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Sharp Maximum Norm Error Estimates for Finite Element Approximations of the Stokes Problem in $2-D$

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Abstract. This paper deals with finite element approximations of the Stokes equations in a plane bounded domain Ω, using the so-called velocity-pressure mixed formulation. Quasi-optimal error estimates in the maximum norm are derived for the velocity, its gradient and the pressure fields. The analysis relies on abstract properties which are in turn a consequence of the eixstence of a local projection operator Π_h satisfying

$$\int_\Omega \operatorname{div}(\mathbf{v}-\Pi_h\mathbf{v})q\,d\mathbf{x}=0, \quad \forall \mathbf{v}\in[H_0^1(\Omega)]^2,\ \forall q\in M_h,$$

where M_h is the finite element space associated with the pressure. Several examples for which this operator can be constructed locally illustrate the theory.

1. Introduction. We consider the Stokes problem arising in fluid dynamics, which describes the flow of a viscous incompressible fluid. In its simplest form, we have to solve

$$\text{(1.1)} \qquad \begin{cases} -\Delta\mathbf{u}+\nabla p=\mathbf{f} & \text{in } \Omega,\\ \operatorname{div}\mathbf{u}=0 & \text{in } \Omega,\\ \mathbf{u}=0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $\mathbf{u}$ represents the velocity of the fluid, p its pressure and $\mathbf{f}$ a given external force.

Several finite element spaces have been considered to approximate the solution of problem (1.1) using the following velocity-pressure formulation: find $\mathbf{u}\in[H_0^1(\Omega)]^2$, $p\in L_0^2(\Omega)$, such that

$$\text{(1.2)} \qquad \begin{cases} \langle\nabla u,\nabla v\rangle-\langle p,\operatorname{div}\mathbf{v}\rangle=\langle\mathbf{f},\mathbf{v}\rangle, & \forall\mathbf{v}\in[H_0^1(\Omega)]^2,\\ \langle q,\operatorname{div}\mathbf{u}\rangle=0, \quad \forall q\in L_0^2(\Omega),\end{cases}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$ and $L_0^2(\Omega)$ is the space of L^2-functions having mean value zero. It is known that this weak formulation is equivalent to a saddle point problem.

The approximation by finite elements of this kind of problems has been studied in an abstract form by F. Brezzi [4], M. Fortin [13] and R. S. Falk and J. E. Osborn [12].

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Given a family $\{\tau_h\}$ $(0 < h < h_0)$ of partitions of Ω, let $X_h \subset [H_0^1(\Omega)]^2$ and $M_h \subset L_0^2(\Omega)$ denote the finite element spaces. Then the corresponding discrete problem reads as follows: find $\mathbf{u}_h \in X_h$ and $p_h \in M_h$ such that

$$\begin{aligned} \langle \nabla \mathbf{u}_h, \nabla \mathbf{v} \rangle - \langle p_h, \operatorname{div} \mathbf{v} \rangle &= \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in X_h, \\ \langle q, \operatorname{div} \mathbf{u}_h \rangle &= 0, \quad \forall q \in M_h. \end{aligned} \tag{1.3}$$

It is well known that X_h and M_h cannot be chosen independently. It was proven in [4] that existence of the discrete solution and stability of the scheme follow from the condition

$$\sup_{\mathbf{v} \in X_h} \frac{\langle q, \operatorname{div} \mathbf{v} \rangle}{\|\mathbf{v}\|_{H_0^1}} \geq \beta \|q\|_{L^2}, \quad \forall q \in M_h, \tag{1.4}$$

where β is a positive number independent of h. Moreover, if (1.4) is satisfied, optimal error estimates in L^2 for the gradient of the velocity and for the pressure hold; namely,

$$\|\mathbf{u} - \mathbf{u}_h\|_{H_0^1} + \|p - p_h\|_{L^2} \leq C \left\{ \inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_{H_0^1} + \inf_{q \in M_h} \|p - q\|_{L^2} \right\}. \tag{1.5}$$

Optimal error estimates in L^2 for the velocity can be derived by duality arguments under some regularity assumption on the domain (see [15]); namely,

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2} \leq Ch\{\|\mathbf{u} - \mathbf{u}_h\|_{H_0^1} + \|p - p_h\|_{L^2}\}. \tag{1.6}$$

In [12] R. S. Falk and J. E. Osborn proved that the condition (1.4) is equivalent to the existence of a projection operator $\Pi_h : [H_0^1(\Omega)]^2 \to X_h$ such that

$$\|\Pi_h \mathbf{v}\|_{H_0^1} \leq C \|\mathbf{v}\|_{H_0^1}, \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^2 \tag{1.7}$$

and

$$\langle \operatorname{div}(\mathbf{v} - \Pi_h \mathbf{v}), q \rangle = 0, \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^2, \ \forall q \in M_h. \tag{1.8}$$

Here and throughout the paper, C denotes a positive constant independent of h and the functions involved in the estimates, but not necessarily the same at each occurrence.

The aim of this paper is to study convergence in L^∞ for the velocity and its first derivatives and for the pressure. Our analysis is based on the technique of weighted Sobolev norms introduced by F. Natterer [17] and J. A. Nitsche [18], [19], combined with the use of regularized Green's functions as proposed by J. Frehse and R. Rannacher [14] and more recently by R. Rannacher and R. Scott [20] for second-order scalar elliptic operators. Elliptic systems were considered by M. Dobrowolski and R. Rannacher [10], but their analysis did not include the Stokes equation, which has a saddle point structure, and so requires compatibility constraints between the discrete spaces.

In many cases the operator Π_h can be constructed locally, and consequently it satisfies optimal approximation properties in weighted norms. Under this crucial assumption we prove the following quasi-optimal uniform estimates,

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} \leq Ch|\log h| \left\{ \inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_{W^{1,\infty}} + \inf_{q \in M_h} \|p - q\|_{L^\infty} \right\}, \tag{1.9}$$

$$\|p - p_h\|_{L^\infty} \le C|\log h|^{1/2} \left\{ \inf_{\mathbf{v}\in X_h} \|\mathbf{u} - \mathbf{v}\|_{W^{1,\infty}} + \inf_{q\in M_h} \|p - q\|_{L^\infty} \right\}, \tag{1.10}$$

$$\|\mathrm{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^\infty} \le C|\log h|^{1/2} \left\{ \inf_{\mathbf{v}\in X_h} \|\mathbf{u} - \mathbf{v}\|_{W^{1,\infty}} + \inf_{q\in M_h} \|p - q\|_{L^\infty} \right\}, \tag{1.11}$$

where we have denoted by $W^{m,p}$ the usual Sobolev spaces. These estimates may be viewed as the L^∞ analogues of (1.5) and (1.6).

The paper is organized as follows. In Section 2 we state the notation and assumptions and recall some properties about weights and related norms. In Section 3 we introduce the regularized Green's functions and prove some weighted a priori estimates. Section 4 deals with the L^∞-error estimates for the solution. A weighted-norm error estimate for Green's functions is proved there as well; this is the key result and involves some technical calculations. Finally, in Section 5 we show several examples of known finite element spaces satisfying our assumptions and state the corresponding rates of convergence in $L^\infty(\Omega)$.

2. Notation and Assumptions. Let Ω be a convex polygon in $\mathbb{R}^2$ and let $\{\tau_h\}$ be a regular and quasi-uniform family of decompositions of Ω, where $h > 0$ denotes the mesh size [8, pp. 132, 140].

We will work with finite element spaces $M_h^k \subset L_0^2(\Omega)$ and $X_h^k \subset [H_0^1(\Omega)]^2$, with k a positive integer such that, for every $T \in \tau_h$,

$$P_{k-1}(T) \subset M_h^k|_T \subset P_{k+m}(T), \tag{2.1}$$

$$[P_k(T)]^2 \subset X_h^k|_T \subset [P_{k+m}(T)]^2, \tag{2.2}$$

where $P_k(T)$ denotes the space of polynomials of degree less than or equal to k restricted to T, and m is some natural number independent of T and h.

We define the weight function σ by

$$\sigma(\mathbf{x}) := (|\mathbf{x} - \mathbf{x}_0|^2 + \theta^2)^{1/2}, \qquad \mathbf{x}, \mathbf{x}_0 \in \Omega, \tag{2.3}$$

where $\theta = Kh$, with $K \ge 1$ a constant to be specified later on. Let us recall the following elementary properties of the weights (see [8]):

$$\max_{\mathbf{x}\in T} \sigma(\mathbf{x}) \le C \min_{\mathbf{x}\in T} \sigma(\mathbf{x}), \quad \forall T \in \tau_h, \tag{2.4}$$

$$|D^j \sigma^\alpha(\mathbf{x})| \le C(j, \alpha)\sigma^{\alpha - j}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{2.5}$$

where $\alpha \in R$ and $D^j f$ denotes the tensor of derivatives of order j of f, and

$$\int_\Omega \sigma^{-2}(k\mathbf{x})\, d\mathbf{x} \le C|\log \theta| \tag{2.6}$$

for θ small enough.

For $\alpha \in \mathbb{R}$ and j a nonnegative integer, we define the weighted seminorms by

$$\|D^j q\|_{\sigma^\alpha}^2 := \sum_{|\beta| = j} \int_\Omega |\partial^\beta q|^2 \sigma^\alpha \, d\mathbf{x}, \qquad q \in H^j(\Omega),$$

and the same notation will be used for vector-valued functions. The following assumptions will be made:

$$\|D^j(\mathbf{v} - \Pi_h^k \mathbf{v})\|_{\sigma^\alpha} \le C h^{k+1-j} \|D^{k+1}\mathbf{v}\|_{\sigma^\alpha}, \quad \forall \mathbf{v} \in [H^{k+1}(\Omega)]^2,\ j = 0, 1; \tag{2.7}$$

$$\|q - P_h^k q\|_{\sigma^\alpha} \le Ch^k \|D^k q\|_{\sigma^\alpha}, \quad \forall q \in H^k(\Omega); \tag{2.8}$$

$$\|\mathbf{v} - \Pi_h^k \mathbf{v}\|_{L^\infty} \le Ch \|\nabla \mathbf{v}\|_{L^\infty}, \quad \forall \mathbf{v} \in [W^{1,\infty}(\Omega)]^2; \tag{2.9}$$

$$\|\nabla \Pi_h^k \mathbf{v}\|_{\sigma^2} \le C \|\nabla \mathbf{v}\|_{\sigma^2}, \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^2, \tag{2.10}$$

where P_h^k is an auxiliary operator having values in M_h. In many cases the operator Π_h^k can be constructed locally and, therefore, properties (2.7), (2.9) and (2.10) are an easy consequence of standard interpolation error estimates. The operator P_h^k is usually either a local L^2-projection or an interpolation operator; thus, the assumption (2.8) holds. Further approximation properties on Π_h^k and P_h^k will be required, namely,

$$\|\sigma^2 q - P_h^k(\sigma^2 q)\|_{\sigma^{-2}} \le Ch \|q\|_{L^2}, \quad \forall q \in M_h^k; \tag{2.11}$$

$$\|\nabla(\sigma^2 \mathbf{v} - \Pi_h^k(\sigma^2 \mathbf{v}))\|_{\sigma^{-2}} \le Ch \|\mathbf{v}\|_{\sigma^{-2}} + Ch \|\nabla \mathbf{v}\|_{L^2}, \quad \forall \mathbf{v} \in X_h^k. \tag{2.12}$$

If the operators P_h^k and Π_h^k are local, then the estimates (2.11) and (2.12) can be proved using a generalized Bramble-Hilbert lemma of T. Dupont and R. Scott [11]. Moreover, a similar estimate to these two was proved by R. Scholz in [21].

Since no confusion is possible, we shall remove the subscript k in the notation of both discrete spaces and interpolant operators.

3. Regularized Green's Functions. In this section we introduce and analyze the so-called regularized Green's functions, which are solutions of the Stokes equations with a right-hand side being a suitable regularization of the Dirac measure. This technique was first used by J. Frehse and R. Rannacher [14] and further developed by R. Rannacher and R. Scott [20], in both cases for a scalar second-order elliptic equation.

Let us start by recalling some regularity results for the following generalized Stokes problem:

$$\begin{cases} -\Delta \mathbf{v} + \nabla q = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = g & \text{in } \Omega, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

Let Ω be a convex polygon in $\mathbb{R}^2$ and let $d(\mathbf{x})$ denote the distance from $\mathbf{x} \in \Omega$ to the closest vertex of Ω. In addition, let $\mathbf{f} \in [L^2(\Omega)]^2$, $g \in H^1(\Omega)$ and $d^{-1}g \in L^2(\Omega)$ hold, as well as the following compatibility condition,

$$\int_\Omega g \, d\mathbf{x} = 0. \tag{3.2}$$

Then the solution $(\mathbf{v}, q)$ of (3.1) satisfies (see R. B. Kellogg and J. E. Osborn [16])

$$\|\mathbf{v}\|_{H^2} + \|q\|_{H^1} \le C\{\|\mathbf{f}\|_{L^2} + \|\nabla g\|_{L^2} + \|d^{-1}g\|_{L^2}\}. \tag{3.3}$$

Moreover, the requirement $d^{-1}g \in L^2(\Omega)$ cannot be eliminated for the solution to satisfy an estimate like (3.3). As an easy calculation reveals, if $g \in H_0^1(\Omega)$ then $d^{-1}g \in L^2(\Omega)$, and we get the estimate

$$\|\mathbf{v}\|_{H^2} + \|q\|_{H^1} \le C\{\|\mathbf{f}\|_{L^2} + \|\nabla g\|_{L^2}\}. \tag{3.4}$$

This is the basic regularity result required in our error analysis. However, stronger regularity will be needed in deriving sharp rates of convergence according to the approximation theory. For these estimates to hold, we have to modify the assumptions on the data. Namely, suppose now

$$\partial\Omega \in C^{k+1} \qquad (\Omega \text{ is not necessarily convex}), \tag{3.5}$$

$$\mathbf{f} \in [W^{k-1,\infty}(\Omega)]^2, \qquad g \in W^{k,\infty}(\Omega), \tag{3.6}$$

where $k \in N$ stands for the order of the approximation as stated in (2.1)–(2.2). A consequence of the results in [1] for general elliptic systems is the a priori estimate

$$\|\mathbf{v}\|_{W^{k+1,s}} + \|q\|_{W^{k,s}} \leq Cs(\|\mathbf{f}\|_{W^{k-1,s}} + \|g\|_{W^{k,s}}), \tag{3.7}$$

where $2 \leq s < \infty$ and $C > 0$ is a constant independent of s (see also [7] and R. Temam [22, p. 33]). The dependence on s follows from [1] by tracing constants in the singular integrals involved.

Let us now introduce the regularizations of the Dirac mass that we shall deal with. For $1 \leq i \leq 3$ let $\mathbf{x}_i$ denote a fixed point in Ω which will be specified later on, and let $T_i \in \tau_h$ be such that $\mathbf{x}_i \in T_i$. Then, let $\delta_i \in C_0^\infty(\Omega)$ satisfy

$$\operatorname{supp}\delta_i \subset B_i, \tag{3.8}$$

$$\int_\Omega \delta_i \, d\mathbf{x} = 1, \qquad \delta_i \geq 0, \tag{3.9}$$

$$\|D^j\delta_i\|_{L^\infty} \leq Ch^{-2-j}, \qquad j = 0, 1, \tag{3.10}$$

where B_i is a ball of radius αh contained in T_i and $\alpha > 0$ is a suitable constant, both to be determined as follows. Let χ be any discrete function; so χ is a piecewise polynomial and possibly discontinuous across interelement boundaries. Assume that the maximum norm of χ is attained at $\mathbf{x}_i$; χ is here extended to the closure of T_i by continuity. Then, since the T_i satisfy a minimum angle property because the partition τ_h is regular, we can always find a ball B_i with center $\mathbf{y}_i$ so that $|\mathbf{x}_i - \mathbf{y}_i| = C\alpha h$. Then a straightforward application of the weighted mean value theorem, combined with (3.9), yields

$$\int_\Omega \chi\delta \, d\mathbf{x} = \chi(\mathbf{z}) \quad \text{for some } \mathbf{z} \in B_i.$$

Using now the mean value theorem together with an inverse inequality leads to

$$\|\chi\|_{L^\infty(\Omega)} = |\chi(\mathbf{x}_i)| \leq |\chi(\mathbf{z})| + C\alpha h\|\nabla\chi\|_{L^\infty(T_i)} \leq C\alpha\|\chi\|_{L^\infty(\Omega)} + \left|\int_\Omega \chi\delta \, d\mathbf{x}\right|.$$

Thus, choosing α so that $C\alpha = \frac{1}{2}$, the previous inequality can be rewritten as

$$\|\chi\|_{L^\infty(\Omega)} \leq 2\left|\int_\Omega \chi\delta \, d\mathbf{x}\right|. \tag{3.11}$$

This inequality will be often used in the next section. In addition, according to (2.3), we define the weight function $\boldsymbol{\sigma}_i$ by $\sigma_i(\mathbf{x}) = (|\mathbf{x} - \mathbf{x}_i|^2 + \theta^2)^{1/2}$ for each $1 \leq i \leq 3$.

Let us now introduce the regularized Green's functions. The first one will be useful in analyzing the velocity field, and is defined by

$$\text{(3.12)} \qquad \begin{cases} -\Delta \mathbf{G}_1 + \nabla \lambda_1 = \boldsymbol{\delta}_1, & \text{in } \Omega, \\ \operatorname{div} \mathbf{G}_1 = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{G}_1 = \mathbf{0}, & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{\delta}_1$ stands for either $(\delta_1, 0)$ or $(0, \delta_1)$. We then have the following a priori estimate in weighted norms.

LEMMA 3.1. *There holds*

$$\|D^2 \mathbf{G}_1\|_{\sigma_1^2} + \|\nabla \lambda_1\|_{\sigma_1^2} \le C|\log h|^{1/2}.$$

Proof. Following P. Ciarlet [8, p. 148], it suffices to deal with the components $\mu_j = \mathbf{x}^j - \mathbf{x}_i^j$ $(j = 1, 2)$ of $\mathbf{x} - \mathbf{x}_i$ instead of with σ_1. Moreover, it is easy to see that

$$\begin{aligned} &\|\mu_j D^2 \mathbf{G}_1\|_{L^2} + \|\mu_j \nabla \lambda_1\|_{L^2} \\ &\qquad \le C\{\|D^2(\mu_j \mathbf{G}_1)\|_{L^2} + \|\nabla(\mu_j \lambda_1)\|_{L^2} + \|\nabla \mathbf{G}_1\|_{L^2} + \|\lambda_1\|_{L^2}\}. \end{aligned}$$

In order to estimate the first two terms on the right-hand side, we shall make use of the Stokes equations in conjunction with (3.4). Namely, since

$$\begin{aligned} -\Delta(\mu_j \mathbf{G}_1) + \nabla(\mu_j \lambda_1) &= \mu_j \boldsymbol{\delta}_1 - 2\nabla \mu_j \cdot \nabla \mathbf{G}_1 + \lambda_1 \nabla \mu_j, \\ \operatorname{div}(\mu_j \mathbf{G}_1) &= \nabla \mu_j \cdot \mathbf{G}_1, \end{aligned}$$

and $\nabla \mu_j \cdot \mathbf{G}_1 \in H_0^1(\Omega)$, (3.4) together with the fact that μ_j is linear yields

$$\text{(3.13)} \qquad \|D^2(\mu_j \mathbf{G}_1)\|_{L^2} + \|\nabla(\mu_j \lambda_1)\|_{L^2} \le C\{\|\mu_j \boldsymbol{\delta}_1\|_{L^2} + \|\nabla \mathbf{G}_1\|_{L^2} + \|\lambda_1\|_{L^2}\}.$$

Since $\|\mu_j \boldsymbol{\delta}_1\|_{L^2} = O(1)$, it only remains to estimate the last two terms on the right-hand side of (3.13). To do so, observe first that the inf-sup condition implies

$$\|\nabla \mathbf{G}_1\|_{L^2} + \|\lambda_1\|_{L^2} \le C\|\boldsymbol{\delta}_1\|_{H^{-1}}$$

and, furthermore,

$$\|\boldsymbol{\delta}_1\|_{H^{-1}} = \sup_{\mathbf{v} \in S} \langle \boldsymbol{\delta}_1, \mathbf{v} \rangle \le \sup_{\mathbf{v} \in S} \langle \boldsymbol{\delta}_1, \mathbf{v} - I_h \mathbf{v} \rangle + \sup_{\mathbf{v} \in S} \langle \boldsymbol{\delta}_1, I_h \mathbf{v} \rangle \le C(1 + |\log h|^{1/2}),$$

where S stands for the unit ball of $[H_0^1(\Omega)]^2$ and I_h denotes the local average interpolant [15, p. 109]. Here we have used the inverse inequality between H^1 and L^∞ in the finite element subspace. Now the assertion of the lemma is a consequence of the above inequalities and the following expressions

$$\|D^2 \mathbf{G}_1\|_{\sigma_1^2}^2 = \theta^2 \|D^2 \mathbf{G}_1\|_{L^2}^2 + \sum_{j=1}^{2} \|\mu_j D^2 \mathbf{G}_1\|_{L^2}^2,$$

$$\|\nabla \lambda_1\|_{\sigma_1^2}^2 = \theta^2 \|\nabla \lambda_1\|_{L^2}^2 + \sum_{j=1}^{2} \|\mu_j \nabla \lambda_1\|_{L^2}^2,$$

combined with (3.4) and the fact that $\theta = Kh$. □

In dealing with the vorticity, and so with derivatives of the velocity, we need the following regularized Green's function:

$$\text{(3.14)} \qquad \begin{cases} -\Delta \mathbf{G}_2 + \nabla \lambda_2 = D\boldsymbol{\delta}_2, & \text{in } \Omega, \\ \operatorname{div} \mathbf{G}_2 = 0, & \text{in } \Omega, \\ \mathbf{G}_2 = 0, & \text{on } \partial\Omega, \end{cases}$$

where $D\boldsymbol{\delta}_2$ stands for any directional derivative of either $\boldsymbol{\delta}_2 = (\delta_2, 0)$ or $\boldsymbol{\delta}_2 = (0, \delta_2)$.

LEMMA 3.2. *We have*

$$\|D^2\mathbf{G}_2\|_{\sigma_2^2} + \|\nabla\lambda_2\|_{\sigma_2^2} \leq Ch^{-1}.$$

Proof. The proof proceeds along the same lines as the previous one. Consequently, one is led to estimate $\|\nabla\mathbf{G}_2\|_{L^2}$ and $\|\lambda_2\|_{L^2}$. To do so, we use the following a priori estimate, which is a trivial by-product of the inf-sup conditon (1.4); namely,

$$\|\nabla\mathbf{G}_2\|_{L^2} + \|\lambda_2\|_{L^2} \leq C\|D\boldsymbol{\delta}_2\|_{H^{-1}} \leq C\|\boldsymbol{\delta}_2\|_{L^2} \leq Ch^{-1},$$

where we have used (3.8) and (3.10). This completes the argument. □

The analysis of the pressure field requires a different regularized Green's function from those above. Indeed, for $\phi \in C_0^\infty(\Omega)$ satisfying $\int_\Omega \phi\, dx = 1$, let us consider the problem:

$$\text{(3.15)}\qquad \begin{cases} -\Delta\mathbf{G}_3 + \nabla\lambda_3 = 0, & \text{in } \Omega, \\ \operatorname{div}\mathbf{G}_3 = \delta_3 - \phi, & \text{in } \Omega, \\ \mathbf{G}_3 = 0, & \text{on } \partial\Omega. \end{cases}$$

Since the compatibility condition (3.2) holds, problem (3.15) actually has a unique solution. Moreover, the a priori estimate (3.4) also holds, because $\delta_3 - \phi \in H_0^1(\Omega)$. This is why we take $\phi \in C_0^\infty(\Omega)$ rather than a constant. However, if Ω were smooth enough, say $\partial\Omega \in C^2$, ϕ might be constant, as asserted by (3.7).

LEMMA 3.3. *We have*

$$\|D^2\mathbf{G}_3\|_{\sigma_3^2} + \|\nabla\lambda_3\|_{\sigma_3^2} \leq Ch^{-1}.$$

Proof. As in Lemma 3.1, we are now led to estimate $\|\nabla(\delta_3 - \phi)\|_{\sigma_3^2}$, $\|\nabla\mathbf{G}_3\|_{L^2}$, and $\|\lambda_3\|_{L^2}$. The first term satisfies $\|\nabla(\delta_3 - \phi)\|_{\sigma_3^2} = O(h^{-1})$ on account of (3.8) and (3.10). The remaining terms can be bounded by making use of a stability estimate which is a trivial consequence of the inf-sup condition (1.4), namely

$$\|\nabla\mathbf{G}_3\|_{L^2} + \|\lambda_3\|_{L^2} \leq C\|\delta_3 - \phi\|_{L^2} \leq Ch^{-1}. \quad \square$$

4. The Error Analysis.

4.1. *L^∞-Error Estimates.* This section is devoted to the error analysis in L^∞ for the velocity, its first derivatives, and the pressure.

From (1.2) and (1.3) the following error equations follow:

$$\text{(4.1)}\qquad \begin{cases} \langle\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v}\rangle - \langle p - p_h, \operatorname{div}\mathbf{v}\rangle = 0, & \forall\mathbf{v} \in X_h, \\ \langle q, \operatorname{div}(\mathbf{u}-\mathbf{u}_h)\rangle = 0, & \forall q \in M_h. \end{cases}$$

Analogous error equations hold for the regularized Green's functions and their approximations defined in Section 3.

Let us start by analyzing the error in the velocity field. Clearly, it is enough to bound $\Pi_h\mathbf{u} - \mathbf{u}_h$. To this aim, let $\mathbf{x}_1 \in \Omega$ be a point where

$$\|\mathbf{u}_h - \Pi_h\mathbf{u}\|_{L^\infty} = \operatorname*{Max}_{1\leq i\leq 2} \|\mathbf{u}_h^i - (\Pi_h\mathbf{u})^i\|_{L^\infty}$$

is attained. In view of (3.11) we have

$$\text{(4.2)}\qquad \begin{aligned} \|\Pi_h\mathbf{u} - \mathbf{u}_h\|_{L^\infty} &\leq 2|\langle\Pi_h\mathbf{u} - \mathbf{u}_h, \boldsymbol{\delta}_1\rangle| \\ &= 2|\langle\nabla\mathbf{G}_1^h, \nabla(\Pi_h\mathbf{u} - \mathbf{u}_h)\rangle - \langle\operatorname{div}(\Pi_h\mathbf{u} - \mathbf{u}_h), \lambda_1^h\rangle|. \end{aligned}$$

Using the error equations (4.1) and the assumption (1.8) on the operator Π_h, we see that the expression in (4.2) is equal to

$$\begin{aligned}(4.3)\qquad & 2|\langle \nabla \mathbf{G}_1^h, \nabla(\Pi_h \mathbf{u} - \mathbf{u})\rangle + \langle p - p_h, \operatorname{div} \mathbf{G}_1^h\rangle| \\ &= 2|\langle \nabla(\mathbf{G}_1^h - \mathbf{G}_1), \nabla(\Pi_h \mathbf{u} - \mathbf{u})\rangle + \langle p - P_h p, \operatorname{div}(\mathbf{G}_1^h - \mathbf{G}_1)\rangle \\ &\qquad - \langle \Delta \mathbf{G}_1, \Pi_h \mathbf{u} - \mathbf{u}\rangle|.\end{aligned}$$

Thus, by the Hölder inequality we have that

$$\begin{aligned}(4.4)\qquad \|\Pi_h \mathbf{u} - \mathbf{u}_h\|_{L^\infty} \le C\{&\|\nabla(\mathbf{u} - \Pi_h \mathbf{u})\|_{L^\infty} + \|p - P_h p\|_{L^\infty}\}\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1} \\ &+ \|\mathbf{u} - \Pi_h \mathbf{u}\|_{L^\infty} \|\Delta \mathbf{G}_1\|_{L^1}.\end{aligned}$$

Therefore, we have reduced the problem to that of estimating $\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1}$ and $\|\Delta \mathbf{G}_1\|_{L^1}$. Using again the Hölder inequality, we see that

$$(4.5)\qquad \|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1} \le C|\log h|^{1/2} \|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{\sigma_1^2}$$

and, then, we have to estimate $\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{\sigma_1^2}$. Similarly, when dealing with the pressure and with the first derivatives of the velocity, we will have to estimate $\|\nabla(\mathbf{G}_i - \mathbf{G}_i^h)\|_{\sigma_i^2}$ for $i = 2, 3$.

The following theorem states a bound for errors like those above, irrespective of the type of Green's functions involved. Moreover, since no confusion is possible, we remove the subscript i.

THEOREM 4.1. *Let Ω be a convex polygon in $\mathbb{R}^2$. Let $(\mathbf{G}, \lambda)$ be the solution of one of the problems introduced in Section 3, and let $(\mathbf{G}^h, \lambda^h) \in X_h \times M_h$ be the approximate solution. Then for K large enough, there exists a constant $C = C(K)$ such that*

$$(4.6)\quad \|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2} \le Ch\{\|D^2 \mathbf{G}\|_{\sigma^2} + \|\nabla \lambda\|_{\sigma^2}\} + Ch^2\{\|D^2 \mathbf{G}\|_{L^2} + \|\nabla \lambda\|_{L^2}\},$$

where K is the parameter introduced in (2.3).

Proof. The proof of this crucial result is rather technical and will be postponed to Subsection 4.2. Instead, we now apply Theorem 4.1 to derive quasi-optimal error estimates in the maximum norm. To begin with, we first consider the velocity field.

THEOREM 4.2. *Let Ω be a convex polygon in $\mathbb{R}^2$ and let $(\mathbf{u}, p)$ be the solution of the Stokes problem* (1.1). *Let $(\mathbf{u}_h, p_h) \in X_h \times M_h$ be the finite element approximations defined by* (1.3). *Assume that $\mathbf{u} \in [W^{1,\infty}(\Omega)]^2$ and $p \in L^\infty(\Omega)$. Then there exists a constant $C > 0$ such that*

$$(4.7)\qquad \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} \le Ch|\log h|\{\|\nabla(\mathbf{u} - \Pi_h \mathbf{u})\|_{L^\infty} + \|p - P_h p\|_{L^\infty}\}.$$

Proof. As we pointed out before, it is enough to bound $\Pi_h \mathbf{u} - \mathbf{u}_h$. In view of (4.4) we have to estimate $\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1}$. To this end, note first that Theorem 4.1 together with (4.5) yields

$$\begin{aligned}\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1} \le\ & C|\log h|^{1/2} h\{\|D^2 \mathbf{G}_1\|_{\sigma^2} + \|\nabla \lambda_1\|_{\sigma^2}\} \\ &+ C|\log h|^{1/2} h^2\{\|D^2 \mathbf{G}_1\|_{L^2} + \|\nabla \lambda_1\|_{L^2}\}.\end{aligned}$$

Then, using the a priori estimate (3.3) and Lemma 3.1, we obtain

$$\|\nabla(\mathbf{G}_1 - \mathbf{G}_1^h)\|_{L^1} \le C|\log h|h.$$

Analogously, using again Lemma 3.1, we get

$$\|\Delta \mathbf{G}_1\|_{L^1} \le C|\log h|.$$

Thus, the theorem follows from the two previous inequalities combined with (4.4) and the assumption (2.9). □

As an easy corollary of Theorem 4.2 we have the following estimate for the vorticity,

$$\|\operatorname{curl}(\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \le C|\log h|\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\}.$$

However, this estimate can be improved, as the following theorem shows.

THEOREM 4.3. *Under the assumptions of Theorem* 4.2 *there exists a constant* $C>0$ *such that*

$$\|\operatorname{curl}(\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \le C|\log h|^{1/2}\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\}. \tag{4.8}$$

Proof. Our present goal is to demonstrate the following estimate,

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \le C|\log h|^{1/2}\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\},$$

which, in particular, implies the assertion. Using the notation of Section 3, with $\mathbf{x}_2$ chosen to maximize $|D(\Pi_h\mathbf{u}-\mathbf{u}_h)(\mathbf{x})|$, and the inequality (3.11), we have

$$\begin{aligned}
\|D(\Pi_h\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} &\le 2|\langle D(\Pi_h\mathbf{u}-\mathbf{u}_h), \boldsymbol{\delta}_2\rangle| = 2|\langle \Pi_h\mathbf{u}-\mathbf{u}_h, D\boldsymbol{\delta}_2\rangle| \\
&= 2|\langle \nabla\mathbf{G}_2^h, \nabla(\Pi_h\mathbf{u}-\mathbf{u}_h)\rangle - \langle \operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h), \lambda_2^h\rangle| \\
&= 2|\langle \nabla(\mathbf{G}_2^h-\mathbf{G}_2), \nabla(\Pi_h\mathbf{u}-\mathbf{u})\rangle + \langle \operatorname{div}(\mathbf{G}_2^h-\mathbf{G}_2), p-P_hp\rangle \\
&\qquad - \langle \Delta\mathbf{G}_2, \Pi_h\mathbf{u}-\mathbf{u}\rangle|.
\end{aligned}$$

This last expression is similar to that in (4.3), where $\mathbf{G}_2$ and $\mathbf{G}_2^h$ are replaced by $\mathbf{G}_1$ and $\mathbf{G}_1^h$; therefore, we can proceed as in the proof of Theorem 4.2. Indeed, using Lemma 3.2, we obtain

$$\|D(\Pi_h\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \le C|\log h|^{1/2}\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\}.$$

Thus, the theorem follows by applying the triangular inequality. □

We conclude this subsection with a theorem concerning the uniform approximation of the pressure.

THEOREM 4.4. *Under the assumptions of Theorem* 4.2 *there exists a constant* $C>0$ *such that*

$$\|p-p_h\|_{L^\infty} \le C|\log h|^{1/2}\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\}. \tag{4.9}$$

Proof. It is sufficient to estimate P_hp-p_h. Let $\mathbf{x}_3$ be a point where the maximum of $|P_hp-p_h|$ is attained. With the notation of Section 3, we have from (3.11)

$$\begin{aligned}
\|P_hp-p_h\|_{L^\infty} &\le 2|\langle P_hp-p_h, \delta_3\rangle| \\
&\le 2|\langle P_hp-p_h, \delta_3-\phi\rangle| + 2|\langle P_hp-p_h, \phi\rangle|.
\end{aligned} \tag{4.10}$$

The second term in (4.10) can be bounded by using the L^2-estimate (1.5) in the following way:

$$\begin{aligned}
|\langle P_hp-p_h, \phi\rangle| &\le \|P_hp-p_h\|_{L^2}\|\phi\|_{L^2} \le C\|P_hp-p_h\|_{L^2} \\
&\le C\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^2} + \|p-P_hp\|_{L^2}\}.
\end{aligned}$$

To bound the first term in (4.10) we use the regularized Green's functions $(\mathbf{G}_3, \lambda_3)$ defined in (3.15). Then,

$$\begin{aligned}\langle P_h p - p_h, \delta_3 - \phi\rangle &= \langle P_h p - p_h, \operatorname{div} \mathbf{G}_3\rangle = \langle P_h p - p_h, \operatorname{div} \mathbf{G}_3^h\rangle \\ &= \langle P_h p - p, \operatorname{div} \mathbf{G}_3^h\rangle + \langle p - p_h, \operatorname{div} \mathbf{G}_3^h\rangle.\end{aligned}$$

Now, the error equations for $(\mathbf{u}, p)$ and $(\mathbf{G}_3, \lambda_3)$, as well as the property (1.8) of Π_h, yield

$$\begin{aligned}&\langle P_h p - p_h, \delta_3 - \phi\rangle \\ &\quad = \langle P_h p - p, \operatorname{div}(\mathbf{G}_3^h - \mathbf{G}_3)\rangle + \langle P_h p - p, \delta_3 - \phi\rangle \\ &\qquad + \langle \nabla(\mathbf{u} - \mathbf{u}_h), \nabla(\mathbf{G}_3^h - \mathbf{G}_3)\rangle + \langle \operatorname{div}(\mathbf{u} - \mathbf{u}_h), \lambda_3 - \lambda_3^h\rangle \\ &\quad = \langle P_h p - p, \operatorname{div}(\mathbf{G}_3^h - \mathbf{G}_3)\rangle + \langle P_h p - p, \delta_3 - \phi\rangle \\ &\qquad + \langle \nabla(\mathbf{u} - \Pi_h \mathbf{u}), \nabla(\mathbf{G}_3^h - \mathbf{G}_3)\rangle + \langle \operatorname{div}(\mathbf{u} - \Pi_h \mathbf{u}), \lambda_3 - P_h \lambda_3\rangle.\end{aligned}$$

Then, applying the Hölder inequality, we get

$$\begin{aligned}|\langle P_h p - p_h, \delta_3 - \phi\rangle| \le C\{&\|\nabla(\mathbf{G}_3 - \mathbf{G}_3^h)\|_{L^1} + \|\delta_3 - \phi\|_{L^1} + h\|\nabla\lambda_3\|_{L^1}\} \\ &\cdot \{\|\nabla(\mathbf{u} - \Pi_h \mathbf{u})\|_{L^\infty} + \|p - P_h p\|_{L^\infty}\}.\end{aligned}$$

Again, by the Hölder inequality, we have

$$\|\nabla(\mathbf{G}_3 - \mathbf{G}_3^h)\|_{L^1} \le C|\log|^{1/2}\|\nabla(\mathbf{G}_3 - \mathbf{G}_3^h)\|_{\sigma^2}$$

and

$$h\|\nabla\lambda_3\|_{L^1} \le Ch|\log h|^{1/2}\|\nabla\lambda_3\|_{\sigma^2}.$$

Finally, the estimate (4.9) follows from the last three inequalities in conjunction with Theorem 4.1 and Lemma 3.3. □

4.2. *Proof of Theorem* 4.1. To complete the error analysis, it remains to prove the crucial and rather technical Theorem 4.1. Let us start by defining

$$\psi := \sigma^2(\mathbf{G} - \mathbf{G}^h). \tag{4.11}$$

Thus,

$$\begin{aligned}\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}^2 &= \langle \nabla(\mathbf{G} - \mathbf{G}^h), \sigma^2 \nabla(\mathbf{G} - \mathbf{G}^h)\rangle \\ &= \langle \nabla(\mathbf{G} - \mathbf{G}^h), \nabla\psi\rangle + \tfrac{1}{2}\langle \mathbf{G} - \mathbf{G}^h, \Delta\sigma^2 \cdot (\mathbf{G} - \mathbf{G}^h)\rangle \\ &\le \langle \nabla(\mathbf{G} - \mathbf{G}^h), \nabla\psi\rangle + Ch^4\{\|D^2\mathbf{G}\|_{L^2} + \|\nabla\lambda\|_{L^2}\}^2,\end{aligned} \tag{4.12}$$

where we have used the property (2.5) and the L^2-estimate (1.6). To bound the first term in (4.12), we make use of the error equation for $(\mathbf{G}, \lambda)$ and the basic assumption (1.8) on the operator Π_h. We thus have

$$\begin{aligned}\langle \nabla(\mathbf{G} - \mathbf{G}^h), \nabla\psi\rangle = \langle &\nabla(\mathbf{G} - \mathbf{G}^h), \nabla(\psi - \Pi_h\psi)\rangle \\ &+ \langle \operatorname{div}(\Pi_h\psi - \psi), \lambda - P_h\lambda\rangle + \langle \operatorname{div}\psi, \lambda - \lambda^h\rangle.\end{aligned}$$

Inserting this expression into (4.12) results in

$$\begin{aligned}\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}^2 \le\ & Ch^4\{\|D^2\mathbf{G}\|_{L^2} + \|\nabla\lambda\|_{L^2}\}^2 + \varepsilon\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}^2 \\ &+ Ch^2\|\nabla\lambda\|_{\sigma^2}^2 + C\|\nabla(\psi - \Pi_h\psi)\|_{\sigma^{-2}}^2 + |\langle \operatorname{div}\psi, \lambda - \lambda^h\rangle|,\end{aligned}$$

where we have used the interpolation error bound (2.8); here, ε denotes a small positive number independent of h to be determined later on. To proceed, we have to evaluate the last two terms in the previous expression. These are a consequence of Lemmas 4.1 and 4.2 below, which imply

$$\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}^2 \le C_K h^4\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}^2 + C_K h^2\{\|D^2\mathbf{G}\|_{\sigma^2}+\|\nabla\lambda\|_{\sigma^2}\} + \left(\varepsilon+\frac{C}{K}\right)\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}^2.$$

Consequently, a proper choice of the constants $\varepsilon > 0$ and $K \ge 1$ allows the last term to be hidden into the left-hand side. The assertion is then obtained. □

Let us now prove the auxiliary results mentioned above.

LEMMA 4.1. *Let K be as in* (2.3) *and $\boldsymbol{\psi}$ as in* (4.11). *Then there exist constants $C_k > 0$ and $C > 0$ independent of K such that*

$$\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|_{\sigma^{-2}} \le C_K h^2\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\} + C_K h\{\|D^2\mathbf{G}\|_{\sigma^2}+\|\nabla\lambda\|_{\sigma^2}\} + \frac{C}{K}\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}. \tag{4.13}$$

Proof. Let us rewrite $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} = \sigma^2(\mathbf{G}-\mathbf{G}^h) = \sigma^2(\mathbf{G}-\Pi_h\mathbf{G}) + \sigma^2(\Pi_h\mathbf{G}-\mathbf{G}^h) =: \boldsymbol{\psi}_1+\boldsymbol{\psi}_2.$$

Then

$$\|\nabla(\boldsymbol{\psi}-\Pi_h\boldsymbol{\psi})\|_{\sigma^{-2}} \le \|\nabla(\boldsymbol{\psi}_1-\Pi_h\boldsymbol{\psi}_1)\|_{\sigma^{-2}} + \|\nabla(\boldsymbol{\psi}_2-\Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}}. \tag{4.14}$$

In view of (2.10), the first term on the right-hand side of (4.14) is bounded by $\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}}$. Using now the properties (2.5) and (2.7) yields

$$\begin{aligned}\|\nabla\boldsymbol{\psi}_1\|_{\sigma^{-2}} &\le C\{\|\mathbf{G}-\Pi_h\mathbf{G}\|_{L^2}+\|\nabla(\mathbf{G}-\Pi_h\mathbf{G})\|_{\sigma^2}\}\\ &\le Ch^2\|D^2\mathbf{G}\|_{L^2}+Ch\|D^2\mathbf{G}\|_{\sigma^2}.\end{aligned}$$

For the second term in (4.14) we use (2.12) to get

$$\begin{aligned}\|\nabla(\boldsymbol{\psi}_2-\Pi_h\boldsymbol{\psi}_2)\|_{\sigma^{-2}} &\le Ch\|\Pi_h\mathbf{G}-\mathbf{G}^h\|_{\sigma^{-2}}+Ch\|\nabla(\Pi_h\mathbf{G}-\mathbf{G}^h)\|_{L^2}\\ &\le \frac{C}{K}\|\Pi_h\mathbf{G}-\mathbf{G}^h\|_{L^2}+\frac{C}{K}\|\nabla(\Pi_h\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}\\ &\le \frac{C}{K}h^2\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}+\frac{C}{K}h\|D^2\mathbf{G}\|_{\sigma^2}+\frac{C}{K}\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}.\end{aligned}$$

Here we have employed standard L^2-error estimates together with the estimate (2.7). Finally, substituting the bounds corresponding to $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ into (4.14) implies the desired result. □

LEMMA 4.2. *Let $\boldsymbol{\psi}$ be as in* (4.11). *Then there exists a constant $C > 0$ such that*

$$|\langle \operatorname{div}\boldsymbol{\psi}, \lambda-\lambda^h\rangle| \le Ch^2\{\|D^2\mathbf{G}\|_{\sigma^2}+\|\nabla\lambda\|_{\sigma^2}\}^2 + Ch^4\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}^2+\varepsilon\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}^2. \tag{4.15}$$

Proof. Since

$$\operatorname{div}\boldsymbol{\psi} = \nabla\sigma^2\cdot(\mathbf{G}-\mathbf{G}^h)+\sigma^2\operatorname{div}(\mathbf{G}-\mathbf{G}^h),$$

we have

$$\begin{aligned}\langle \operatorname{div}\boldsymbol{\psi}, \lambda-\lambda^h\rangle &= \langle \nabla\sigma^2\cdot(\mathbf{G}-\mathbf{G}^h), \lambda-\lambda^h\rangle + \langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \sigma^2(\lambda-\lambda^h)\rangle\\ &= \gamma\langle\phi, \lambda-\lambda^h\rangle+\langle g, \lambda-\lambda^h\rangle+\langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \sigma^2(\lambda-\lambda^h)\rangle.\end{aligned} \tag{4.16}$$

Here we have set

$$\gamma := \int_\Omega \nabla\sigma^2 \cdot (\mathbf{G} - \mathbf{G}^h)\, dx \tag{4.17}$$

and

$$g := \nabla\sigma^2 \cdot (\mathbf{G} - \mathbf{G}^h) - \gamma\phi, \tag{4.18}$$

where $\phi \in C_0^\infty(\Omega)$ stands for the function defined in Section 3. As ϕ is chosen independent of h, so are $\|\phi\|_{L^2}$ and $\|\nabla\phi\|_{L^2}$. The contribution due to the first term in (4.16) is now easily evaluated on account of the bound for $|\gamma|$ proved in Lemma 4.3 below. Indeed, we can write

$$\begin{aligned}\gamma\langle\phi, \lambda - \lambda^h\rangle &\le Ch\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}\|\lambda - \lambda^h\|_{L^2}\\ &\le Ch^4\{\|\nabla\lambda\|_{L^2}^2 + \|D^2\mathbf{G}\|_{L^2}^2\} + \varepsilon\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}^2.\end{aligned}$$

The contribution due to the remaining terms in (4.16) is analyzed below in Lemmas 4.4 and 4.5. This completes the proof. □

In order to complete the whole argument it only remains to derive some auxiliary estimates related to expression (4.16). To begin with, let us first obtain a bound for $|\gamma|$.

LEMMA 4.3. *Let γ be defined by* (4.17). *Then there exists a constant $C > 0$ such that*

$$|\gamma| \le Ch\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}. \tag{4.19}$$

Proof. Integrating by parts yields

$$|\gamma| = |\langle \operatorname{div}(\mathbf{G} - \mathbf{G}^h), \sigma^2\rangle| = |\langle \operatorname{div}(\mathbf{G} - \mathbf{G}^h), \sigma^2 - P_h\sigma^2\rangle| \le Ch\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2},$$

where we have used the second error equation in (4.1) for the function $\mathbf{G}$. This is the desired bound. □

The second step consists of analyzing the middle term in (4.16):

LEMMA 4.4. *Let g be defined by* (4.18). *Then there exists a constant $C > 0$ such that*

$$|\langle g, \lambda - \lambda^h\rangle| \le Ch^4\{\|D^2\mathbf{G}\|_{L^2} + \|\nabla\lambda\|_{L^2}\}^2 + \varepsilon\|\nabla(\mathbf{G} - \mathbf{G}^h)\|_{\sigma^2}^2. \tag{4.20}$$

Proof. We use a duality argument. Consider the following auxiliary Stokes problem,

$$\begin{cases} -\Delta\mathbf{w} + \nabla\eta = 0 & \text{in } \Omega,\\ \operatorname{div}\mathbf{w} = g & \text{in } \Omega,\\ \mathbf{w} = 0 & \text{on } \partial\Omega.\end{cases} \tag{4.21}$$

Since $g \in H_0^1(\Omega)$ and $\int_\Omega g\, dx = 0$, we can apply the regularity result (3.4) for the problem (4.21) in a plane polygon to get

$$\mathbf{w} \in [H^2(\Omega) \cap H_0^1(\Omega)]^2, \qquad \eta \in H^1(\Omega) \cap L_0^2(\Omega)$$

and

$$\|D^2\mathbf{w}\|_{L^2} + \|\nabla\eta\|_{L^2} \le C\|\nabla g\|_{L^2}. \tag{4.22}$$

We thus have

$$\begin{aligned}|\langle g, \lambda-\lambda^h\rangle| &\leq |\langle \operatorname{div}\mathbf{w}, \lambda-\lambda^h\rangle| \\ &= |\langle \operatorname{div}(\mathbf{w}-\mathbf{w}_h), \lambda-\lambda^h\rangle + \langle \nabla(\mathbf{G}-\mathbf{G}^h), \nabla(\mathbf{w}-\mathbf{w}_h)\rangle \\ &\qquad + \langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \eta_h-\eta\rangle|\end{aligned}$$

as a consequence of the error equations for $(\mathbf{G},\lambda)$. Therefore, by (4.22) and the error estimate (1.5) we can write

$$|\langle g, \lambda-\lambda^h\rangle| \leq Ch^2\|\nabla g\|_{L^2}\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}. \tag{4.23}$$

Now, using the definition (4.18) of g and the L^2-estimate (1.6), we get

$$\|\nabla g\|_{L^2} \leq Ch^2\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\} + C\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2} + C|\gamma|. \tag{4.24}$$

Finally, combining (4.23) and (4.24) with (4.19) yields the assertion. □

The third and last step deals with the remaining term in (4.16).

LEMMA 4.5. *There exists a constant $C>0$ such that*

$$\begin{aligned}&|\langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \sigma^2(\lambda-\lambda^h)\rangle| \\ &\quad \leq Ch^2\|\nabla\lambda\|_{\sigma^2}^2 + Ch^4\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}^2 + \varepsilon\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}^2.\end{aligned} \tag{4.25}$$

Proof. Let $\varsigma := \sigma^2(P_h\lambda-\lambda^h)$. Using the error equations for $(\mathbf{G},\lambda)$, we have

$$\begin{aligned}&|\langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \sigma^2(\lambda-\lambda^h)\rangle| \\ &\quad = |\langle \operatorname{div}(\mathbf{G}-\mathbf{G}^h), \sigma^2(\lambda-P_h\lambda)+\varsigma-P_h\varsigma\rangle| \\ &\quad \leq Ch\|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}\|\nabla\lambda\|_{\sigma^2} + \|\nabla(\mathbf{G}-\mathbf{G}^h)\|_{\sigma^2}\|\varsigma-P_h\varsigma\|_{\sigma^{-2}}.\end{aligned}$$

Applying (1.5) and (2.11), we can further write

$$\|\varsigma-P_h\varsigma\|_{\sigma^{-2}} \leq Ch\|P_h\lambda-\lambda^h\|_{L^2} \leq Ch^2\{\|D^2\mathbf{G}\|_{L^2}+\|\nabla\lambda\|_{L^2}\}.$$

Therefore, (4.25) follows from the two previous inequalities. The lemma is thus proved. □

5. Applications. The aim of this section is to apply our general results to some low-order finite element approximations to the Stokes equations. To begin with, let us first consider *continuous pressure* finite elements which are those preferred by engineers.

Example 5.1. MINI ELEMENT. It was introduced by D. Arnold, F. Brezzi and M. Fortin [2] as a remedy for the unstable P_1-P_1 element. The key idea was to enrich the velocity space $\mathbf{P}_1$ by adding bubble functions; this new space is denoted by $\mathbf{P}_1^+$. Then the discrete spaces are:

$$X_h|_T := \mathbf{P}_1^+(T), \qquad M_h|_T := P_1(T), \quad \forall T\in\tau_h. \tag{5.1}$$

Therefore, the order of this approximation is $k=1$, whereas $m=2$ according to (2.1) and (2.2). The local operator Π_h was explicitly built in [2] as a way to show the inf-sup condition (1.4).

The mini element is the simplest one of this class. The next one is the popular Taylor-Hood element, for which the existence of a local Π_h is not known. However, the above trick led D. Arnold, F. Brezzi and M. Fortin to a slightly bigger element, namely

Example 5.2. ENRICHED TAYLOR-HOOD ELEMENT. Now the discrete spaces are

$$X_h|_T := \mathbf{P}_2^+(T), \qquad M_h|_T := P_1(T), \quad \forall T \in \tau_h. \tag{5.2}$$

We thus have $k = 2$ and $m = 1$. The computational labor involved is comparable to that for the classical Taylor-Hood element, because the internal nodes in each element can be easily eliminated by the process of static condensation. On the other hand, the computational results seem to be better [5].

Let us now turn our attention to *discontinuous pressure* approximations.

Example 5.3. BERNARDI-RAUGEL ELEMENT. Let the discrete spaces be defined by

$$X_h|_T := \mathbf{P}_1(T) \oplus \operatorname{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}, \qquad M_h|_T := P_0(T) \quad \forall T \in \tau_h. \tag{5.3}$$

Here, $\mathbf{p}_i$ stands for the polynomials $\mathbf{p}_1 = \lambda_2\lambda_3\boldsymbol{\nu}_1$, $\mathbf{p}_2 = \lambda_1\lambda_3\boldsymbol{\nu}_2$, $\mathbf{p}_3 = \lambda_1\lambda_2\boldsymbol{\nu}_3$, where λ_i are the barycentric coordinates and $\boldsymbol{\nu}_i$ are normal vectors to the edges opposite to vertices i. We then have $k = 1$ and $m = 1$. This element was presented in [3] and may be regarded as a simplification of the classical $P_2 - P_0$ element [5], [15], which also fits in our theory.

In the latter case, the velocity space is much bigger than necessary for a first-order approximation to be stable. However, taking discontinuous P_1-polynomials for pressure, results in an unstable scheme. The difficulty is circumvented by simply augmenting the velocity space with bubble functions; we refer to [9], [5], [15] for more details.

Example 5.4. CROUZEIX-RAVIART ELEMENT. Let now X_h and M_h be defined by

$$X_h|_T := \mathbf{P}_2^+(T), \qquad M_h|_T := P_1(T) \quad \forall T \in \tau_h; \tag{5.4}$$

thus $k = 2$ and $m = 1$. The idea behind this choice was further exploited by F. Brezzi and J. Pitkäranta [6] who suggested a general stabilization technique; in particular, (5.4) is a consequence of their results.

The families already mentioned are all defined over triangular decompositions of Ω. Let us now consider some stable quadrilateral elements. The simplest case is the couple $Q_2 - Q_0$, for which $k = 1$ and $m = 3$; here and below, Q_i denotes the set of polynomials of degree at most i in each variable separately. The existence of a local operator Π_h can be proved along the same lines as for $P_2 - P_0$ elements. Since Q_2 is actually too big, we might expect the pressure space to be enriched without loosing stability. This is what indeed happens.

Example 5.5. $Q_2 - P_1$ ELEMENT. Let us set

$$X_h|_T := \mathbf{Q}_2(T), \qquad M_h|_T := P_1(T), \quad \forall T \in \tau_h, \tag{5.5}$$

which yields $k = 2$ and $m = 2$. The analysis of this element is much the same as that of the Crouzeix-Raviart element [5].

We now conclude the paper by establishing the rates of convergence in L^∞ of the finite element approximations described above.

COROLLARY 5.1. *Assume that Ω is a plane convex polygonal domain. Let $\mathbf{u} \in [W^{k+1,\infty}(\Omega)]^2$ and $p \in W^{k,\infty}(\Omega)$ be the solutions of the Stokes problem* (1.1).

Let $\mathbf{u}_h$ and p_h be the discrete solutions corresponding to any of the above families. Then

$$
\begin{aligned}
h^{-1}|\log h|^{-1/2}\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty} + \|p-p_h\|_{L^\infty} + \|\operatorname{curl}(\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \\
\leq Ch^k|\log h|^{1/2}(\|\mathbf{u}\|_{W^{k+1,\infty}(\Omega)} + \|p\|_{W^{k,\infty}(\Omega)}).
\end{aligned} \tag{5.6}
$$

Proof. The proof of (5.6) is an easy application of (4.7), (4.8) and (4.9). We thus omit the details. □

For (5.6) to hold, the continuous solutions must satisfy an a priori regularity which is difficult to check. An attempt to weaken this constraint is the following result, which holds only under a proper regularity of $\partial\Omega$.

COROLLARY 5.2. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,p_h)$ be the continuous and discrete solutions of the Stokes problem. For $\partial\Omega \in C^{k+1}$ we have*

$$
\begin{aligned}
h^{-1}|\log h|^{-1/2}\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty} + \|p-p_h\|_{L^\infty} + \|\operatorname{curl}(\mathbf{u}-\mathbf{u}_h)\|_{L^\infty} \\
\leq Ch^k|\log h|^{3/2}\|\mathbf{f}\|_{W^{k-1,\infty}(\Omega)}.
\end{aligned} \tag{5.7}
$$

Proof. In view of Theorems 4.2, 4.3 and 4.4, the left-hand side of (5.7) is bounded by

$$I = C|\log h|^{1/2}\{\|\nabla(\mathbf{u}-\Pi_h\mathbf{u})\|_{L^\infty} + \|p-P_hp\|_{L^\infty}\}.$$

As $\mathbf{f} \in [W^{k-1,\infty}(\Omega)]^2$, the a priori estimate (3.7) combined with standard interpolation error estimates yields

$$I \leq Ch^k|\log h|^{1/2}sh^{-2/s}\|\mathbf{f}\|_{W^{k-1,\infty}}, \qquad 2 \leq s < \infty.$$

Taking now $s = |\log h|$ implies the desired result. □

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