MATHEMATICS OF COMPUTATION
VOLUME 51, NUMBER 184
OCTOBER 1988, PAGES 451-476

# An Iterative Finite Element Method for Approximating the Biharmonic Equation

By P. B. Monk*

**Abstract.** A mixed finite element method for the biharmonic model of the simply supported and clamped plate is analyzed and error estimates are obtained. We show that the discrete problem may be solved efficiently by using the conjugate gradient method and a sequence of Dirichlet problems for Poisson's equation.

**1. Introduction.** Let $\Omega$ be a smooth bounded domain in $R^2$. Denote by $\Gamma$ the boundary of $\Omega$, by $\nu$ the unit outward normal to $\Gamma$, by $s$ the unit tangent to $\Gamma$, and by $\kappa$ the curvature of $\Gamma$. Finally, let $\tau$ be a constant with $1/2 < \tau < 1$, and let $f, g_1$ and $g_2$ be given functions. This paper will concern approximating the solution $W$ of the biharmonic equation

$$\Delta^2 W = f \quad \text{in } \Omega \tag{1.1}$$

subject to either *simply supported* boundary conditions

$$\left.\begin{aligned} W &= g_1 \\ \Delta W &= \tau\left(\kappa\frac{\partial W}{\partial \nu} + \frac{\partial^2 W}{\partial s^2}\right) + g_2 \end{aligned}\right\} \quad \text{on } \Gamma, \tag{1.2}$$

or *clamped plate* boundary conditions

$$\left.\begin{aligned} W &= g_1 \\ \frac{\partial W}{\partial \nu} &= g_2 \end{aligned}\right\} \quad \text{on } \Gamma. \tag{1.3}$$

In the remainder of this paper we will refer to (1.1) with boundary conditions (1.2) as the *simply supported plate problem*, and refer to (1.1) with (1.3) as the *clamped plate problem.* These names reflect the fact that these boundary value problems are simple models for a thin plate under different support conditions on the boundary of the plate.

The direct discretization of the biharmonic equation usually involves the construction of finite element subspaces of $H_0^1(\Omega) \cap H^2(\Omega)$ (cf. [4], [9]). However, by adopting the mixed method approach, we can reformulate the biharmonic equation as a system of lower-order equations. In particular, if we introduce the variable $\tilde{v} = -\Delta W$, we may rewrite (1.1) to obtain

$$\left.\begin{aligned} -\Delta W &= \tilde{v} \\ -\Delta \tilde{v} &= f \end{aligned}\right\} \quad \text{in } \Omega. \tag{1.4}$$

Received September 28, 1987; revised March 11, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 65N15, 65N30.
*Key words and phrases.* Iterative method, mixed method, error estimate, biharmonic equation.
*Supported in part by grants from AFOSR and the NSF.

This system, together with the boundary conditions

$$\left.\begin{aligned} W &= g_1 \\ \tilde{v} &= -\tau\left(\kappa\frac{\partial W}{\partial \nu} + \frac{\partial^2 W}{\partial s^2}\right) - g_2 \end{aligned}\right\} \quad \text{on } \Gamma, \tag{1.5}$$

is equivalent to the simply supported plate problem, while (1.4) and (1.3) are equivalent to the clamped plate problem. To discretize (1.4) we need only consider subspaces of $H_0^1(\Omega)$ and $H^1(\Omega)$.

For the clamped plate problem, (1.4) and (1.3) have been used by Ciarlet and Raviart [11] to formulate a mixed finite element method when $\Omega$ is polygonal. In [15], Glowinski and Pironneau suggest a rearrangement of the discrete problem arising from the Ciarlet-Raviart method and solve the problem iteratively by a sequence of discrete Poisson problems. Following Glowinski and Pironneau, we shall further rewrite the plate problems to obtain a formulation suitable for iterative solution.

Let us define solution operators $G$ and $T$ for the Dirichlet problem for Poisson's equation as follows. Given a function $\lambda$ defined on $\Gamma$, define $G\lambda$ to be the function such that

$$\begin{aligned} -\Delta G\lambda &= 0 \quad \text{in } \Omega, \\ G\lambda &= \lambda \quad \text{on } \Gamma, \end{aligned} \tag{1.6}$$

and given $f$ defined on $\Omega$, define $Tf$ to be such that

$$\begin{aligned} -\Delta Tf &= f \quad \text{in } \Omega, \\ Tf &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.7}$$

Now define the pair of functions $(u(\lambda), v(\lambda))$ by

$$v(\lambda) = Tf - G\lambda, \qquad u(\lambda) = Tv + Gg_1. \tag{1.8}$$

Clearly, $(u, v)$ solves Eq. (1.4) (take $W = u$ and $\tilde{v} = v$) together with the boundary condition $u = g_1$ and $v = -\lambda$ on $\Gamma$. Using $(u, v)$ we can reformulate the simply supported plate problem as an equation for $\lambda$ (i.e., for $\Delta W$ on $\Gamma$). We seek the function $\lambda$ such that

$$\lambda = \tau\left(\kappa\frac{\partial u(\lambda)}{\partial \nu} + \frac{\partial^2 u(\lambda)}{\partial s^2}\right) + g_2. \tag{1.9}$$

In the same way, the clamped plate problem becomes the problem of finding $\lambda$ such that

$$\frac{\partial u(\lambda)}{\partial \nu} = g_2. \tag{1.10}$$

The plate problems have now been reduced to problems involving the function $\lambda$ supported on the boundary. Once we have found $\lambda$, the respective boundary value problems are solved. We can find $u$ and $v$ via (1.8) and make the identification $W = u$ and $-\Delta W = v$.

There is one remaining difficulty: $\partial u/\partial \nu$ is difficult to approximate using discrete Dirichlet problems. Instead we obtain variational problems equivalent to (1.9) and (1.10) by multiplying the equations by a smooth function $\phi$, integrating over $\Gamma$, and using Green's formula. The simply supported plate problem is then equivalent to finding $\lambda$ such that

$$\langle \lambda, \phi\rangle + \tau(v(\lambda), G(\kappa\phi)) = \tau(\nabla Gg_1, \nabla G(\kappa\phi)) + \tau\langle g_{1ss}, \phi\rangle + \langle g_2, \phi\rangle \tag{1.11}$$

for every $\phi \in C^\infty(\Gamma)$. Here $\langle \cdot, \cdot \rangle$ represents the $L^2$ inner product on $\Gamma$, and $(\cdot, \cdot)$ represents the $L^2$ inner product on $\Omega$. Later it will prove useful to write (1.11) as an operator equation, so we define the operator $M$ acting on functions on $\Gamma$ by

$$M\lambda = \lambda + \tau\kappa \frac{\partial TG\lambda}{\partial \nu}. \tag{1.12}$$

Again, using $G$ and Green's theorem as in the derivation of (1.11), we find that if $\lambda$ is smooth enough,

$$\langle M\lambda, \phi \rangle = \langle \lambda, \phi \rangle - \tau(G\lambda, G(\kappa\phi)) \quad \forall \phi \in C^\infty(\Gamma), \tag{1.13}$$

and thus we may write (1.11) as $M\lambda = F^{ss}$ where $F^{ss}$ is the function such that

$$\langle F^{ss}, \phi \rangle = \tau(-(Tf, G(\kappa\phi)) + (\nabla G g_1, \nabla G(\kappa\phi)) + \langle g_{1ss}, \phi \rangle) + \langle g_2, \phi \rangle$$

for every $\phi \in C^\infty(\Gamma)$.

In the same way, the clamped plate problem (1.10) is equivalent to finding $\lambda$ such that

$$(v(\lambda), G\phi) = (\nabla G g_1, \nabla G\phi) - \langle g_2, \phi \rangle \tag{1.14}$$

for every $\phi \in C^\infty(\Gamma)$. Again, it will prove useful to cast this as an operator equation. We define the operator $A$ acting on functions on $\Gamma$ by

$$A\lambda = \frac{\partial TG\lambda}{\partial \nu}; \tag{1.15}$$

this operator satisfies

$$\langle A\lambda, \phi \rangle = (G\lambda, G\phi) \quad \forall \phi \in C^\infty(\Gamma). \tag{1.16}$$

Thus (1.14) is equivalent to the equation $A\lambda = F^c$ where $F^c$ satisfies

$$\langle F^c, \phi \rangle = (Tf, G\phi) - (\nabla G g_1, \nabla G\phi) + \langle g_2, \phi \rangle \quad \forall \phi \in C^\infty(\Gamma).$$

At this stage we can easily obtain a finite element discretization of either boundary value problem for the biharmonic equation. Let $\dot{S}_k$ and $S_h^B$ be suitable finite element subspaces on $\Gamma$, let $G_h$ and $T_h$ be discrete operators approximating $G$ and $T$, and let $[g_1]_I \in S_h^B$ be a particular interpolant of $g_1$ on $\Gamma$ (to be detailed in Section 2). Then the finite-dimensional simply supported plate problem is to find $\lambda_k \in \dot{S}_k$ such that

$$\begin{aligned} \langle \lambda_k, \phi_k \rangle + \tau(v_h(\lambda_k), G_h(\kappa\phi_k)) &= \tau(\nabla G_h[g_1]_I, \nabla G_h(\kappa\phi_k)) \\ &\quad + \tau\langle g_{1ss}, \phi_k \rangle + \langle g_2, \phi_k \rangle \quad \forall \phi_k \in \dot{S}_k, \end{aligned} \tag{1.17}$$

where

$$v_h(\lambda_k) = T_h f - G_h\lambda_k, \qquad u_h(\lambda_k) = T_h v_h(\lambda_k) + G_h[g_1]_I. \tag{1.18}$$

Similarly, we can discretize the clamped plate problem by seeking $\lambda_k \in \dot{S}_k$ such that

$$(v_h(\lambda_k), G_h\phi_k) = (\nabla G_h[g_1]_I, \nabla G_h\phi_k) - \langle g_2, \phi_k \rangle \quad \forall \phi_k \in \dot{S}_k. \tag{1.19}$$

Let us discuss the relationship of our method to other methods for approximating the biharmonic problem. Much work has been devoted to using finite element methods to compute an approximation to the displacement $W$ in the clamped plate problem using variational principles based directly on (1.1). A review of the literature on displacement finite element methods, as well as a detailed presentation of the theory, can be found in [9]. As pointed out previously, displacement methods require the construction of subspaces of $H^2(\Omega)$, which results in complex finite element spaces. To avoid this problem, a number of investigators have tried to write (1.1) as a system of lower-order equations by introducing auxiliary variables. The mixed methods that result can then be discretized more easily by methods appropriate for lower-order problems.

Since the literature on the clamped plate problem is more extensive than on the simply supported plate problem, we shall discuss mixed finite element methods for the clamped plate problem first. The Herrmann-Johnson method [16], [17] and Hermann-Miyoshi method [16], [18] both use as auxiliary variables the vector of second partial derivatives of $W$. These methods differ in that they use different variational principles to construct the discrete problem, but both methods produce approximations to the displacement $W$ and the moments $\partial^2 W/\partial x_i x_j$ directly. An alternative method, which we have already mentioned in this introduction, is to use the single auxiliary variable $-\Delta W$. This approach yields a smaller discrete problem than either the Herrmann-Johnson or Herrmann-Miyoshi method. The first analysis of a mixed finite element method based on adding $-\Delta W$ as the auxiliary variable was presented by Ciarlet and Raviart [11] for polygonal regions, and a unified analysis of the Herrmann methods and the Ciarlet-Raviart method was given by Falk and Osborn [14]. For smooth domains, the Ciarlet-Raviart method has been analyzed in [19]. Computational aspects of the Ciarlet-Raviart method are discussed in [10] and [15]. In the latter paper, Glowinski and Pironneau show how to rearrange the discrete problem arising from the Ciarlet-Raviart method and solve the problem by computing an approximation to $-\Delta W$ on $\Gamma$. If an iterative method is used, the biharmonic problem is reduced to solving a sequence of Dirichlet problems for Poisson's equation. However, the conditioning of the problem becomes worse as the mesh is refined, and so Glowinski and Pironneau suggest a preconditioner to speed convergence. Another iterative mixed method for the clamped plate problem using a sequence of Neumann problems for Poisson's equation to approximate $W$ and $-\Delta W$ has been proposed by Falk [13].

Our method for the clamped plate problem, which is not the main focus of our paper, is motivated by [11] and [15], but differs from previous methods mentioned above in that we explicitly discretize $-\Delta W$ on $\Gamma$ using a space of functions on $\Gamma$. The introduction of this space allows us to prove estimates for the approximation of $-\Delta W$ on $\Gamma$ (in applications to fluid flow problems $-\Delta W$ is the vorticity) and to give conditions under which the preconditioner suggested in [15] is effective. We are also able to suggest a new preconditioner that may be more effective if the boundary mesh is nonuniform. Compared to the method of Falk [13], the advantage of our method is that we only approximate one function on $\Gamma$, whereas Falk must use two functions, thus increasing the dimension of the discrete problem.

Mixed methods for the simply supported plate problem, which are the main focus of this paper, have received less attention than methods for the clamped plate problem. On a polygonal domain the problem is simple (since $\kappa = 0$), however on a smooth domain some care is necessary. Babuška [3] has shown that no convergent approximation may be found if the curved boundary is replaced by a polygonal boundary (since again $\kappa = 0$ on the polygon). Thus methods for the simply supported plate problem must deal carefully with a curved boundary. Bramble and Falk [7] have investigated two mixed methods for the simply supported plate problem. Their most general method is based on (1.4) and (1.5), using Neumann problems for Poisson's equation as the underlying problem. As a result, they must use two unknown functions on $\Gamma$ and precondition the iteration in a complex way. Bramble and Falk's second method, which is much simpler than the first, is limited to the case when $\kappa$ is positive. In [19], a method similar to the Ciarlet-Raviart method for the clamped plate problem, but based on Bramble and Falk's second method is analyzed. This method is also restricted to the case of positive $\kappa$.

The main focus of our paper is the simply supported plate problem, and the method we propose is a new method for this problem. Our method for the simply supported plate problem using Dirichlet problems for Poisson's equation is simpler than the general Bramble-Falk method discussed above, since our method involves only one unknown function on $\Gamma$ and no preconditioning is necessary. Furthermore, compared to Bramble and Falk's second method, our method is not restricted to positive $\kappa$.

An outline of the paper is as follows. In the remainder of the introduction we shall define some notation. In Section 2 we will collect some results concerning the finite element spaces used in this paper and define the operators $G_h$ and $T_h$ via Scott's method [22]. Then we will give some approximation properties of these operators. In Section 3 we will investigate the operator $M$ defined by (1.12) and the finite element analogue of this operator. In Section 4 we will derive error estimates for the method for the simply supported plate problem given by (1.17). Section 5 starts our analysis of the convergence properties of the method given by (1.19) for approximating the clamped plate problem. We analyze the operator $A$ defined by (1.15), and then extend these results to a discrete analogue of this operator. In Section 6 we derive error estimates for the clamped plate problem. Our analysis of both the simply supported and clamped plate problems is based on the analysis of Lagrange multiplier methods due to Bramble [6], Bramble and Falk [7], and Falk [13]. Finally, in Section 7 we discuss the numerical implementation of the methods described above and show how both the clamped plate problem and the simply supported plate problem may be solved by the conjugate gradient algorithm.

Now let us define some notation. Let $S$ be a Lipschitz bounded open set in $R^2$ with boundary $\partial R$, and let $T$ be a $C^\infty$ curve in the plane. Then $H^s(S)$ and $H^s(T)$ denote the usual Sobolev spaces of functions on $S$ and $T$, respectively. Let $\|\cdot\|_{s,S}$ denote the norm on $H^s(S)$ and $|\cdot|_{s,T}$ denote the norm on $H^s(T)$. We will also write $L^2(S) \equiv H^0(S)$. If $S = \Omega$ or $T = \Gamma$, we will omit the specification of the domain as a subscript in the norms and inner products. Recall that if $s \geq 0$ and if $s$ is an integer, then

$$\|u\|_{s,S} = \left\{ \sum_{|\alpha| \leq s} \|D^\alpha u\|_{0,S}^2 \right\}^{1/2},$$

where we use the standard notation $\alpha = (i, j)$ for a vector of nonnegative integers, $|\alpha| = i + j$, and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^i \left(\frac{\partial}{\partial x_2}\right)^j.$$

If $s < 0$, then

$$\|u\|_{s,S} = \sup_{\phi \in H^{-s}(S)} \frac{(u, \phi)_S}{\|\phi\|_{-s,S}}.$$

The norm $|\cdot|_{s,T}$ is defined in the same way. We shall also need to consider the spaces $C^m(S)$ and $C^m(T)$ of $m$ times continuously differentiable functions on $S$ and $T$, respectively. We shall denote by $\|\cdot\|_{m,\infty,S}$ and $|\cdot|_{m,\infty,T}$ the norms on $C^m(S)$ and $C^m(T)$, respectively. If $m$ is an integer and $m \geq 0$, then

$$\|u\|_{m,\infty,S} = \sup_{x \in S, |\alpha| \leq m} |D^\alpha u|.$$

Here, $|D^\alpha u|$ has its usual meaning as Euclidean length. Finally, we shall use the standard Sobolev spaces

$$H_0^1(S) = \{u \in H^1(S) | u = 0 \text{ on } \partial S\},$$
$$H_0^2(S) = \{u \in H^2(S) | u = 0 \text{ and } \partial u/\partial \nu = 0 \text{ on } \partial S\}.$$

For a detailed discussion of Sobolev spaces the reader can consult [1].

**2. Finite Element Spaces and the Dirichlet Problem for Poisson's Equation.** We start by describing Scott's method [22] for constructing $S_h$. This begins by dividing $\overline{\Omega}$ into a collection $\tau_h$ of closed subdomains of maximum diameter $h$. The elements of $\tau_h$ are of two types. In the interior of $\Omega$, the elements are triangles, while at the boundary the elements have two straight sides (in $\overline{\Omega}$) and a third possibly curved edge consisting of a segment of $\Gamma$. These latter elements will be referred to as *boundary elements.*

We assume that the triangulation $\tau_h$ satisfies the usual finite element geometric restrictions [9]. In addition, we require the triangulation to be *regular*, by which we mean that the ratio of the radii $r_1$ and $r_2$ of the circumscribed and inscribed circles of each element is bounded. That is, there is a constant $K$ independent of $h$ such that

$$\frac{r_1}{r_2} < K$$

for each element in $\tau_h$ and each $h$. (For a boundary element, the inscribed circle is the largest circle contained in that element and in the triangle formed by joining its boundary vertices by a straight line.) We also assume that $S_h$ satisfies an *inverse assumption*, that is, there is a constant $K$ independent of $h$ such that

$$\frac{h}{r_2} \leq K$$

for each triangle in $\tau_h$, and each $h > 0$. We shall discuss where this assumption is used after we define discrete solution operators for Laplace's equation at the end of this section.

Having defined $\tau_h$, we define $S_h \subset H^1(\Omega)$ to be the set of all continuous piecewise $(r-1)$-degree polynomials on $\tau_h$ (of course, $r > 1$). Since we are interested in approximating the Dirichlet problem for Laplace's equation, we must also define a subspace of $H^1(\Omega)$ that corresponds in a suitable sense to $H_0^1(\Omega)$. We use Scott's definition [22], which proceeds by defining the degrees of freedom of $S_h$. For a triangular element with no edge on $\Gamma$, the degrees of freedom are the standard Lagrange degrees of freedom [9]. For a boundary element, denoted $\tau_i^h$, we use Lagrange degrees consisting of the function values at the vertices of $\tau_i^h$, at $(r-2)$ interpolation points uniformly spaced on each straight edge of $\tau_i^h$ not on $\Gamma$, and at $(r-3)(r-2)/2$ points in the interior of the element chosen so that if a polynomial of degree $r-4$ vanishes at the points, it vanishes identically. Finally, the remaining $(r-2)$ interpolation points are positioned along the edge of $\tau_i^h$ on $\Gamma$ as follows. Choose a local coordinate system for the boundary element as shown in Figure 1.

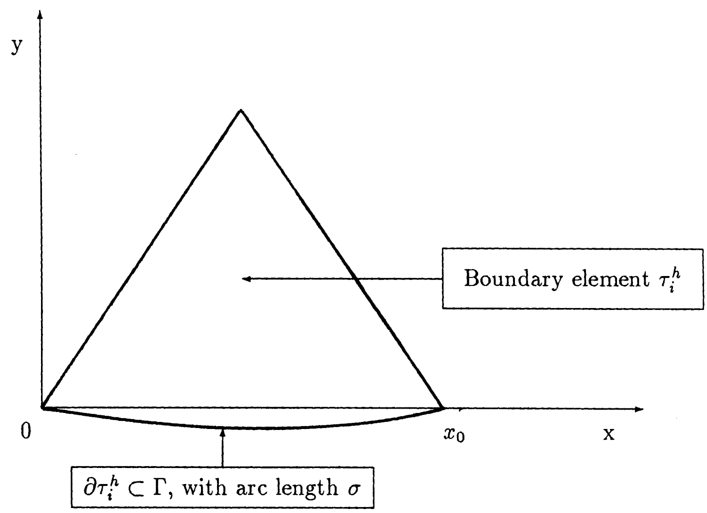


FIGURE 1

If $h$ is small enough, $\partial\tau_i^h \cap \Gamma$ is the graph of a function $\rho$,

$$\partial\tau_i^h \cap \Gamma = \{(x, \rho(x)) : 0 \le x \le x_0\}. \tag{2.1}$$

We place the remaining $(r-2)$ interpolation points at $(\eta_i x_0, \rho(\eta_i x_0))$, $i = 1, \dots, r-2$, where $0 < \eta_1 < \eta_2 < \cdots < \eta_{r-2} < 1$ are the Lobatto quadrature points in $(0,1)$ (cf. [12], [23]).

Using the degrees of freedom defined above, we define the space $S_h^0$ by

$$S_h^0 = \{u_h \in S_h : u_h = 0 \text{ at interpolation points on } \Gamma\}.$$

Notice that in general $S_h^0$ is not a subspace of $H_0^1(\Omega)$. For a continuous function $g$, we define the set $S_h^g$ by

$$S_h^g = \{u_h \in S_h : u_h = g \text{ at interpolation points on } \Gamma\}.$$

We also need a space of functions on $\Gamma$ associated with $S_h$, which we shall define next. The triangulation $\tau_h$ of $\Omega$ induces a mesh $M_h$ on $\Gamma$ where every mesh point of $M_h$ is a triangle vertex of $\tau_h$, and every triangle vertex on $\Gamma$ is a point in $M_h$. Then we define the boundary space $S_h^B \subset H^1(\Gamma)$ to be the space of continuous piecewise $(r-1)$-degree polynomials in arc length on the mesh $M_h$.

Again the degrees of freedom of $S_h^B$ must be chosen with care, and we use the degrees due to Blair [5]. Let $[\sigma_i, \sigma_{i+1}]$ be a mesh interval on $\Gamma$; then any $(r-1)$-degree polynomial on that interval is uniquely specified by the following degrees of freedom:

1. The function value at $\sigma_i$ and $\sigma_{i+1}$.
2. The $r-2$ moments $y_n$, $n = 1, \dots, r-2$, on $[\sigma_i, \sigma_{i+1}]$, where for a function $v$,

$$y_n = \frac{1}{|\sigma_{i+1} - \sigma_i|^{n+1}} \int_{\sigma_i}^{\sigma_{i+1}} v\sigma^n \, d\sigma.$$

We may interpolate with these degrees of freedom, and we shall term this *interpolation in the sense of Blair.* Note that interpolation in the sense of Blair is equivalent to a local $H^1$ norm projection on each subinterval on $\Gamma$, since if $\lambda_I$ interpolates $\lambda$ on $[\sigma_i, \sigma_{i+1}]$, $\lambda_I = \lambda$ at $\sigma_i$ and $\sigma_{i+1}$, and

$$\int_{\sigma_i}^{\sigma_{i+1}} \lambda_I' \mu' \, d\sigma = \int_{\sigma_i}^{\sigma_{i+1}} \lambda' \mu' \, d\sigma \quad \forall \mu \in S_h^B,$$

where prime denotes the derivative with respect to arc length. The following lemma, which can be found in [5], is proved using the above orthogonality property.

LEMMA 2.1. *Let $\lambda \in H^m(\Gamma)$ and let $\lambda_I \in S_h^B$ interpolate $\lambda$ in the sense of Blair. Then for $1 \le m \le r$ and $-r+2 \le s \le 1$,*

$$|\lambda - \lambda_I|_s \le Ch^{m-s} |\lambda|_m.$$

Finally, we need a space of functions on $\Gamma$ in which to compute the unknown function $\lambda_k$. We take $\dot{S}_k \subset H^{r-3}(\Gamma)$ to be the space of piecewise polynomials of degree less than $r-2$ on $\Gamma$ with $r-4$ continuous derivatives. Note in particular, if $r=4$, $\dot{S}_k$ is just a standard space of continuous piecewise linear polynomials on $\Gamma$. We assume that $\dot{S}_k$ is *compatible* with $S_h$, by which we mean that $\dot{S}_k \subset S_h^B$. This implies that the mesh points of $\dot{S}_k$ are contained in $M_h$. We shall assume that the mesh for $\dot{S}_k$ is sufficiently regular so that the following estimates hold:

1. If $\phi \in H^l(\Gamma)$, and $j \le r-3 \le l \le r-2$, there is a constant $C_j$ such that

$$\inf_{\mu \in \dot{S}_k} |\phi - \mu|_j \le C_j k^{l-j} |\phi|_l.$$

2. For $j \le i \le r-3$ there is a constant $C_j$ such that

$$|\phi|_i \le C_j k^{j-i} |\phi|_j \quad \forall \phi \in \dot{S}_k.$$

From the results in [8], [6], we know that the above assumptions imply that there is an operator $\pi_k \colon H^{j_0}(\Gamma) \to \dot{S}_k$ such that, if $j_0 \le j \le r-3$ and $j \le l \le r-2$, there is a constant $C_{j_0}$ with

$$|\phi - \pi_k \phi|_j \le C_{j_0} k^{l-j} |\phi|_l. \tag{2.2}$$

Let us also denote by $P_0$ the $L_2(\Gamma)$ orthogonal projection operator onto $\dot{S}_k$. Thus, for $\phi \in L^2(\Gamma)$, $P_0\phi \in \dot{S}_k$ satisfies

$$\langle P_0\phi, \theta \rangle = \langle \phi, \theta \rangle \quad \forall \theta \in \dot{S}_k.$$

The following results concerning $P_0$ follow from the approximation properties for $\dot{S}_k$ and can be found in [6] and [7].

LEMMA 2.2. 1. *For* $-r+2 \leq j \leq r-3$ *and* $\max(-r+3, j) \leq l \leq r-2$, *there is a constant* $C$ *such that for all* $\phi \in H^l(\Gamma)$

$$|(I-P_0)\phi|_j \leq Ck^{l-j}|\phi|_l.$$

2. *There is a constant* $C$ *such that if* $|s| \leq r-3$, *and* $\phi \in H^s(\Gamma)$, *then*

$$|P_0\phi|_s \leq C|\phi|_s.$$

Having defined the spaces to be used in this paper, we can now define the discrete solution operators for Laplace's equation. Suppose $f \in H^{-1}(\Omega)$ and $g \in C(\Gamma)$, and let $u_h \in S_h^g$ solve

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in S_h^0. \tag{2.3}$$

Then we define $T_h\colon H^{-1} \to S_h^0$ by requiring $T_h f$ to solve (2.3) with $g \equiv 0$, and $G_h\colon C(\Gamma) \to S_h^g$ by requiring $G_h g$ to solve (2.3) with $f \equiv 0$. The main results of this section give the approximation properties of $G_h$ and $T_h$.

THEOREM 2.1. 1. *If* $Tf \in H^m(\Omega)$, *then for* $-r+2 \leq s \leq 1 \leq m \leq r$,

$$\|(T-T_h)f\|_s \leq Ch^{m-s}\|Tf\|_m. \tag{2.4}$$

2. *If* $G\lambda \in H^m(\Omega)$, *then for* $3/2 \leq m \leq r$,

$$\|(G-G_h)\lambda\|_1 \leq Ch^{m-1}\|G\lambda\|_m. \tag{2.5}$$

3. *If* $\lambda \in H^{m-1/2}(\Gamma)$ *and* $\lambda_I \in S_h^B$ *interpolates* $\lambda$ *in the sense of Blair, then for* $-r+5/2 \leq s \leq 1$ *and* $3/2 \leq m \leq r$,

$$\|G\lambda - G_h\lambda_I\|_s \leq Ch^{m-s}\|G\lambda\|_m. \tag{2.6}$$

The above theorem was proved in [19]. The inverse hypothesis was used in this proof to prove the results for $3/2 \leq m \leq 2$, and this is the only place in the present paper where the inverse hypothesis is used. The results for $m = 3/2$ are only needed in the proofs in the following sections when $r = 4$. Thus, if $r > 4$, all the results for the biharmonic problem in this paper are valid without the assumption of an inverse hypothesis on $S_h$. However, in order not to further complicate the statement and proof of the theorems, we will not point this out again.

The final lemma of this section measures the difference between the finite element solution and the boundary data for some special data. The proof can be found in the appendix.

LEMMA 2.3. *Suppose* $S_h, G_h$ *and* $\dot{S}_k$ *are as defined in this section. Further, suppose* $\lambda, \mu \in C^\infty(\Omega)$, $h \leq \tilde{c}k$ *for some constant* $\tilde{c}$, *and* $r \geq 4$. *Then for* $-r+3 \leq m \leq r-3/2$ *and* $0 \leq s \leq r-2$, *the following estimate holds* (*with constant independent of* $\lambda$ *but depending on* $\mu$):

$$|\mu P_0\lambda - G_h(\mu P_0\lambda)|_{-s} \leq C\{h^{m+s+1/2} + h^{s+r-5/2}k^{m-r+3}\}|\lambda|_m.$$

*Remark.* Note that if $\mu = 1$ and $\Omega$ is polygonal, $P_0\lambda = G_h(P_0\lambda)$. This lemma shows that curvature of the boundary has a reasonable effect.

**3. The Simply Supported Plate Problem—Preliminaries.** In this section we shall establish some estimates for the operator $M$ defined in (1.12) and show that these results carry over to a finite-dimensional approximation of $M$. Let us recall the following a priori estimates. If $u$ solves the biharmonic equation (1.1), then

$$\|u\|_{4+s} \le C\left\{\|f\|_s + |u|_{s+7/2} + \left|\Delta u - \tau\left(\kappa\frac{\partial u}{\partial \nu} + \frac{\partial^2 u}{\partial s^2}\right)\right|_{s+3/2}\right\}. \tag{3.1}$$

Alternatively,

$$\|u\|_{4+s} \le C\left\{\|f\|_s + |u|_{s+7/2} + \left|\frac{\partial u}{\partial \nu}\right|_{s+5/2}\right\}. \tag{3.2}$$

These estimates are just a priori estimates for the simply supported plate and clamped plate problem, respectively (cf. [21]). The first theorem of this section shows that the operator $M$ is coercive. From this we can conclude that (1.11) has a unique solution.

THEOREM 3.1. *Let $M$ be defined by* (1.12) *and suppose $\lambda \in H^{-1/2}(\Gamma)$. Then there exist positive constants $C_1$ and $C_2$ independent of $\lambda$ such that for any $s$,*

$$C_1|\lambda|_{-s} \le |M\lambda|_{-s} \le C_2|\lambda|_{-s}. \tag{3.3}$$

To prove this theorem, we first prove a lemma.

LEMMA 3.1. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy $\Delta^2 u = 0$ in $\Omega$; then for any real $s$,*

$$\left|\frac{\partial u}{\partial \nu}\right|_s \le C\|u\|_{s+3/2} \le C|\Delta u|_{s-1}.$$

*Proof of Lemma* 3.1. If $s > 0$, the left-hand inequality is just the trace theorem. For $s \le 0$, let $\phi \in C^\infty(\Gamma)$, and define $\psi = TG\phi$. Then we can write

$$\left\langle \frac{\partial u}{\partial \nu}, \phi\right\rangle = (\Delta u, \Delta\psi) \le \|\Delta u\|_{s-1/2}\|\Delta u\|_{-s+1/2}.$$

The proof is completed using a priori estimates for Poisson's equation. □

*Proof of Theorem* 3.1. Let $u = -TG\lambda$. The right-hand inequality in (3.3) follows from Lemma 3.1. To prove the left-hand inequality, let $\phi \in C^\infty(\Gamma)$. Define $v_1$ to solve the clamped plate problem (1.1) and (1.3) with $g_1 = 0$ and $g_2 = \phi$. Then by estimate (3.2), $\|v_1\|_s \le C|\phi|_{s-3/2}$ for any $s$. Furthermore, using Green's theorem,

$$\langle \lambda, \phi\rangle = \left\langle \Delta u, \frac{\partial v_1}{\partial \nu}\right\rangle = \left\langle \frac{\partial u}{\partial \nu}, \Delta v_1\right\rangle. \tag{3.4}$$

Now define $v_2$ to solve the simply supported plate problem (1.1) and (1.2) with $g_1 = 0$ and $g_2 = \Delta v_1$. Clearly, $v_2 \in H^2(\Omega)$, and hence using (3.4), together with the fact that

$$\left\langle M\lambda, \frac{\partial v_2}{\partial \nu}\right\rangle = \left\langle \frac{\partial u}{\partial \nu}, \Delta v_2 - \tau\kappa\frac{\partial v_2}{\partial n}\right\rangle, \tag{3.5}$$

which is proved in [7], we find that

$$\langle \lambda, \phi\rangle = \left\langle \frac{\partial u}{\partial \nu}, \Delta v_1\right\rangle = \left\langle M\lambda, \frac{\partial v_2}{\partial \nu}\right\rangle.$$

Hence, by Lemma 3.1, (3.1) and the trace theorem,

$$\begin{aligned}\langle \lambda, \phi \rangle &\le |M\lambda|_{-s} \left| \frac{\partial v_2}{\partial \nu} \right|_s \le C|M\lambda|_{-s} \|v_2\|_{s+3/2} \\ &\le C|M\lambda|_{-s}\|v\|_{s+3/2} \le C|M\lambda|_{-s}|\phi|_s.\end{aligned}$$

This completes the proof of the theorem. □

Now we turn to the finite-dimensional problem (1.17). First we introduce a finite-dimensional analogue to $M$. Define the operator $M_k\colon \dot{S}_k \to \dot{S}_k$ such that for $\lambda_k \in \dot{S}_k, M_k\lambda_k \in \dot{S}_k$ is the unique solution of

$$\langle M_k\lambda_k, \phi_k \rangle = \langle \lambda_k, \phi_k \rangle - \tau(G_h\lambda_k, G_h(\kappa\phi_k)) \quad \forall \phi_k \in \dot{S}_k. \tag{3.6}$$

Define the vector $F_k^{ss} \in \dot{S}_k$ to be the unique vector such that

$$\begin{aligned}\langle F_k^{ss}, \phi_k \rangle = \tau\{&-(T_hf, G_h(\kappa\phi_k)) + (\nabla G_h[g_1]_I, \nabla G_h(\kappa\phi_k)) + \langle g_{1s}, \phi_{ks}\rangle\} \\ &+ \langle g_2, \phi_k \rangle \quad \forall \phi_k \in \dot{S}_k.\end{aligned} \tag{3.7}$$

With these definitions, the solution $\lambda_k$ of (1.17) is just the solution of the linear system

$$M_k\lambda_k = F_k^{ss}. \tag{3.8}$$

Our main theorem of this section shows that $M_k$ is nonsingular and hence that (1.17) or (3.8) have a unique solution.

THEOREM 3.2. *Let $r \ge 4$ and let $G_h, T_h, S_h, S_h^0$ and $\dot{S}_k$ be constructed as in Section 2. Suppose, in addition, that $h \le \tilde{c}k$ for some constant $\tilde{c}$. Then there exists a positive constant $k_0$ and positive constants $C_0$ and $C_1$ independent of $h, k$, and $\lambda_k \in \dot{S}_k$ such that*

$$C_0|\lambda_k|_{-s} \le |M_k\lambda_k|_{-s} \le C_1|\lambda_k|_{-s}, \tag{3.9}$$

*for $0 < k \le k_0$ and $0 \le s \le r - 5/2$.*

In order to prove this theorem, we first prove a lemma.

LEMMA 3.2. *Let $\dot{S}_k$ and $P_0$ be defined as in Section 2. Then for $k$ small enough, there exist positive constants $C_0$ and $C_1$ such that for every $\phi_k \in \dot{S}_k$ and $0 \le s \le r - 2$,*

$$C_0|\lambda_k|_{-s} \le |P_0M\lambda_k|_{-s} \le C_1|\lambda_k|_{-s}.$$

*Proof of Lemma* 3.2. By Theorem 3.1, we know that

$$C_0|\lambda_k|_{-s} - |(I - P_0)M\lambda_k|_{-s} \le |P_0M\lambda_k|_{-s} \le C_1|\lambda_k|_{-s} + |(I - P_0)M\lambda_k|_{-s}.$$

Hence, if we can estimate $|(I - P_0)M\lambda_k|_{-s}$, we will be done. Let $u = -TG\lambda_k$. By the definition of $M$ and the estimates for $P_0$ in Lemma 2.2 we obtain

$$\begin{aligned}|(I - P_0)M\lambda_k|_{-s} &= \tau \left| (I - P_0)\kappa \frac{\partial u(\lambda_k)}{\partial \nu} \right|_{-s} \\ &\le Ck^{s+1} \left| \frac{\partial u}{\partial \nu} \right|_1 \le Ck|\lambda_k|_{-s}.\end{aligned}$$

Clearly, if we take $k$ small enough, the lemma is proved. □

*Proof of Theorem* 3.2. From Lemma 3.2 we know that

$$\begin{aligned}C_0|\lambda_k|_{-s} - |P_0M\lambda_k - M_k\lambda_k|_{-s} &\le |M_k\lambda_k|_{-s} \\ &\le C_1|\lambda_k|_{-s} + |P_0M\lambda_k - M_k\lambda_k|_{-s}.\end{aligned} \tag{3.10}$$

To complete the proof, we must analyze $|P_0M\lambda_k - M_k\lambda_k|_{-s}$ for $s \ge 0$ and show that this term may be made small. Let $\phi \in C^\infty(\Gamma)$; then by (1.13) and (3.6),

$$\begin{aligned}&\langle P_0M\lambda_k - M_k\lambda_k, \phi\rangle \\ &\quad = \langle M\lambda_k, P_0\phi\rangle - \langle M_k\lambda_k, P_0\phi\rangle \\ &\quad = \tau[(G_h\lambda_k, G_h(\kappa P_0\phi)) - (G\lambda_k, G(\kappa P_0\phi))] \\ &\quad = \tau[((G_h - G)\lambda_k, G(\kappa\phi)) + ((G_h - G)\lambda_k, G(\kappa(P_0 - I)\phi)) \\ &\qquad + ((G_h - G)\lambda_k, (G_h - G)(\kappa P_0\phi)) + (G\lambda_k, (G_h - G)(\kappa P_0\phi))].\end{aligned} \tag{3.11}$$

We may estimate each term in (3.11) separately. Using (2.6) and the inverse assumption on $\dot{S}_k$, we can show that

$$\begin{aligned}((G_h - G)\lambda_k, G(\kappa\phi)) &\le C\|(G_h - G)\lambda_k\|_{-s-1/2}\|G(\kappa\phi)\|_{s+1/2} \\ &\le Ch^{\hat{s}+3/2}k^{-s-1}|\lambda_k|_{-s}|\phi|_s,\end{aligned} \tag{3.12}$$

where $\hat{s} = \min(s + 1/2, r - 5/2)$. In the same way, using in addition Lemma (2.2),

$$((G_h - G)\lambda_k, G(\kappa(P_0 - I)\phi)) \le Ch^{3/2}k^{-1/2}|\lambda_k|_{-s}|\phi|_{-s}. \tag{3.13}$$

The remaining terms in (3.11) must be estimated separately for different $s$. The techniques are similar to those used above. First we do the case when $0 \le s \le 1$:

$$((G_h - G)\lambda_k, (G_h - G)(\kappa P_0\phi)) \le Ch^3k^{-2}|\lambda_k|_{-s}|\phi|_s, \tag{3.14}$$

$$(G\lambda_k, (G_h - G)(\kappa P_0\phi)) \le Ch^2k^{-1}|\lambda_k|_{-s}|\phi|_s. \tag{3.15}$$

Next we consider the case when $1 \le s \le r - 5/2$, using arguments similar to those used above, and in addition (2.5):

$$\begin{aligned}&((G_h - G)\lambda_k, (G_h - G)(\kappa P_0\phi)) \\ &\quad = ((G_h - G)\lambda_k, (G_h - G)(\kappa(P_0 - I)\phi)) + ((G_h - G)\lambda_k, (G_h - G)(\kappa\phi)) \\ &\quad \le C[h^3k^{-2} + h^{s+2}k^{-s-1}]|\lambda_k|_{-s}|\phi|_s,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&(G\lambda_k, (G_h - G)(\kappa P_0\phi)) \\ &\quad = (G\lambda_k, (G_h - G)(\kappa(P_0 - I)\phi)) + (G\lambda_k, (G_h - G)(\kappa\phi)) \\ &\quad \le C[h^{3/2}k^{-1/2} + h^sk^{-s+1}]|\lambda_k|_{-s}|\phi|_{-s}.\end{aligned} \tag{3.17}$$

Now if we combine (3.12) through (3.17) and use the definition of the negative norm, we obtain the estimate

$$\begin{aligned}&|P_0M\lambda_k - M_k\lambda_k|_{-s} \\ &\quad \le C[h^{\hat{s}+3/2}k^{-s-1} + h^{3/2}k^{-1/2} + h^3k^{-2} \\ &\qquad + h^{s+2}k^{-s-1} + h^2k^{-1} + h^sk^{s+1}]|\lambda_k|_{-s}.\end{aligned} \tag{3.18}$$

Next we use the assumptions that $h \le \tilde{c}k$ and $0 \le s \le r - 5/2$ to show that

$$|P_0M\lambda_k - M_k\lambda_k|_{-s} \le Ck^{1/2}|\lambda_k|_{-s}.$$

The right-hand side can be made arbitrarily small, so combining this estimate with (3.10) proves the theorem. □

**4. Estimates for the Simply Supported Plate Problem.** In this section we shall derive error estimates for the simply supported plate problem. We shall assume that

$$f \in H^{r-4}(\Omega), \quad g_1 \in H^{r-1/2}(\Gamma), \quad g_2 \in H^{r-5/2}(\Gamma). \tag{4.1}$$

This implies $W \in H^r(\Omega)$, which is exactly the smoothness required by the interior finite element methods (i.e., for $G_h$ and $T_h$). The first theorem is the fundamental result, and subsequent estimates are derived from that theorem.

THEOREM 4.1. *Suppose $\lambda$ is the solution of problem* (1.11), *and $\lambda_k$ solves problem* (1.17). *Suppose that $T_h, G_h, S_h, S_h^0$ and $\dot{S}_k$ are constructed as detailed in Section* 2, *with $r \geq 4$ and $h \leq \tilde{c}k$ for some constant $\tilde{c}$. The following estimate holds for $-r+3 \leq s \leq r-5/2$:*

$$|\lambda-\lambda_k|_{-s} \leq C\{k^{r-5/2+s}+h^{r-2+\hat{s}}+h^{2r-9/2}k^{s-r+3}\}(\|f\|_{r-4}+|g_1|_{r-1/2}+|g_2|_{r-5/2}),$$

*where $\hat{s} = \min(s+1/2, r-5/2)$.*

*Remark.* From the smoothness assumptions (4.1), $\lambda \in H^{r-5/2}(\Gamma)$, and so the power of $k$ in the first term in the above estimate is correct for the given smoothness.

COROLLARY 4.2. *Suppose all the hypotheses of Theorem* 4.1 *hold*; *then for $-r+3 \leq s \leq r-3$,*

$$|\lambda - \lambda_k|_{-s} \leq C\{k^{r-5/2+s} + h^{r-3/2+s}\}(\|f\|_{r-4} + |g_1|_{r-1/2} + |g_2|_{r-5/2}).$$

THEOREM 4.3. *Suppose all the hypotheses of Theorem* 4.1 *hold, and in addition let $W$ satisfy the biharmonic equation* (1.1) *with simply supported boundary conditions* (1.2). *Let $v_h(\lambda_k)$ be defined by* (1.18); *then for $-r+3 \leq j \leq 1$,*

$$\|-\Delta W - v_h\|_j \leq C\{k^{r-2-j} + h^{r-2-j}\}(\|f\|_{r-4} + |g_1|_{r-1/2} + |g_2|_{r-5/2}).$$

THEOREM 4.4. *Suppose all the hypotheses of Theorem* 4.1 *hold, and in addition let $W$ satisfy the biharmonic equation* (1.1) *with simply supported boundary conditions* (1.2). *Let $u_h(\lambda_k)$ be defined by* (1.18); *then for $-r+5 \leq j \leq 1$,*

$$\|W - u_h\|_j \leq C\{k^{r-j} + h^{r-j}\}(\|f\|_{r-4} + |g_1|_{r-1/2} + |g_2|_{r-5/2}).$$

*Remark.* The theorems suggest that a good choice for the mesh for $\dot{S}_k$ would be the mesh $M_h$ induced by $S_h$ on $\Gamma$. In this case, there is a constant $C_1$ such that $C_1 k \leq h \leq \tilde{c}k$, and our estimate for $W$ and $\Delta W$ are of optimal order in $h$. Here an inverse assumption on the interior mesh seems natural.

To prove these theorems, and the corresponding theorems for the clamped plate problem, we will prove three lemmas.

LEMMA 4.1. *Suppose all the hypotheses of Theorem* 4.1 *hold. In addition, let $\mu \in C^\infty(\Gamma)$ be a fixed function and let $[g_1]_I \in S_h^B$ interpolate $g_1$ in the sense of Blair. Then the following estimate holds for $0 \leq s \leq r-5/2$, and for every $\phi \in C^\infty(\Gamma)$* (*with constant independent of $\phi$ but dependent on $\mu$*):

$$\begin{aligned}
&|(Tf, G(\mu P_0\phi)) - (T_h f, G_h(\mu P_0\phi))| \\
&\qquad + |(\nabla G g_1, \nabla G(\mu P_0\phi)) - (\nabla G_h[g_1]_I, \nabla G_h(\mu P_0\phi))| \\
&\qquad \leq C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}(\|f\|_{r-4} + |g_1|_{r-1/2})|\phi|_s.
\end{aligned}$$

*Proof of Lemma* 4.1. First we expand the two parts of the expression to be estimated:

$$\begin{aligned}(4.2)\qquad &(Tf, G(\mu P_0\phi)) - (T_h f, G_h(\mu P_0\phi))\\ &\quad = ((T-T_h)f, G(\mu P_0\phi)) + ((T_h - T)f, (G-G_h)(\mu P_0\phi))\\ &\qquad + (Tf, (G-G_h)(\mu P_0\phi)),\end{aligned}$$

$$\begin{aligned}(4.3)\qquad &(\nabla G g_1, \nabla G(\mu P_0\phi)) - (\nabla G_h[g_1]_I, \nabla G_h(\mu P_0\phi))\\ &\quad = (\nabla G g_1, \nabla(G-G_h)(\mu P_0\phi)) + (\nabla(G g_1 - G_h[g_1]_I), \nabla G(\mu P_0\phi))\\ &\qquad + (\nabla(G_h[g_1]_I - G g_1), \nabla(G-G_h)(\mu P_0\phi)).\end{aligned}$$

Next we estimate the first two terms on the right-hand side of (4.2) and the last two terms in (4.3). We use the estimates for $G_h$ and $T_h$ in (2.4)–(2.6) and the inverse properties of $P_0$ in Lemma (2.2), and consider two cases. The first is $0 \le s \le r-3$:

$$\begin{aligned}(4.4)\qquad &((T-T_h)f, G(\mu P_0\phi)) + ((T_h - T)f, (G-G_h)(\mu P_0\phi))\\ &\quad \le \|(T-T_h)f\|_{-r+5/2}\|G(\mu P_0\phi)\|_{r-5/2}\\ &\qquad + \|(T_h-T)f\|_{-1}\|(G-G_h)(\mu P_0\phi)\|_1\\ &\quad \le Ch^{2r-9/2}k^{s-r+3}\|f\|_{r-4}|\phi|_s,\end{aligned}$$

$$\begin{aligned}(4.5)\qquad &(\nabla(G g_1 - G_h[g_1]_I), \nabla G(\mu P_0\phi)) + (\nabla(G_h[g_1]_I - G g_1), \nabla(G-G_h)(\mu P_0\phi))\\ &\quad \le \|G g_1 - G_h[g_1]_I\|_{-r+9/2}\|G(\mu P_0\phi)\|_{r-5/2}\\ &\qquad + \|G_h[g_1]_I - G g_1\|_1\|(G-G_h)(\mu P_0\phi)\|_1\\ &\quad \le Ch^{2r-9/2}k^{s-r+3}|g_1|_{r-1/2}|\phi|_s.\end{aligned}$$

Next we consider the case when $r-3 \le s \le r-5/2$. In this case, we expand the terms still further by writing $P_0\phi = (P_0 - I)\phi + \phi$ and estimate terms in the same way as above, but now also using the accuracy properties of $P_0$ from Lemma 2.2:

$$\begin{aligned}(4.6)\qquad &((T-T_h)f, G(\mu P_0\phi)) + ((T_h - T)f, (G-G_h)(\mu P_0\phi))\\ &\quad = ((T-T_h)f, G(\mu(P_0 - I)\phi)) + ((T-T_h)f, G(\mu\phi))\\ &\qquad + ((T_h-T)f, (G-G_h)(\mu(P_0-I)\phi)) + ((T_h-T)f, (G-G_h)(\mu\phi))\\ &\quad \le C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}\|f\|_{r-4}|\phi|_s,\end{aligned}$$

$$\begin{aligned}(4.7)\qquad &(\nabla(G g_1 - G_h[g_1]_I), \nabla G(\mu P_0\phi)) + (\nabla(G_h[g_1]_I - G g_1), \nabla(G-G_h)(\mu P_0\phi))\\ &\quad = (\nabla(G g_1 - G_h[g_1]_I), \nabla G(\mu(P_0 - I)\phi))\\ &\qquad + (\nabla(G g_1 - G_h[g_1]_I), \nabla G(\mu\phi))\\ &\qquad + (\nabla(G_h[g_1]_I - G g_1), \nabla(G-G_h)(\mu(P_0-I)\phi))\\ &\qquad + (\nabla(G_h[g_1]_I - G g_1), \nabla(G-G_h)(\mu\phi))\\ &\quad \le C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}|g_1|_{r-1/2}|\phi|_s.\end{aligned}$$

The remaining terms in (4.2) and (4.3) must be estimated more carefully. Using Green's theorem and the properties of the operator $T$, we obtain

$$\begin{aligned}(4.8)\qquad (Tf, (G-G_h)(\mu P_0\phi)) &= (\nabla T^2 f, \nabla(G-G_h)(\mu P_0\phi))\\ &\quad - \left\langle \frac{\partial T^2 f}{\partial\nu}, (G-G_h)(\mu P_0\phi)\right\rangle,\end{aligned}$$

$$(4.9)\qquad (\nabla G g_1, \nabla(G-G_h)(\mu P_0\phi)) = \left\langle \frac{\partial G g_1}{\partial\nu}, (G-G_h)(\mu P_0\phi)\right\rangle.$$

Now we can estimate the final term in (4.8) and (4.9), using Lemma 2.3:

$$\left\langle \frac{\partial T^2 f}{\partial \nu}, (G - G_h)(\mu P_0 \phi) \right\rangle \le C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}\|f\|_{r-4}|\phi|_s, \tag{4.10}$$

$$\left\langle \frac{\partial G g_1}{\partial \nu}, (G - G_h)(\mu P_0 \phi) \right\rangle \le C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}|g_1|_{r-1/2}|\phi|_s. \tag{4.11}$$

Finally, we must estimate the first term on the right-hand side of (4.8). Now we use the properties of $G$ and $G_h$ and expand the resulting term:

$$\begin{aligned}(\nabla T^2 f, \nabla(G - G_h)(\mu P_0\phi)) &= -(\nabla T^2 f, \nabla G_h(\mu P_0 \phi)) \\ &= (\nabla(T - T_h)Tf, \nabla(G - G_h)(\mu P_0\phi)) - (\nabla(T - T_h)Tf, \nabla G(\mu P_0\phi)).\end{aligned} \tag{4.12}$$

If $0 \le s \le r-3$, we use techniques similar to those used previously in this lemma and obtain

$$(\nabla T^2 f, \nabla(G - G_h)(\mu P_0\phi)) \le Ch^{2r-9/2}k^{s-r+3}\|f\|_{r-4}|\phi|_s. \tag{4.13}$$

If $r-3 \le s \le r-5/2$, we expand (4.12) still further by writing $P_0\phi = (P_0 - I)\phi + \phi$ and use the estimates for $P_0$ in Lemma 2.2 to obtain

$$(\nabla T^2 f, \nabla(G - G_h)(\mu P_0\phi)) \le C\{h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3}\}\|f\|_{r-4}|\phi|_s. \tag{4.14}$$

Combining (4.2), (4.4), (4.6), (4.8), (4.10), (4.12), (4.13), and (4.14) proves the first part of the desired estimate. The second is proved by combining (4.3), (4.5), (4.7), (4.9), and (4.11).□

LEMMA 4.2. *Suppose all the hypotheses of Theorem* 4.1 *hold. In addition, let* $\mu \in C^\infty(\Gamma)$ *be a fixed function, let* $\pi_k$ *be the approximation operator obeying* (2.2), *and assume* $h \le \tilde{c}k$ *for some constant* $\tilde{c}$. *Then the following estimates hold for* $0 \le s \le r-3$, *for* $r - 5/2 \le l \le r-2$, *and for every* $\phi \in C^\infty(\Gamma)$ (*with constant independent of* $\phi$ *but dependent on* $\mu$):

$$(G_h(\pi_k\lambda), G_h(\mu P_0\phi)) - (G\lambda, G(\mu P_0\phi)) \le C\{k^{l+1+s} + h^{r-3/2+s}\}|\lambda|_l|\phi|_s.$$

*Proof of Lemma* 4.2. First we use the operator $T$, Green's Theorem, and the definition of $G$ to expand the term to be estimated:

$$\begin{aligned}&(G_h(\pi_k\lambda), G_h(\mu P_0\phi)) - (G\lambda, G(\mu P_0\phi)) \\ &\quad= (\nabla T G_h(\pi_k\lambda), \nabla G_h(\mu P_0\phi)) + \left\langle \frac{\partial T G\lambda}{\partial \nu}, G(\mu P_0\phi)\right\rangle \\ &\qquad - \left\langle \frac{\partial T G_h(\pi_k\lambda)}{\partial \nu}, G_h(\mu P_0\phi)\right\rangle.\end{aligned} \tag{4.15}$$

Next we estimate the interior term in (4.15). We expand the term and then use estimates for $G_h$ and $T_h$ in (2.4)–(2.6) together with the inverse estimate for $P_0$ in

Lemma 2.2:

$$
\begin{aligned}
(\nabla T G_h(\pi_k\lambda), \nabla G_h(\mu P_0\phi)) &= (\nabla(T-T_h)G_h(\pi_k\lambda), \nabla G_h(\mu P_0\phi)) \\
&= (\nabla(T-T_h)(G_h-G)(\pi_k\lambda-\lambda), \nabla(G_h-G)(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)(G_h-G)\lambda, \nabla(G_h-G)(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)G(\pi_k\lambda-\lambda), \nabla(G_h-G)(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)G\lambda, \nabla(G_h-G)(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)(G_h-G)(\pi_k\lambda-\lambda), \nabla G(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)(G_h-G)\lambda, \nabla G(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)G(\pi_k\lambda-\lambda), \nabla G(\mu P_0\phi)) \\
&\quad + (\nabla(T-T_h)G\lambda, \nabla G(\mu P_0\phi)) \\
&\le C\{k^{l+1+s}|\lambda|_l + h^{2r-9/2}k^{s-r+3}|\lambda|_{r-5/2}\}|\phi|_s.
\end{aligned} \tag{4.16}
$$

Now we estimate the boundary terms in (4.15). We start by expanding the term and using Lemma 2.3:

$$
\begin{aligned}
&\left\langle \frac{\partial TG\lambda}{\partial\nu}, G(\mu P_0\phi)\right\rangle - \left\langle \frac{\partial TG_h(\pi_k\lambda)}{\partial\nu}, G_h(\mu P_0\phi)\right\rangle \\
&\quad = \left\langle \frac{\partial TG\lambda}{\partial\nu}, \mu P_0\phi - G_h(\mu P_0\phi)\right\rangle \\
&\quad\quad - \left\langle \frac{\partial}{\partial\nu}TG(\lambda-\pi_k\lambda), \mu P_0\phi - G_h(\mu P_0\phi)\right\rangle \\
&\quad\quad + \left\langle \frac{\partial}{\partial\nu}TG(\lambda-\pi_k\lambda), \mu P_0\phi - \mu\phi\right\rangle + \left\langle \frac{\partial}{\partial\nu}TG(\lambda-\pi_k\lambda), \mu\phi\right\rangle \\
&\quad\quad + \left\langle \frac{\partial}{\partial\nu}T(G_h-G)(\lambda-\pi_k\lambda), \mu P_0\phi - G_h(\mu P_0\phi)\right\rangle \\
&\quad\quad + \left\langle \frac{\partial}{\partial\nu}T(G_h-G)\lambda, \mu P_0\phi - G_h(\mu P_0\phi)\right\rangle \\
&\quad\quad + \left\langle \frac{\partial}{\partial\nu}T(G-G_h)\pi_k\lambda, \mu P_0\phi - G_h(\mu P_0\phi)\right\rangle \\
&\quad \le C\{k^{l+1+s}|\lambda|_l + (h^{r-3/2+s} + h^{2r-9/2}k^{s-r+3})|\lambda|_{r-5/2}\}|\phi|_s \\
&\quad\quad + \left|\left\langle \frac{\partial}{\partial\nu}T(G-G_h)\pi_k\lambda, \mu P_0\phi\right\rangle\right|.
\end{aligned} \tag{4.17}
$$

It remains to estimate the last term in (4.17). Let $\lambda_I \in S_h^B$ interpolate $\lambda$ in the sense of Blair. Then using (2.6), we obtain

$$
\begin{aligned}
&\left\langle \frac{\partial}{\partial\nu}T(G-G_h)\pi_k\lambda, \mu P_0\phi\right\rangle \\
&\quad \le \left\langle \frac{\partial}{\partial\nu}T(G-G_h)(\pi_k\lambda-\lambda_I), \mu P_0\phi\right\rangle \\
&\quad\quad + \left\langle \frac{\partial}{\partial\nu}TG(\lambda_I-\lambda), \mu P_0\phi\right\rangle + \left\langle \frac{\partial}{\partial\nu}T(G\lambda-G_h\lambda_I), \mu P_0\phi\right\rangle \\
&\quad \le C\{k^{l+1+s}|\lambda|_l + h^{s+r-3/2}|\lambda|_{r-5/2}\}|\phi|_s.
\end{aligned} \tag{4.18}
$$

Combining (4.15) through (4.18) proves the lemma. □

LEMMA 4.3. *Suppose $T_h$ and $G_h$ are constructed as detailed in Section 2. Let $(u,v)$ be defined by (1.8) and $(u_h,v_h)$ be defined by (1.18). Then the following estimates hold for $-r+5/2 \le j \le 1$ and $r-5/2 \le l \le r-2$:*

$$\|v-v_h\|_j \le C\Big\{h^{l+1/2-j}(\|f\|_{l-3/2}+|\lambda|_l) + h^{3/2-j}k^{l-1}|\lambda|_l + h^{3/2-j}|\lambda-\lambda_k|_1 + |\lambda-\lambda_k|_{j-1/2}\Big\},$$

$$\|u-u_h\|_j \le C\Big\{h^{r-j}(\|f\|_{r-4}+|g_1|_{r-1/2}+|\lambda|_{r-5/2}) + h^{2-j}\|v-v_h\|_0 + \|v-v_h\|_{j-2}\Big\}.$$

*Proof of Lemma* 4.3. These results follow in a straightforward way from the definitions of $(u,v)$ and $(u_h,v_h)$ by using the estimates (2.4)–(2.6). We will prove only the first estimate. Let $\lambda_I \in S_h^B$ interpolate $\lambda$ in the sense of Blair; then from the definitions of $v$ and $v_h$,

$$\begin{aligned}\|v-v_h\|_j &\le \|(T-T_h)f\|_j + \|G\lambda - G_h\lambda_k\|_j\\ &\le \|(T-T_h)f\|_j + \|G\lambda - G_h\lambda_I\|_j\\ &\quad + \|(G-G_h)(\lambda_I-\lambda_k)\|_j + \|G(\lambda_I-\lambda_k)\|_j\\ &\le C\{h^{l+1/2-j}(\|Tf\|_{l+1/2}+\|G\lambda\|_{l+1/2})\\ &\qquad + h^{3/2-j}|\lambda_I-\lambda_k|_1 + |\lambda_I-\lambda_k|_{j-1/2}\}.\end{aligned}$$

Application of the approximation properties of the Blair interpolant from Lemma 2.1 and the a priori estimates for $T$ and $G$ completes the proof. The second estimate is proved in the same way. □

*Proof of Theorem* 4.1. Let $\pi_k$ be the operator obeying estimate (2.2); then, using Theorem 3.2,

$$\begin{aligned}|\lambda-\lambda_k|_{-s} &\le |\lambda-\pi_k\lambda|_{-s} + |\pi_k\lambda-\lambda_k|_{-s}\\ &\le Ck^{r-5/2+s}|\lambda|_{r-5/2} + C|M_k(\pi_k\lambda-\lambda_k)|_{-s}\\ &\le Ck^{r-5/2+s}|\lambda|_{r-5/2}\\ &\quad + C(|M_k(\pi_k\lambda)-P_0M\lambda|_{-s} + |P_0M\lambda - M_k\lambda_k|_{-s}).\end{aligned}\tag{4.19}$$

We now estimate the last two terms in (4.19). Let $\phi \in C^\infty(\Gamma)$; using (1.12) and (3.6) and the equations satisfied by $\lambda$ (1.11) and $\lambda_k$ (1.17), we find that the last term in (4.19) can be estimated as follows:

$$\begin{aligned}\langle P_0M\lambda - M_k\lambda_k, \phi\rangle &= \langle M\lambda, P_0\phi\rangle - \langle M_k\lambda_k, P_0\phi\rangle\\ &= \tau\{-(Tf, G(\kappa P_0\phi)) + (\nabla G g_1, \nabla G(\kappa P_0\phi))\\ &\quad + (T_hf, G_h(\kappa P_0\phi)) - (\nabla G_h[g_1]_I, \nabla G_h(\kappa P_0\phi))\}.\end{aligned}\tag{4.20}$$

This is estimated using Lemma 4.1 with $\mu=\kappa$. To estimate the remaining term in (4.19), we use the properties of $M$ (1.13) and $M_k$ (3.6) to write

$$\begin{aligned}\langle M_k(\pi_k\lambda) - P_0M\lambda, \phi\rangle &= \langle M_k(\pi_k\lambda), P_0\phi\rangle - \langle M\lambda, P_0\phi\rangle\\ &= \langle \pi_k\lambda-\lambda, P_0\phi\rangle + \tau\{(G\lambda, G(\kappa P_0\phi)) - (G_h(\pi_k\lambda), G_h(\kappa P_0\phi))\}.\end{aligned}\tag{4.21}$$

This can be estimated by Lemma 4.2 with $\mu=\kappa$ and $l=r-5/2$. The combination of (4.21), (4.20), and (4.19) proves the theorem for $s\ge 0$. For $s<0$ the result follows from the estimate for $s=0$ using the inverse property of $\dot S_k$. □

*Proof of Theorems* 4.3 *and* 4.4. By Lemma 4.3, we may estimate $v - v_h$ and $u - u_h$ in terms of estimates for $\lambda - \lambda_k$, and we use Theorem 4.1 to estimate the terms in $\lambda - \lambda_k$. □

**5. The Clamped Plate Problem—Preliminaries.** In this section we shall analyze the finite-dimensional clamped plate problem (1.19) in a way similar to the analysis of the simply supported plate problem in Section 3. First, we shall derive a priori estimates for the operator $A$ defined by (1.15); then we derive similar results for a finite-dimensional approximation to $A$.

THEOREM 5.1. *Suppose $\lambda \in H^{-1/2}(\Gamma)$. Then there exist positive constants $C_0$ and $C_1$ independent of $u$ such that for all $s$,*

$$C_0|\lambda|_{-s-1} \le |A\lambda|_{-s} \le C_1|\lambda|_{-s-1}.$$

*Proof of Theorem* 5.1. The right-hand side follows from Lemma 3.1. To prove the left-hand inequality, let $u = -TG\lambda$, take $\phi \in C^\infty(\Gamma)$, and define $v$ to be the solution of the clamped plate problem (1.1) and (1.3) with $g_1 = 0$ and $g_2 = \phi$. Then, using (3.4), (3.5) and the a priori estimate (3.2), we can show that

$$\langle \lambda, \phi\rangle = \left\langle \frac{\partial u}{\partial \nu}, \Delta v\right\rangle \le C\left|\frac{\partial u}{\partial \nu}\right|_{-s} |\phi|_{s+1} = C|A\lambda|_{-s}|\phi|_{s+1}.$$

This, together with the definition of the negative norm, completes the proof. □

Now let us define the operator $A_k\colon \dot{S}_k \to \dot{S}_k$. Given $\lambda_k \in \dot{S}_k, A_k\lambda_k$ satisfies

$$\langle A_k\lambda_k, \phi_k\rangle = (G_h\lambda_k, G_h\phi_k) \quad \forall \phi_k \in \dot{S}_k. \tag{5.1}$$

Also define the finite-dimensional data $F_k^c \in \dot{S}_k$ to satisfy

$$\langle F_k^c, \phi_k\rangle = (T_h f, G_h\phi_k) - (\nabla G_h[g_1]_I, \nabla G_h\phi_k) + \langle g_2, \phi_k\rangle \quad \forall \phi_k \in \dot{S}_k. \tag{5.2}$$

With these definitions, the solution $\lambda_k$ of the finite-dimensional clamped plate problem (1.19) is just the solution of the linear system

$$A_k\lambda_k = F_k^c. \tag{5.3}$$

Note that $A_k$ is related to the operator $A_h$ appearing in [15]. The main theorem of this section states that under certain conditions on $S_h$ and $\dot{S}_k, A_k$ is positive definite. Hence the finite-dimensional problem (5.3) has a unique solution.

THEOREM 5.2. *Suppose $r \ge 4$ and $G_h, T_h, S_h, S_h^0$ and $\dot{S}_k$ are constructed as in Section 2. Then, if $h \le \tilde{\varepsilon}k$ for some positive $\tilde{\varepsilon}$ small enough, there exist positive constants $C_0$ and $C_1$ independent of $h, k$ and $\lambda_k$ such that for $0 \le s \le r-3$,*

$$C_0|\lambda_k|_{-s-1} \le |A_k\lambda_k|_{-s} \le C_1|\lambda_k|_{-s-1} \quad \forall \lambda_k \in \dot{S}_k.$$

In order to prove this theorem, we shall use the following lemma from [7].

LEMMA 5.1. *For $u \in H^2(\Omega)$, define $E(u,u) \in R$ by*

$$E(u,u) = (\Delta u, \Delta u) - (u_{x_1,x_1}, u_{x_2,x_2}) + (u_{x_1x_2}, u_{x_1x_2}).$$

1. *For every $u \in H^2(\Omega)$,*

$$\sum_{|\alpha|=2} \|D^\alpha u\|_0^2 \le 2E(u,u).$$

2. *If $u$ satisfies $\Delta^2 u = 0$ in $\Omega$ and $u = 0$ on $\Gamma$, then*

$$E(u,u) = \left\langle \Delta u - \frac{1}{2}\kappa \frac{\partial u}{\partial \nu}, \frac{\partial u}{\partial \nu} \right\rangle .$$

We also need a lemma analogous to Lemma 3.2.

LEMMA 5.2. *Let $\dot{S}_k$ and $P_0$ be as defined in Section 2; then there exist positive constants $C_0$ and $C_1$ independent of $k$ and $\lambda_k$ such that for $0 \le s \le r-2$,*

$$C_0|\lambda_k|_{-s-1} \le |P_0 A\lambda_k|_{-s} \le C_1|\lambda_k|_{-s-1}.$$

*Proof of Lemma* 5.2. Let $u = -TG\lambda_k$. By Theorem 5.1,

$$C_0|\lambda_k|_{-s-1} - |(I-P_0)A\lambda_k|_{-s} \le |P_0 A\lambda_k|_{-s} \le C_1|\lambda_k|_{-s-1} + |(I-P_0)A\lambda_k|_{-s}. \tag{5.4}$$

It remains to estimate $|(I-P_0)A\lambda_k|_{-s}$. Using Lemma 2.2 and (3.2),

$$\begin{aligned} |(I-P_0)A\lambda_k|_{-s} &\le Ck^{s+1/2}\|u\|_2 \\ &= Ck^{s+1/2}\left\{ \sum_{|\alpha|=2} \|D^\alpha u\|_0^2 + \|u\|_1^2 \right\}^{1/2} \\ &\le Ck^{s+1/2}\left\{ \sum_{|\alpha|=2} \|D^\alpha u\|_0^2 + |\lambda_k|_{-3/2}^2 \right\}^{1/2} . \end{aligned} \tag{5.5}$$

However, by Lemma 5.1,

$$\begin{aligned} \sum_{|\alpha|=2} \|D^\alpha u\|_0^2 &\le 2E(u,u) = 2\left\langle \Delta u - \frac{1}{2}\kappa\frac{\partial u}{\partial \nu}, \frac{\partial u}{\partial \nu}\right\rangle \\ &\le C\left\{ |\lambda_k|_0 |P_0 A\lambda_k|_0 + \left|\frac{\partial u}{\partial \nu}\right|_0^2 \right\} \\ &\le C\left\{ k^{-2s-1}|\lambda_k|_{-s-1}|P_0 A\lambda_k|_{-s} + \left|\frac{\partial u}{\partial \nu}\right|_0^2 \right\} . \end{aligned} \tag{5.6}$$

Combining (5.6) and (5.5), and using Lemma 3.1, we obtain

$$|(I-P_0)A\lambda_k|_{-s} \le C|\lambda_k|_{-s-1}|P_0 A\lambda_k|_{-s} + Ck|\lambda_k|_{-s-1}.$$

Hence, for any $\delta > 0$,

$$|(I-P_0)A\lambda_k|_{-s} \le C(k+\delta)\ |\lambda_k|_{-s-1}^2 + \frac{C}{\delta}|P_0 A\lambda_k|_{-s}^2.$$

Taking $\delta$ and $k$ small enough, and using this estimate in (5.4), proves the left-hand inequality in the lemma. Taking $\delta$ large enough proves the right-hand inequality. $\square$

*Proof of Theorem* 5.2. Using Lemma 5.2,

$$C_0|\lambda_k|_{-s-1} - |P_0 A\lambda_k - A_k\lambda_k|_{-s} \le |A_k\lambda_k|_{-s} \le C_1|\lambda_k|_{-s-1} + |P_0 A\lambda_k - A_k\lambda_k|_{-s}.$$

We must estimate $|P_0 A\lambda_k - A_k\lambda_k|_{-s}$; so, letting $\phi \in C^\infty(\Gamma)$, using (1.16) and (5.1), then using (2.6) and Lemma 2.2, we find that

$$
\begin{aligned}
\langle P_0 A\lambda_k - A_k\lambda_k, \phi\rangle &= \langle A\lambda_k, P_0\phi\rangle - \langle A_k\lambda_k, P_0\phi\rangle \\
&= (G\lambda_k, G(P_0\phi)) - (G_h\lambda_k, G_h(P_0\phi)) \\
&= ((G_h - G)\lambda_k, G(P_0\phi - \phi)) + ((G - G_h)\lambda_k, G\phi) \\
&\quad + ((G_h - G)\lambda_k, (G_h - G)(P_0\phi)) + (G\lambda_k, (G - G_h)(P_0\phi)) \\
&\le C(h^2k^{-2} + h^{s+2}k^{-s-2})|\lambda_k|_{-s-1}|\phi|_s \\
&\quad + ((G_h - G)\lambda_k, (G_h - G)(P_0\phi)) + (G\lambda_k, (G - G_h)(P_0\phi)).
\end{aligned} \tag{5.7}
$$

The remaining terms in (5.7) must be estimated in two cases depending on $s$. The first case is $0 \le s \le 1$:

$$
\begin{aligned}
((G_h - G)\lambda_k, (G_h - G)(P_0\phi)) &\le Ch^3k^{-3}|\lambda_k|_{-s-1}|\phi|_s, \\
(G\lambda_k, (G - G_h)(P_0\phi)) &\le Ch^{3/2}k^{-3/2}|\lambda_k|_{-s-1}|\phi|_s.
\end{aligned} \tag{5.8}
$$

The second case is $1 \le s$:

$$
\begin{aligned}
((G_h - G)\lambda_k, (G_h - G)(P_0\phi)) &= ((G_h - G)\lambda_k, (G_h - G)(P_0\phi - \phi)) \\
&\quad + ((G_h - G)\lambda_k, (G_h - G)\phi) \\
&\le C(h^3k^{-3} + h^{s+2}k^{-s-2})|\lambda_k|_{-s-1}|\phi|_s, \\
(G\lambda_k, (G - G_h)(P_0\phi)) &= (G\lambda_k, (G - G_h)(P_0\phi - \phi)) + (G\lambda_k, (G - G_h)\phi) \\
&\le C(h^{3/2}k^{-3/2} + h^{s-1/2}k^{-s+1/2})|\lambda_k|_{-s-1}|\phi|_s.
\end{aligned} \tag{5.9}
$$

Combining (5.7), (5.8), and (5.9), and using $h \le \tilde{\varepsilon}k$, we obtain

$$|P_0 A\lambda_k - A_k\lambda_k|_{-s} \le C\tilde{\varepsilon}^{1/2}|\lambda_k|_{-s-1}.$$

Hence, using this estimate with $\tilde{\varepsilon}$ small enough proves the result. □

The next lemma will be of use in Section 7.

LEMMA 5.3. *Suppose $A_k$ is defined by* (5.1) *and $G_h$ is constructed as in Section* 2. *Then, if $h \le \tilde{\varepsilon}k$, with $\tilde{\varepsilon}$ small enough, there exist positive constants $C_0$ and $C_1$ such that*

$$C_0|\lambda_k|^2_{-1/2} \le \langle A_k\lambda_k, \lambda_k\rangle \le C_1|\lambda_k|^2_{-1/2} \quad \forall \lambda_k \in \dot{S}_k.$$

To prove this lemma, we recall the following lemma which may be found in [15].

LEMMA 5.4. *There exist positive constants $C_0$ and $C_1$ independent of $\lambda$, such that for every $\lambda \in H^{-1/2}(\Gamma)$,*

$$C_0|\lambda|^2_{-1/2} \le \langle A\lambda, \lambda\rangle \le C_1|\lambda|^2_{-1/2}.$$

*Proof of Lemma* 5.3. By Lemma 5.4, we know that

$$C_0|\lambda_k|^2_{-1/2} - \langle A\lambda_k - A_k\lambda_k, \lambda_k\rangle \le \langle A_k\lambda_k, \lambda_k\rangle \le C_1|\lambda_k|^2_{-1/2} + \langle A\lambda_k - A_k\lambda_k, \lambda_k\rangle.$$

It remains to estimate $\langle A\lambda_k - A_k\lambda_k, \lambda_k\rangle$, using methods similar to those used to prove Theorem 5.2:

$$
\begin{aligned}
\langle A\lambda_k - A_k\lambda_k, \lambda_k\rangle &= (G\lambda_k, G\lambda_k) - (G_h\lambda_k, G_h\lambda_k) \\
&= 2((G - G_h)\lambda_k, G\lambda_k) + ((G_h - G)\lambda_k, (G - G_h)\lambda_k) \\
&\le C(h^{3/2}k^{-3/2} + k^3k^{-3})|\lambda_k|^2_{-1/2} \le C\tilde{\varepsilon}^{3/2}|\lambda_k|^2_{-1/2}.
\end{aligned}
$$

Taking $\tilde{\varepsilon}$ small enough and combining the above estimates proves the lemma. □

**6. Estimates for the Clamped Plate Problem.** In this section we shall assume the following smoothness for the data:

$$f \in H^{r-7/2}(\Omega), \quad g_1 \in H^r(\Gamma), \quad g_2 \in H^{r-2}(\Gamma). \tag{6.1}$$

This is more smoothness than is needed for the interior finite element problems alone. However, from Theorem 5.2 we must take $h \leq \tilde{\varepsilon} k$ for some sufficiently small $\tilde{\varepsilon}$, and so wish to take $k$ as large as possible. The extra smoothness helps this slightly. Our main error estimate is contained in Theorem 6.1, and the remaining estimates follow from that result.

THEOREM 6.1. *Suppose $r \geq 4$ and $G_h, T_h, S_h, S_h^0$ and $\dot{S}_k$ are constructed as detailed in Section* 2. *Let $\lambda$ solve* (1.14), *and let $\lambda_k \in \dot{S}_k$ solve* (1.19). *Then, if $h \leq \tilde{\varepsilon} k$, with $\tilde{\varepsilon}$ small enough, the following estimate holds for $-r+2 \leq s \leq r-3$:*

$$|\lambda - \lambda_k|_{-s-1} \leq C\{k^{r-1+s} + h^{r-3/2+s}\}(\|f\|_{r-7/2} + |g_1|_r + |g_2|_{r-2}).$$

THEOREM 6.2. *Suppose all the hypotheses of Theorem* 6.1 *are satisfied. Let $W$ solve the biharmonic problem* (1.1) *with clamped plate boundary conditions* (1.3), *and let $v_h(\lambda_k)$ be defined by* (1.18); *then the following estimate holds for $-r+5/2 \leq j \leq 1$:*

$$\|-\Delta W - v_h\|_j \leq C\{k^{r-3/2-j} + h^{r-2-j}\}(\|f\|_{r-7/2} + |g_1|_r + |g_2|_{r-2}).$$

THEOREM 6.3. *Let all the hypotheses of Theorems* 6.1 *and* 6.2 *hold, and let $u_h(\lambda_k)$ be defined by* (1.18). *Then, for $-r+9/2 \leq l \leq 1$, the following estimate holds:*

$$\|W - u_h\|_l \leq C\{k^{r+1/2-l} + h^{r-l}\}(\|f\|_{r-7/2} + |g_1|_r + |g_2|_{r-2}).$$

*Remarks.* Consider the case $l = 1$ in Theorem 6.3. Then

$$\|W - u_h\|_1 \leq C\{k^{r-1/2} + h^{r-1}\}(\|f\|_{r-7/2} + |g_1|_r + |g_2|_{r-2}).$$

We may balance terms in the estimate by taking $k = h^{(r-1)/(r-1/2)}$. Obviously, one can satisfy this equality at least approximately with compatible meshes. This choice of $h$ and $k$ has the additional advantage that for any fixed $\tilde{\varepsilon}, h \leq \tilde{\varepsilon} k$ if $k$ is small enough.

The proofs of the preceding theorems, which we outline next, use the lemmas from Section 4.

*Proof of Theorem* 6.1. We use Theorem 5.2:

$$\begin{aligned}
|\lambda - \lambda_k|_{-s-1} &\leq |\lambda - \pi_k\lambda|_{-s-1} + |\pi_k\lambda - \lambda_k|_{-s-1} \\
&\leq C(k^{r-1+s}|\lambda|_{r-2} + |A_k(\pi_k\lambda - \lambda_k)|_{-s}) \\
&\leq C(k^{r-1+s}|\lambda|_{r-2} + |A_k\pi_k\lambda - P_0A\lambda|_{-s} + |P_0A\lambda - A_k\lambda_k|_{-s}).
\end{aligned}$$

It remains to estimate the two final terms in the above expression. Let $\phi \in C^\infty(\Gamma)$; then

$$\begin{aligned}
\langle A_k\pi_k\lambda - P_0A\lambda, \phi\rangle &= (G_h(\pi_k\lambda_k), G_h(P_0\phi)) + (G\lambda, G_h(P_0\phi)), \\
\langle P_0A\lambda - A_k\lambda_k, \phi\rangle &= (Tf, G(P_0\phi)) - (\nabla Gg_1, \nabla G(P_0\phi)) \\
&\quad - (T_hf, G_h(P_0\phi)) + (\nabla G_h[g_1]_I, \nabla G_h(P_0\phi)).
\end{aligned}$$

The right-hand side of these expressions is estimated using Lemmas 4.1 and 4.2 with $\mu = 1$ and $l = r - 2$. This completes the proof of Theorem 6.1 for $s \geq 0$, and the result for $s < 0$ follows by the inverse property of $\dot{S}_k$. $\square$

*Proof of Theorems* 6.2 *and* 6.3. These follow from Theorem 6.1 by applying Lemma 4.3. $\square$

**7. Implementation of the Algorithms.** In this section we shall discuss how to implement (1.17) and (1.19), using the conjugate gradient algorithm.

7.1. *The Simply Supported Plate Problem.* To solve the simply supported plate problem, we seek to compute $\lambda_k \in \dot{S}_k$ which satisfies the linear equation (3.8). Once we have chosen a basis for $\dot{S}_k$, (3.8) is a matrix problem. Unfortunately, the matrix $M_k$ is not symmetric, so we must solve instead

$$M_k^T M_k \lambda_k = M_k^T F_k^{ss}. \tag{7.1}$$

A more detailed examination of (3.6) shows that the matrix representing $M_k$ is costly to compute, since to find the matrix, we must solve many Dirichlet problems for Laplace's equation. Fortunately, if we solve (7.1) using the conjugate gradient algorithm, we can avoid computing the matrix for $M_k$ and need only compute its action on vectors in $\dot{S}_k$. To make the action of $M_k$ cheaper to compute, we use the following result.

LEMMA 7.1. *Let $G_h$ and $T_h$ be constructed via Scott's method. Given $\gamma \in C^\infty(\Gamma)$ and any function $\phi \in \dot{S}_k$, define $\mu_h(\gamma\phi)$ to be the function in $S_h$ that interpolates $\gamma\phi$ at interpolation points on $\Gamma$ and which interpolates zero at points in the interior of $\Omega$. Then the following equality holds for all $\phi \in \dot{S}_k$:*

$$(G_h\lambda_k, G_h(\gamma\phi_k)) = (G_h\lambda_k, \mu_h(\gamma\phi_k)) - (\nabla T_h G_h \lambda_k, \nabla \mu_h(\gamma\phi_k)).$$

*Remark.* Note that the left-hand side in the above equality involves integration only over elements along $\Gamma$.

*Proof of Lemma* 7.1. Taking $\mu_h$ as defined above,

$$(G_h\lambda_k, G_h(\gamma\phi_k)) = (G_h\lambda_k, G_h(\gamma\phi_k) - \mu_h(\gamma\phi)) + (G_h\lambda_k, \mu_h(\gamma\phi)).$$

Note that $G_h(\gamma\phi_k) - \mu_h(\gamma\phi) \in S_h^0$; hence, using the properties of $T_h$, we obtain

$$(G_h\lambda_k, G_h(\gamma\phi_k)) = (\nabla T_h G_h\lambda_k, \nabla(G_h(\gamma\phi_k) - \mu_h(\gamma\phi))) + (G_h\lambda_k, \mu_h(\gamma\phi)).$$

Using the definition of $G_h$ and the fact that $T_h G_h \lambda_k \in S_h^0$ completes the proof. $\square$

Lemma 7.1 can be applied to compute the action of $M_k$ on any function in $\dot{S}_k$ by solving only two discrete Dirichlet problems for Poisson's equation. Similar results also hold for $M_k^T$. This makes the solution of (7.1) by conjugate gradients feasible, provided (7.1) does not become badly conditioned as $h$ and $k$ decrease. However, by Theorem 3.2,

$$C_0|\lambda_k|_0^2 \leq \langle M_k^T M_k \lambda_k, \lambda_k\rangle \leq C_1|\lambda_k|_0^2.$$

Thus, provided the hypotheses of Theorem 3.2 are satisfied, we know $M_k^T M_k$ has a condition number bounded independent of $h$ or $k$. Hence, we may solve (7.1) to accuracy $O(k^{2r})$ in $O(\ln(1/k))$ iterations of the conjugate gradient algorithm (cf. [2]). Each iteration of the algorithm requires the solution of four discrete Dirichlet problems. Numerical results for this algorithm for the simply supported plate problem can be found in [20].

7.2. *The Clamped Plate Problem.* Now let us turn to solving the clamped plate problem (5.3). In this case, the matrix involved is $A_k$, which is symmetric. As in the case of $M_k$, $A_k$ is costly to compute, but we can use Lemma 7.1 to compute the action of $A_k$ by solving only two Dirichlet problems for Poisson's equation. Unfortunately, Lemma 5.3 shows that the spectral condition number of $A_k$ is $O(k^{-1})$ and hence increases without bound as $k$ decreases to zero. This ill-conditioning will adversely affect the convergence properties of iterative methods applied to (5.3), so we must precondition the problem. We consider two possible preconditioners.

Let the discrete surface Laplacian $l_k\colon \dot{S}_k \to \dot{S}_k$ be defined so that if $\phi \in \dot{S}_k$ then

$$\langle l_k\phi, \theta\rangle = \langle \phi, \theta\rangle + \langle \phi', \theta'\rangle \quad \forall \phi \in \dot{S}_k,$$

where prime denotes derivative with respect to arc length. $l_k$ is estimated in the following lemma (cf. [6]).

LEMMA 7.2. *If $\dot{S}_k \subset H^1(\Gamma)$, then for $|s| < 1$ there are positive constants $C_0$ and $C_1$ such that*

$$C_0|\phi|_s \le |l_k^{s/2}\phi|_0 \le C_1|\phi|_s \quad \forall \phi \in \dot{S}_k.$$

The use of a fractional power of $l_k$ to precondition the clamped plate problem is suggested in [15], and our analysis follows Bramble [6]. Using Lemmas 5.3 and 7.2, we find that

$$C_0|\sigma_k|_0^2 \le \langle l_k^{1/4}A_k l_k^{1/4}\sigma_k, \sigma_k\rangle \le C_1|\sigma_k|_0^2 \quad \forall \sigma_k \in \dot{S}_k.$$

Hence, if we solve

$$l_k^{1/4}A_k l_k^{1/4}\sigma_k = l_k^{1/4}F_k^c, \tag{7.2}$$

we know that the matrix involved is symmetric and has a bounded condition number as $k$ decreases. Thus we can use the conjugate gradient algorithm on (7.2) and must compute $l_k^{1/2}A_k\phi$ for various $\phi \in \dot{S}_k$. This preconditioned problem is useful when $l_k^{1/2}$ can be computed rapidly, for instance if $\dot{S}_k$ consists of smooth splines on a uniform mesh (cf. [6] and [15] for more discussion on this case).

If $l_k^{1/2}$ is difficult to compute, we must use a different preconditioned system. From Theorem 5.2 and Lemma 7.2 we obtain

$$C_0|\sigma_k|_0^2 \le \langle l_k^{1/2}A_k^2 l_k^{1/2}\sigma_k, \sigma_k\rangle \le C_1|\sigma_k|_0^2 \quad \forall \sigma_k \in \dot{S}_k.$$

Hence the matrix $l_k^{1/2}A_k^2 l_k^{1/2}$ is symmetric, positive definite and has a bounded condition number as $k$ decreases to zero. We can thus use the conjugate gradient algorithm on the system

$$l_k^{1/2}A_k^2 l_k^{1/2}\sigma_k = l_k^{1/2}A_k F_k^c$$

in an efficient way. In applying the iterative method to this system, we must be able to compute $l_k A_k^2\phi_k$ for $\phi_k \in \dot{S}_k$. We can easily compute $A_k^2\phi_k$ via Lemma 7.1, and the action of $l_k$ only involves inverting the stiffness matrix for $\dot{S}_k$.

**Acknowledgment.** The author would like to express his extreme debt and gratitude to Dr. R. S. Falk, who directed the writing of the author's Ph.D. thesis on which this paper reports.

## A. Appendix.

*Proof of Lemma* 2.3. Essentially, Lemma 2.3 is an extension of a result in [22], and the proof we give below makes use of many results in that paper. We let $\tau_i^h$ be a boundary element and use the notation of Section 2 (see Figure 1). We start with a slight extension of a lemma in [22]. We first prove that for $0 \le s \le r-2$,

$$\tag{A.1} \begin{aligned} &\int_{\partial\tau_i^h} (\mu P_0\lambda - G_h(\mu P_0\lambda))\phi \\ &\qquad \le C x_0^{r+s+1/2} |\phi|_{s,\partial\tau_i^h} \{|P_0\lambda|_{r-1,\infty,\partial\tau_i^h} + \|G_h(\mu P_0\lambda)\|_{r-1,\infty,\overline{\tau}_i^h}\}. \end{aligned}$$

This is proved by taking $\psi$ to be a polynomial of degree $s-1$ ($\psi = 0$ if $s = 1$) such that

$$\sum_{j=0}^{s} x_0^j \left|\phi\frac{d\sigma}{dx} - \psi\right|_{j,[0,x_0]} \le C x_0^s \left|\phi\frac{d\sigma}{dx}\right|_{s,[0,x_0]},$$

where $\sigma$ is arc length on $\partial\tau_i^h$. Hence,

$$\tag{A.2} \begin{aligned} &\left|\int_{\tau_i^h} (\mu P_0\lambda - G_h(\mu P_0\lambda))\phi\right| \\ &\qquad \le \left|\int_0^{x_0} ((\mu P_0\lambda)(\sigma(x)) - G_h(\mu P_0\lambda)(x,\rho(x)))\psi\right| \\ &\qquad\quad + C x_0^s |\phi|_{s,\partial\tau_i^h} \left|\int_0^{x_0} (\mu P_0\lambda(\sigma(x)) - G_h(\mu P_0\lambda)(x,\rho(x)))^2\right|^{1/2}. \end{aligned}$$

To estimate the first term in (A.2), we recall the error estimates for Lobatto quadrature (cf. [12] and [23]). Then, using the fact that $P_0\phi$ and $G_h(P_0\phi)$ are polynomials, we obtain the following:

$$\tag{A.3} \begin{aligned} &\left|\int_0^{x_0} ((\mu P_0\lambda)(\sigma(x)) - G_h(\mu P_0\lambda)(x,\rho(x)))\psi\right| \\ &\qquad \le C x_0^{2r-3/2} |\phi|_{s,\partial\tau_i^h} \{|P_0\lambda|_{r-1,\infty,\partial\tau_i^h} + \|G_h(\mu P_0\lambda)\|_{r-1,\infty,\overline{\tau}_i^h}\}, \end{aligned}$$

where $\overline{\tau}_i^h$ is the circumscribed circle for this element. To estimate the second term in (A.2), we use standard one-dimensional interpolation theory:

$$\sup_{x\in[0,x_0]} |\mu P_0\lambda - G_h(\mu P_0\lambda)| \le C x_0^r \{|P_0\lambda|_{r-1,\infty,\partial\tau_i^h} + \|G_h(\mu P_0\lambda)\|_{r-1,\infty,\overline{\tau}_i^h}\}.$$

Combining the above estimate and (A.3) in (A.2) proves (A.1). Now we estimate terms on the right-hand side of (A.1). We start with the term in $G_h(\mu P_0\lambda)$ when $r-3 \le m \le r-3/2$. Let $[G(\mu P_0\lambda)]_I \in S_h$ be the interpolant of $G(\mu P_0\lambda)$; then by the regularity of the mesh and using standard bounds on norms of the interpolant (see [22]),

$$\tag{A.4} \begin{aligned} &\|G_h(\mu P_0\lambda)\|_{r-1,\infty,\overline{\tau}_i^h} \\ &\qquad \le C x_0^{-r+1} \{\|(G_h - G)(\mu(P_0 - I)\lambda)\|_{1,\tau_i^h} + \|(G_h - G)(\mu\lambda)\|_{1,\tau_i^h} \\ &\qquad\qquad + \|G(\mu(P_0 - I)\lambda) - [G(\mu(P_0 - I)\lambda)]_I\|_{1,\tau_i^h} \\ &\qquad\qquad\qquad + \|G(\mu\lambda) - [G(\mu\lambda)]_I\|_{1,\tau_i^h}\} \\ &\qquad\quad + C x_0^{-5/2} \|G(\mu(I - P_0)\lambda)\|_{r-5/2,\tau_i^h} + C x_0^{m-r+1/2} \|G(\mu\lambda)\|_{m+1/2,\tau_i^h}. \end{aligned}$$

For $m \leq r-3$ we must adopt a slightly different strategy. Again using the interpolant and the regularity of the mesh,

$$\begin{aligned} &\|G_h(\mu P_0\lambda)\|_{r-1,\infty,\tau_i^h} \\ &\quad \leq C x_0^{-r+1}\{\|(G_h - G)(\mu P_0\lambda)\|_{1,\tau_i^h} + \|G(\mu P_0\lambda) - [G(\mu P_0\lambda)]_I\|_{1,\tau_i^h}\} \\ &\qquad + C x_0^{-5/2}\|G(\mu P_0\lambda)\|_{r-5/2,\tau_i^h}. \end{aligned} \tag{A.5}$$

Now we turn to the first term on the right-hand side of (A.1). For $m \leq r-3$, we use the regularity of the mesh to write

$$|P_0\lambda|_{r-1,\infty,\partial\tau_i^h} \leq C x_0^{-r+m+1/2}|P_0\lambda|_{m,\partial\tau_i^h}. \tag{A.6}$$

For $r-3 \leq m \leq r-3/2$, we let $\lambda_I \in S_h^B$ interpolate $\lambda$ in the Lagrange sense and obtain the following:

$$\begin{aligned} &|P_0\lambda|_{r-1,\infty,\partial\tau_i^h} \\ &\quad \leq |P_0\lambda - \lambda_I|_{r-1,\infty,\partial\tau_i^h} + |\lambda_I|_{r-1,\infty,\partial\tau_i^h} \\ &\quad \leq C\{x_0^{-7/2}|P_0\lambda - \lambda|_{r-3,\partial\tau_i^h} + x_0^{-7/2}|\lambda - \lambda_I|_{r-3,\partial\tau_i^h} \\ &\qquad\qquad + x_0^{m-r+1/2}|\lambda_I|_{m,\partial\tau_i^h}\} \\ &\quad \leq C\{x_0^{-7/2}|P_0\lambda - \lambda|_{r-3,\partial\tau_i^h} + x_0^{m-r+1/2}|\lambda|_{m,\partial\tau_i^h}\}. \end{aligned} \tag{A.7}$$

Using (A.5) and (A.6) in (A.1), summing over boundary elements, and using the approximation properties of $P_0, G_h$ and the interpolant and inverse properties of $\dot{S}_k$ proves the lemma when $m \leq r-3$. In the same way, using (A.4) and (A.7) in (A.1) proves the result when $m \geq r-3$.□

Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716
*E-mail*: monk@vax1.acs.udel.edu

1. R. A. ADAMS, *Sobolev Spaces*, Pure and Appl. Math., Vol. 65, Academic Press, New York, 1975.

2. O. AXELSSON, *Solution of Linear Systems of Equations: Iterative Methods in Sparse Matrix Techniques*, Lecture Notes in Math., Vol. 572, Springer-Verlag, New York, 1977.

3. I. BABUŠKA, "The theory of small changes in the domain of existence in the theory of partial differential equations and its applications," *Differential Equations and Their Applications*, Academic Press, New York, 1963, pp. 13–26.

4. I. BABUŠKA & A. K. AZIZ, "Survey lecture on the mathematical foundations of the finite element method," *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, (A. K. Aziz, ed.), Academic Press, New York, 1972, pp. 5–359.

5. J. J. BLAIR, "Higher order approximations to the boundary conditions for the finite element method," *Math. Comp.*, v. 30, 1976, pp. 250–262.

6. J. H. BRAMBLE, "The Lagrange multiplier method for Dirichlet's problem," *Math. Comp.*, v. 37, 1981, pp. 1–11.

7. J. H. BRAMBLE & R. S. FALK, "Two mixed finite element methods for the simply supported plate problem," *RAIRO Numér. Anal.*, v. 17, 1983, pp. 337–384.

8. J. H. BRAMBLE & R. SCOTT, "Simultaneous approximation in scales of Banach spaces," *Math. Comp.*, v. 32, 1978, pp. 947–954.

9. P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, Studies in Math. and Its Applications, Vol. 4, North-Holland, Amsterdam, 1978.

10. P. G. CIARLET & R. GLOWINSKI, "Dual iterative techniques for solving a finite element approximation to the biharmonic equation," *Comput. Methods Appl. Mech. Engrg.*, v. 5, 1975, pp. 277–295.

11. P. G. CIARLET & A. RAVIART, "A mixed finite element method for the biharmonic equation," *Mathematical Aspects of Finite Elements in Partial Differential Equations*, (C. de Boor, ed.), Academic Press, New York, 1974, pp. 125–145.

12. P. J. DAVIS & P. RABINOWITZ, *Methods of Numerical Integration*, 2nd ed., Academic Press, 1984.

13. R. S. FALK, "Approximation of the biharmonic equation by a mixed finite element method," *SIAM J. Numer. Anal.*, v. 15, 1978, pp. 556–567.

14. R. S. FALK & J. E. OSBORN, "Error estimates for mixed methods," *RAIRO Anal. Numér.*, v. 14, 1980, pp. 249–277.

15. R. GLOWINSKI & O. PIRONNEAU, "Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem," *SIAM Rev.*, v. 21, 1979, pp. 167–212.

16. L. HERRMANN, "Finite element bending analysis for plates," *J. Eng. Mech. Div. A.S.C.E. EM5*, v. 93, 1967, pp. 49–83.

17. C. JOHNSON, "On the convergence of a mixed finite element method for plate bending problems," *Numer. Math.*, v. 21, 1973, pp. 43–62.

18. T. MIYOSHI, "A finite element method for the solution of fourth order partial differential equations," *Kumamoto J. Sci. (Math.)*, v. 9, 1973, pp. 87–116.

19. P. B. MONK, "A mixed finite element method for the biharmonic equation," *SIAM J. Numer. Anal.*, v. 24, 1987, pp. 737–749.

20. P. B. MONK, *Some Finite Element Methods for the Approximation of the Biharmonic Equation*. Ph.D. thesis, Rutgers University, 1983.

21. M. SCHECHTER, "On $L_p$ estimates and regularity. II," *Math. Scand.*, v. 13, 1963, pp. 47–69.

22. R. SCOTT, "Interpolated boundary conditions in the finite element method," *SIAM J. Numer. Anal.*, v. 12, 1975, pp. 404–427.

23. A. H. STROUD & D. SECREST, *Gaussian Quadrature Formulae*, Prentice-Hall, Englewood Cliffs, N.J., 1973.