(\mathbf{curl}(\omega - \omega_h), \nabla\pi_h q) &= \nu(\mathbf{curl}(\omega - \omega_h), \mathbf{v}_h - \mathbf{curl}\ r_h\varphi)\\ &= \nu(\mathbf{curl}(\omega - \omega_h), \mathbf{v}_h - \mathbf{v}) + \nu(\mathbf{curl}(\omega - \omega_h), \mathbf{v})\\ &\quad - v(\mathbf{curl}(\omega - \omega_h), \mathbf{curl}\ r_h\varphi).\end{aligned}$$

Again, the first two terms are easily bounded. For the last term, we take the difference between (6.1) and (6.8):

$$\nu(\mathbf{curl}(\omega - \omega_h), \mathbf{curl}\ r_h\varphi) = ((\omega_h - \omega) \times \mathbf{curl}\ \boldsymbol{\psi}_h, \mathbf{curl}\ r_h\varphi) + (\omega \times \mathbf{curl}(\boldsymbol{\psi}_h - \boldsymbol{\psi}), \mathbf{curl}\ r_h\varphi).$$

This expression is also readily evaluated, and (6.15) follows from the above bounds. □

COROLLARY 6.1. *With the notations and assumptions of Theorem* 6.1, $\{\mathbf{u}_h = (\boldsymbol{\psi}_h, \omega_h), \tilde{p}_h\}$ *converges to* $\{\mathbf{u} = (\boldsymbol{\psi}, \omega), \tilde{p}\}$ *when* $k \geq 2$. *In addition, when the solution is sufficiently smooth, we have the following orders of convergence*:

$$\|\mathbf{curl}(\boldsymbol{\psi} - \boldsymbol{\psi}_h)\|_{0,\Omega} + \|\omega - \omega_h\|_{0,\Omega} \leq Ch^{k-1}\{|\mathbf{curl}\ \boldsymbol{\psi}|_{k,\Omega} + |\omega|_{k,\Omega}\},$$
$$\|\tilde{p} - \tilde{p}_h\|_{0,\Omega} \leq Ch^{k-1}\{|\mathbf{curl}\ \boldsymbol{\psi}|_{k,\Omega} + |\omega|_{k,\Omega}\} + Ch^{k'}|\tilde{p}|_{k',\Omega}.$$

When solved with the same mixed finite element method, the classical Stokes problem has an analogous order of convergence (cf. Nédélec [17]). In particular, we observe here the same loss of one power of h, which is not due to the nonlinearity, but to the coupling of the vorticity ω_h and vector potential $\boldsymbol{\psi}_h$ in formula (6.9). This does not occur when the boundary conditions (1.2a) are discretized because then, ω_h and $\boldsymbol{\psi}_h$ belong to the same space. However, we believe that the results of Corollary 6.1 are not optimal and that, like in the two-dimensional situation, a factor $h^{1/2}$ can be recovered using the argument of Scholz [20]. This requires a sharp (and difficult) L^∞ error estimate for the projection operator P_h on $\mathbf{F}_{0h}$ defined by (4.15), which is not yet established.

Finally, we also observe that in the present situation it is not worth using the complete P_k finite elements of Nédélec: The incomplete P_k give the same order of convergence and are substantially cheaper.

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