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The Stieltjes Function—Definition and Properties

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Abstract. Close to the singular point $s = 1$ the zeta function can be represented as a Laurent series in $(s-1)$. The coefficients in this series are called the Stieltjes constants, and the first ones were computed already 100 years ago. In order to investigate their somewhat unexpected behavior we have defined a related function which we call the Stieltjes function, and examined its properties.

Introduction. From the definition of the zeta function we have

$$(s-1)\zeta(s) = \sum_{k=1}^{\infty}(s-1)k^{-s}.$$

Assuming s real and larger than 1, we subtract

$$\sum_{k=1}^{\infty}(k^{1-s} - (k+1)^{1-s}) = 1$$

to get

$$(s-1)\zeta(s) = 1 + \sum_{k=1}^{\infty}\{(k+1)^{1-s} - k^{1-s} + (s-1)k^{-s}\}. \tag{1}$$

Note that the sum above is 0 when $s = 1$. Hence,

$$\begin{aligned}(s-1)\zeta(s) &= 1 + \sum_{k=1}^{\infty}\{\exp(-(s-1)\ln(k+1)) - \exp(-(s-1)\ln k) \\ &\qquad + (s-1)k^{-1}\exp(-(s-1)\ln k)\} \\ &= 1 + \sum_{k=1}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}[(\ln(k+1))^n - (\ln k)^n]\right. \\ &\qquad \left. + \frac{s-1}{k}\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n(\ln k)^n}{n!}\right\}.\end{aligned}$$

As observed before, the coefficient of $(s-1)^{-1}$ is 1 and that of $(s-1)^0$ is 0. Further, the coefficient of $(s-1)^1$ is easily found to be γ. Dividing by $(s-1)$ we get

$$\zeta(s) = 1/(s-1) + \sum_{n=0}^{\infty}(-1)^n\gamma_n(s-1)^n/n!, \tag{2}$$

where

$$\gamma_n = \sum_{k=1}^{\infty}\left\{\frac{(\ln k)^n}{k} - \frac{(\ln(k+1))^{n+1} - (\ln k)^{n+1}}{n+1}\right\}. \tag{3}$$

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Note that when $n = 0$ we have interpreted the term 0^0 as 1 which gives the value $\gamma_0 = \gamma$. Formula (3) can also be written as

$$\gamma_n = \lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{(\ln k)^n}{k} - \frac{(\ln N)^{n+1}}{n+1}\right\} \tag{4}$$

or

$$\gamma_n = \lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{(\ln k)^n}{k} - \int_1^N \frac{(\ln x)^n}{x}\,dx\right\}. \tag{5}$$

The values $A_n = (-1)^n\gamma_n/n!$ are called *Stieltjes constants* and were computed numerically for $n = 1(1)8$ in 1887 by Jensen [1] to 9 decimals (with the notation $C_{n+1} = A_n$), and in 1895 by Gram [2] to 16 places. The constants γ_n were computed for $n = 0(1)19$ by Liang and Todd [3] to 15 significant digits in 1972, and they were also mentioned by Apostol [4]. Our values displayed in Table 1 agree with those in [3], except for $n = 8$ and $n = 9$, which were given as −0.35212 33538 03039 (−3) and −0.34394 77441 80881 (−4), respectively.

TABLE 1

The constants γ_n, $n = 0(1)20$.

n	γ_n	n	γ_n
0	0.57721 56649 01532 86061 (0)		
1	− 0.72815 84548 36767 24861 (−1)	11	0.27018 44395 43903 52667 (−3)
2	− 0.96903 63192 87231 84845 (−2)	12	0.16727 29121 05140 19335 (−3)
3	0.20538 34420 30334 58662 (−2)	13	− 0.27463 80660 37601 58860 (−4)
4	0.23253 70065 46730 00575 (−2)	14	− 0.20920 92620 59299 94584 (−3)
5	0.79332 38173 01062 70175 (−3)	15	− 0.28346 86553 20241 44664 (−3)
6	− 0.23876 93454 30199 60987 (−3)	16	− 0.19969 68583 08969 77471 (−3)
7	− 0.52728 95670 57751 04607 (−3)	17	0.26277 03710 99183 36699 (−4)
8	− 0.35212 33538 03039 50960 (−3)	18	0.30736 84081 49252 82659 (−3)
9	− 0.34394 77441 80880 48178 (−4)	19	0.50360 54530 47355 62906 (−3)
10	0.20533 28149 09064 79468 (−3)	20	0.46634 35615 11559 44940 (−3)

2. The Stieltjes Function $\gamma(z)$. In order to get a clear idea concerning the behavior of the coefficients γ_n, we define a function $\gamma(z)$ as follows:

$$\gamma(z) = \sum_{k=2}^{\infty}\left\{\frac{(\ln k)^z}{k} - \frac{(\ln k)^{z+1} - (\ln(k-1))^{z+1}}{z+1}\right\}, \tag{6}$$

where we have discarded the dubious first term $(\ln 1)^z$ which will cause trouble when $\mathrm{Re}(z) < 0$. With this definition, we have $\gamma(0) = -1 + \gamma$ while $\gamma_0 = \gamma$. As before, the last term can be replaced by an integral, and we see that the whole expression can be computed by Euler-Maclaurin's formula. Choosing the integer N sufficiently large and putting $p = \ln N$, we obtain the semiconvergent series

$$\begin{aligned}\gamma(z) \sim {}& \sum_{k=2}^{N-1}(\ln k)^z/k - p^{z+1}/(z+1) + p^z/2N - (zp^{z-1} - p^z)/12N^2 \\ &+ (z(z-1)(z-2)p^{z-3} - 6z(z-1)p^{z-2} + 11zp^{z-1} - 6p^z)/720N^4 \\ &- (z(z-1)(z-2)(z-3)(z-4)p^{z-5} - 15z(z-1)(z-2)(z-3)p^{z-4} \\ &\quad + 85z(z-1)(z-2)p^{z-3} - 225z(z-1)p^{z-2} \\ &\quad + 274zp^{z-1} - 120p^z)/30240N^6 + \cdots .\end{aligned} \tag{7}$$

Let us first examine convergence of the series in (6). Denoting the general term by $r_k(z)$ and writing $q = \ln k$, we have for large k ($z \neq -1$):

$$\ln(k-1) = q(1 - (1/k + 1/2k^2 + 1/3k^3 + \dots)/q) = q(1-h),$$

$$(\ln(k-1))^{z+1} = q^{z+1}(1-h)^{z+1} = q^{z+1}\left(1 - (z+1)h + \frac{z(z+1)}{2}h^2 - \cdots\right)$$
$$= q^{z+1}(1 - (z+1)(1/kq + 1/2k^2q - z/2k^2q^2) + \cdots)$$

and

$$r_k(z) = q^z/k - q^{z+1}(1/qk + 1/2k^2q - z/2k^2q^2 + \mathcal{O}(k^{-3}))$$
$$= q^{z-1}(z-q)/2k^2 + \mathcal{O}(k^{-3}).$$

As can be seen by direct inspection of (6), our computation is valid for all z except $z = -1$, and hence we have convergence in the whole complex plane except in $z = -1$.

In the vicinity of the singular point $z = -1$ we have

$$\gamma(-1+\varepsilon) = \lim_{N\to\infty}\left\{\sum_{k=2}^{N-1}(\ln k)^\varepsilon/k\ln k - (\ln N)^\varepsilon/\varepsilon\right\}$$

(8)
$$= \lim_{N\to\infty}\left\{\sum_{k=2}^{N-1}\left(\frac{1}{k\ln k} + \frac{\varepsilon\ln\ln k}{k\ln k}\right)\right.$$
$$\left. -\varepsilon^{-1}\left(1 + \varepsilon\ln\ln N + \frac{1}{2}\varepsilon^2(\ln\ln N)^2\right) + \mathcal{O}(\varepsilon^2)\right\}$$
$$= -\varepsilon^{-1} + a + b\varepsilon + \mathcal{O}(\varepsilon^2).$$

Here the constants a and b are given by

(9)
$$a = \lim_{N\to\infty}\left\{\sum_{k=2}^{N-1}\frac{1}{k\ln k} - \ln\ln N\right\} \simeq 0.79467\ 86454,$$
$$b = \lim_{N\to\infty}\left\{\sum_{k=2}^{N-1}\frac{\ln\ln k}{k\ln k} - \frac{1}{2}(\ln\ln N)^2\right\} \simeq -0.26028\ 76870.$$

We now turn to the numerical computation of the Stieltjes function in general and start with the case when z is real, $z = x$. For large values of x, e.g., $x > 20$, we get enormous cancellations with a loss of 15–30 significant digits. We can get rid of much of this cancellation if we note that the wanted sum is in fact built up by the small pieces between the curve $y(t) = (\ln t)^x/t$ and the consecutive chords. In this way we can gain, say, 10 digits at a price of a considerably more complicated formula, and no real advantage is obtained. So we decided to perform the computations in multiple precision (33 significant digits), and in this way we could proceed as far as $x = 50$ with 10 digit accuracy. Using high-order extrapolation, we could go even further in some respects (determination of the zeros). In Table 2 we present the values for $x = 0(0.1)10(0.5)50$ with 10 significant figures. In addition, the values for $x = 50(5)100$ are presented with decreasing accuracy. Further, ten-digit values are also given for a reasonable selection of negative arguments. The function $y = \operatorname{sign}(\gamma(x))\ln(1 + 10000|\gamma(x)|)$ for $x \geq 0$ (where the factor 10000 was chosen for convenience) is presented in Figure 1.

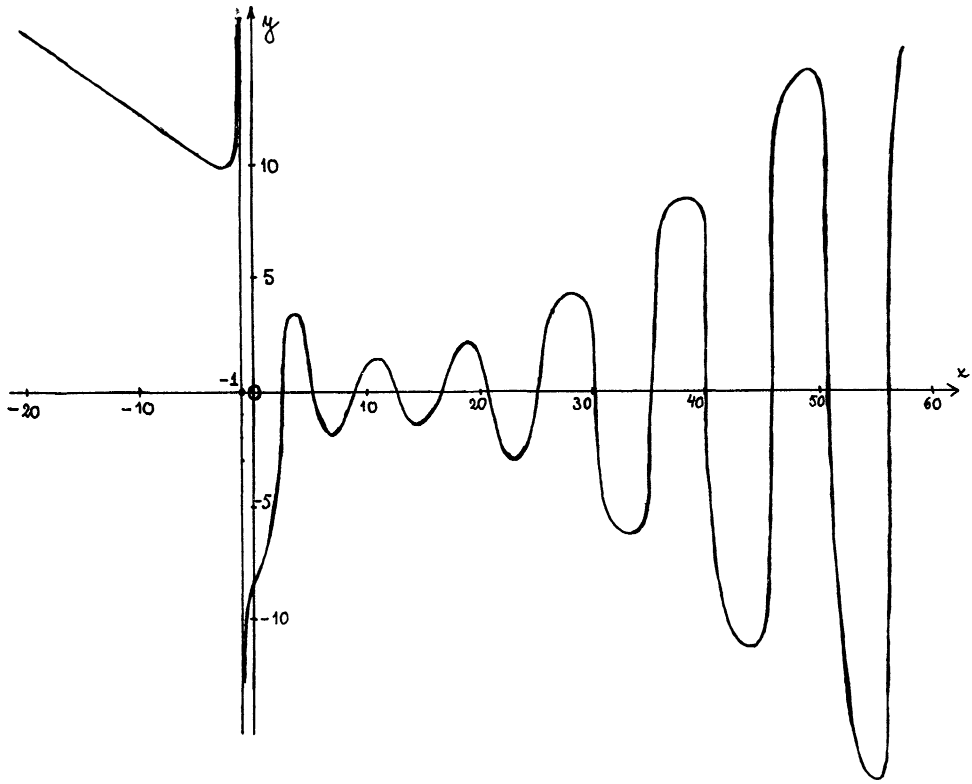

FIGURE 1

$$y = \operatorname{sign}(\gamma(x)) \ln(1 + 10^4 |\gamma(x)|)$$

TABLE 2

$\gamma(x)$, $x = 0(0.1)10(0.5)50(5)100$ *and various negative values.*

x	$\gamma(x)$	x	$\gamma(x)$	x	$\gamma(x)$
0	−0.42278 43351 (0)	1.5	−0.29365 43221 (−1)	3.0	0.20538 34420 (−2)
0.1	−0.34950 59311 (0)	1.6	−0.24087 00361 (−1)	3.1	0.23308 89549 (−2)
0.2	−0.29073 64821 (0)	1.7	−0.19580 94389 (−1)	3.2	0.25255 65122 (−2)
0.3	−0.24300 28902 (0)	1.8	−0.15738 69240 (−1)	3.3	0.26502 92026 (−2)
0.4	−0.20383 02192 (0)	1.9	−0.12468 04475 (−1)	3.4	0.27159 41098 (−2)
0.5	−0.17140 87252 (0)	2.0	−0.96903 63193 (−2)	3.5	0.27320 17767 (−2)
0.6	−0.14438 57596 (0)	2.1	−0.73383 29802 (−2)	3.6	0.27068 30375 (−2)
0.7	−0.12173 11285 (0)	2.2	−0.53541 23329 (−2)	3.7	0.26476 36129 (−2)
0.8	−0.10264 73398 (0)	2.3	−0.36879 29963 (−2)	3.8	0.25607 67985 (−2)
0.9	−0.86508 20111 (−1)	2.4	−0.22967 19054 (−2)	3.9	0.24517 45229 (−2)
1.0	−0.72815 84548 (−1)	2.5	−0.11432 30739 (−2)	4.0	0.23253 70065 (−2)
1.1	−0.61170 04122 (−1)	2.6	−0.19513 41762 (−3)	4.1	0.21858 12170 (−2)
1.2	−0.51245 87783 (−1)	2.7	0.57567 59585 (−3)	4.2	0.20366 82851 (−2)
1.3	−0.42777 27814 (−1)	2.8	0.11936 70272 (−2)	4.3	0.18811 00222 (−2)
1.4	−0.35544 63719 (−1)	2.9	0.16801 83041 (−2)	4.4	0.17217 46581 (−2)

TABLE 2 (*continued*)

x	$\gamma(x)$	x	$\gamma(x)$	x	$\gamma(x)$
4.5	0.15609 19008 (−2)	9.6	0.12765 12234 (−3)	33.5	−0.51079 82975 (−1)
4.6	0.14005 74048 (−2)	9.7	0.14964 64810 (−3)	34.0	−0.51126 92802 (−1)
4.7	0.12423 67240 (−2)	9.8	0.16996 64230 (−3)	34.5	−0.42034 33725 (−1)
4.8	0.10876 88118 (−2)	9.9	0.18854 63547 (−3)	35.0	−0.20373 04360 (−1)
4.9	0.93769 12540 (−3)	10.0	0.20533 28149 (−3)	35.5	0.17044 60151 (−1)
5.0	0.79332 38173 (−3)	10.5	0.26123 90031 (−3)	36.0	0.72482 15881 (−1)
5.1	0.65535 00622 (−3)	11.0	0.27018 44395 (−3)	36.5	0.14634 86129 (0)
5.2	0.52437 31151 (−3)	11.5	0.23610 56107 (−3)	37.0	0.23602 63822 (0)
5.3	0.40085 43707 (−3)	12.0	0.16727 29121 (−3)	37.5	0.33450 88580 (0)
5.4	0.28513 07740 (−3)	12.5	0.75034 22831 (−4)	38.0	0.42896 34467 (0)
5.5	0.17743 02283 (−3)	13.0	−0.27463 80660 (−4)	38.5	0.49942 34605 (0)
5.6	0.77885 34240 (−4)	13.5	−0.12654 21906 (−3)	39.0	0.51792 18402 (0)
5.7	−0.13454 30097 (−4)	14.0	−0.20920 92621 (−3)	39.5	0.44850 75994 (0)
5.8	−0.96611 18973 (−4)	14.5	−0.26429 65032 (−3)	40.0	0.24872 15660 (0)
5.9	−0.17167 00980 (−3)	15.0	−0.28346 86553 (−3)	40.5	−0.12676 66386 (0)
6.0	−0.23876 93454 (−3)	15.5	−0.26206 15173 (−3)	41.0	−0.71957 49013 (0)
6.1	−0.29809 29885 (−3)	16.0	−0.19969 68583 (−3)	41.5	−0.15572 92583 (+1)
6.2	−0.34986 38447 (−3)	16.5	−0.10062 20008 (−3)	42.0	−0.26387 94656 (+1)
6.3	−0.39433 72610 (−3)	17.0	0.26277 03711 (−4)	42.5	−0.39149 22253 (+1)
6.4	−0.43179 55506 (−3)	17.5	0.16783 42461 (−3)	43.0	−0.52649 32297 (+1)
6.5	−0.46254 30260 (−3)	18.0	0.30736 84081 (−3)	43.5	−0.64702 58486 (+1)
6.6	−0.48690 15690 (−3)	18.5	0.42587 05181 (−3)	44.0	−0.71887 56619 (+1)
6.7	−0.50520 66811 (−3)	19.0	0.50360 54530 (−3)	44.5	−0.69345 90001 (+1)
6.8	−0.51780 39670 (−3)	19.5	0.52208 68162 (−3)	45.0	−0.50723 87787 (+1)
6.9	−0.52504 60047 (−3)	20.0	0.46634 35615 (−3)	45.5	−0.83623 90601 (0)
7.0	−0.52728 95671 (−3)	20.5	0.32734 98851 (−3)	46.0	0.66098 44430 (+1)
7.1	−0.52489 31553 (−3)	21.0	0.10443 77698 (−3)	46.5	0.18055 58199 (+2)
7.2	−0.51821 48172 (−3)	21.5	−0.19254 24529 (−3)	47.0	0.34038 88242 (+2)
7.3	−0.50761 02190 (−3)	22.0	−0.54159 95822 (−3)	47.5	0.54568 01487 (+2)
7.4	−0.49343 09474 (−3)	22.5	−0.90784 57710 (−3)	48.0	0.78668 26060 (+2)
7.5	−0.47602 30186 (−3)	23.0	−0.12439 62090 (−2)	48.5	0.10394 43754 (+3)
7.6	−0.45572 55736 (−3)	23.5	−0.14922 39513 (−2)	49.0	0.12579 58628 (+3)
7.7	−0.43286 97425 (−3)	24.0	−0.15885 11279 (−2)	49.5	0.13716 28021 (+3)
7.8	−0.40777 76606 (−3)	24.5	−0.14682 04830 (−2)	50.0	0.12695 51644 (+3)
7.9	−0.38076 16213 (−3)	25.0	−0.10745 91953 (−2)	55	−0.34571 41208 (+4)
8.0	−0.35212 33538 (−3)	25.5	−0.36909 38654 (−3)	60	0.98543 255 (+5)
8.1	−0.32215 34117 (−3)	26.0	0.65680 35186 (−3)	65	−0.28450 765 (+7)
8.2	−0.29113 06636 (−3)	26.5	0.19688 51145 (−2)	70	0.79321 67 (+8)
8.3	−0.25932 18746 (−3)	27.0	0.34778 36914 (−2)	75	−0.19194 8 (+10)
8.4	−0.22698 13701 (−3)	27.5	0.50296 71920 (−2)	80	0.2516 (+11)
8.5	−0.19435 07753 (−3)	28.0	0.64000 68532 (−2)	85	0.1259 (+13)
8.6	−0.16165 88205 (−3)	28.5	0.72965 85702 (−2)	90	−0.147 (+15)
8.7	−0.12912 12091 (−3)	29.0	0.73711 51770 (−2)	95	0.94 (+16)
8.8	−0.96940 53963 (−4)	29.5	0.62462 60468 (−2)	100	−0.41 (+18)
8.9	−0.65306 27786 (−4)	30.0	0.35577 28856 (−2)	−0.1	−0.51559 79234 (0)
9.0	−0.34394 77442 (−4)	30.5	−0.98396 11551 (−3)	−0.2	−0.63549 72753 (0)
9.1	−0.43693 22767 (−5)	31.0	−0.75133 25998 (−2)	−0.3	−0.79436 10381 (0)
9.2	0.24619 74575 (−4)	31.5	−0.15912 66990 (−1)	−0.4	−0.10120 03396 (+1)
9.3	0.52434 93351 (−4)	32.0	−0.25703 72911 (−1)	−0.5	−0.13241 10541 (+1)
9.4	0.78951 41805 (−4)	32.5	−0.35938 66049 (−1)	−0.6	−0.18020 81907 (+1)
9.5	0.10405 68662 (−3)	33.0	−0.45106 73411 (−1)	−0.7	−0.26125 53939 (+1)

TABLE 2 (*continued*)

x	$\gamma(x)$	x	$\gamma(x)$	x	$\gamma(x)$
−0.75	−0.32674 67953 (+1)	−1.5	0.29376 63638 (+1)	− 8	0.95666 92286 (+1)
−0.8	−0.42554 95289 (+1)	−2	0.21097 42801 (+1)	− 9	0.13698 22178 (+2)
−0.85	−0.59099 65179 (+1)	−2.5	0.19834 63547 (+1)	− 10	0.19672 86125 (+2)
−0.9	−0.92308 73446 (+1)	−3	0.20658 86539 (+1)	− 12	0.40764 06333 (+2)
−0.95	−0.19218 21584 (+2)	−3.5	0.22654 89660 (+1)	− 15	0.12214 75189 (+3)
−1.05	0.20807 81441 (+2)	−4	0.25591 19743 (+1)	− 20	0.76293 91946 (+3)
−1.1	0.10821 19596 (+2)	−4.5	0.29447 15937 (+1)	− 30	0.29799 32744 (+5)
−1.15	0.75014 94534 (+1)	−5	0.34298 16260 (+1)	− 40	0.11639 96208 (+7)
−1.2	0.58487 14787 (+1)	−6	0.47583 10818 (+1)	− 50	0.45467 06832 (+8)
−1.25	0.48628 61469 (+1)	−7	0.67159 26333 (+1)		

Note that for large negative values of x, the series $\sum_{k=2}^{\infty}(\ln k)^x/k$ converges very fast. Already for $x \leq -20$, ten-digit accuracy is obtained with 7 terms, for $x \leq -30$ with 3 terms, for $x \leq -40$ with 2 terms, and for $x \leq -50$ with just one term.

We have also computed the zeros, $r_n < 100$, and we found that they behaved in a very regular manner. The highest values could be obtained by using a difference scheme, and we believe that these extrapolated values are correct to about 4 decimals (except, possibly, the last two or three values). Similarly, we have also computed all extrema (u_n, v_n) with abscissas < 50. We have maxima when $u_n v_n > 0$, minima when $u_n v_n < 0$. Using a least squares method, we found approximately (at least for $n \geq 10$)

$$\ln r_n \sim 1.2801 \ln n + 0.7494.$$

TABLE 3

Zeros r_n, $n = 1(1)20$, and extrema (u_n, v_n), $n = 1(1)11$, including also the only one (a minimum) with negative abscissa.

n	r_n	u_n	v_n
1	2.62337 95173	3.48598 38948	0.27324 29390 (−2)
2	5.68468 77241	6.99706 78204	− 0.52729 15647 (−3)
3	9.11483 45138	10.84622 54435	0.27229 35151 (−3)
4	12.86713 39196	14.99167 82643	− 0.28347 43214 (−3)
5	16.90282 16968	19.38703 05535	0.52394 35965 (−3)
6	21.18896 11256	23.99631 82201	− 0.15885 17159 (−2)
7	25.69817 10775	28.79413 77570	0.74639 47075 (−2)
8	30.40797 32610	33.76226 11956	− 0.52038 20137 (−1)
9	35.29985 20762	38.88632 09576	0.52002 58065 (0)
10	40.35837 63576	44.15528 37933	− 0.72407 51927 (+1)
11	45.57051 04169	49.55908 10723	0.13718 85564 (+3)
12	50.92510 260		
13	56.4125	− 2.48809 89869	0.19834 09087 (+1)
14	62.0243		
15	67.7531		
16	73.5922		
17	79.5358		
18	85.5787		
19	91.7162		
20	97.9440		

In a similar way we found for positive u_n

$$\ln u_n \sim 1.202 \ln n + 1.0203.$$

All values r_n, u_n and v_n are displayed in Table 3.

3. Complex Arguments. We have also made extensive calculations for complex arguments. In particular, we first studied the behavior of $\gamma(c+it)$, $c = -\frac{1}{2}$, 0 and $\frac{1}{2}$, and in all these cases we found that the image, at least to start with, was spiraling around the origin for increasing values of t (cf. Figure 2). This led us to explore in considerable detail the curves defined by $\mathrm{Re}(\gamma(z)) = 0$ and $\mathrm{Im}(\gamma(z)) = 0$ (see Figure 3). In this way we could localize 11 zeros $x+iy$, all of them in the strip $-1 < x < 3$, $0 < y < 100$ (see Table 4).

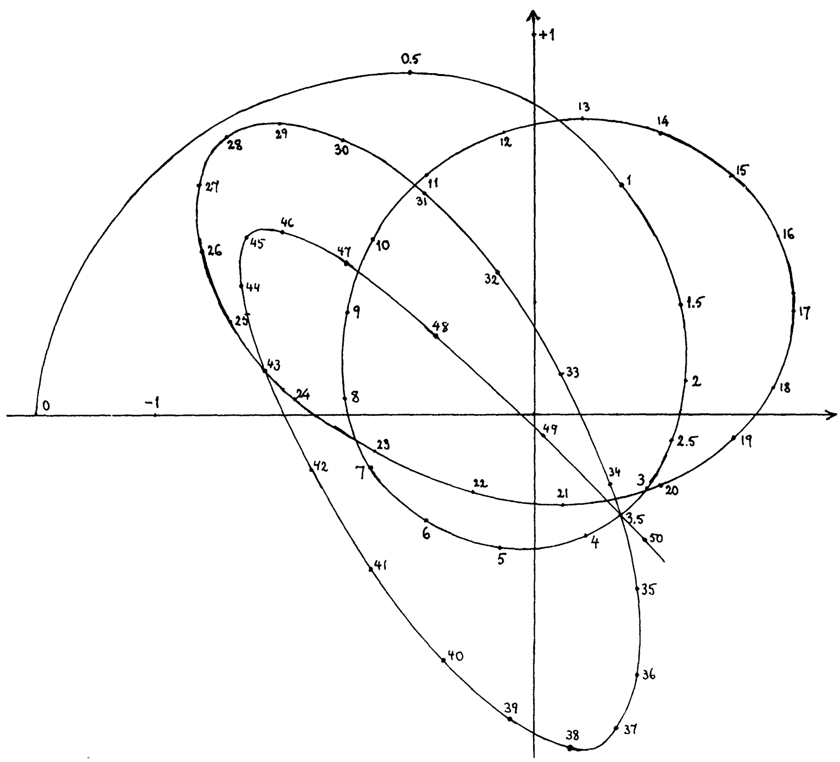

FIGURE 2
$\gamma(-\frac{1}{2}+iy);\ 0 \le y \le 50.$

TABLE 4

Complex zeros $x+iy$.

x	y	x	y
0.5989 0652	21.4580 3709	0.6553 1678	70.2783 3078
−0.1272 3183	33.1970 0002	−0.6703 5953	76.4928 9402
1.3715 3674	41.4157 6303	2.0601 6658	84.1588 1507
−0.5689 2255	48.8826 7161	−0.1121 4635	87.9399 6747
0.9475 3774	58.0415 2000	0.3368 3943	94.5524 9453
0.4724 4064	62.5865 0427		

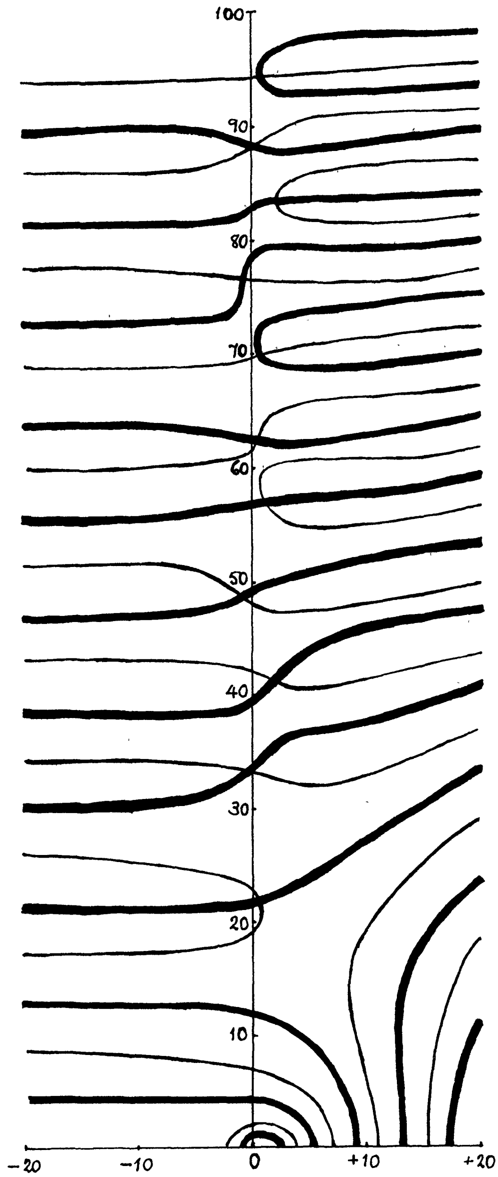

FIGURE 3
$\mathrm{Re}(\gamma(z)) = 0$ (*heavy lines*); $\mathrm{Im}(\gamma(z)) = 0$ (*thin lines*).

Note that $\gamma(-1+r+is) \simeq -(r-is)/(r^2+s^2)+a+b(r+is)$ (cf. (8) and (9)), where r and s are small. Supposing $r \ll s$, we find that the real part is approximately $-r/s^2 + a$, which is zero if we choose $r = as^2$. This explains how we can get a zero for the real part in the pole $z = -1$.

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