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A Curiosity Concerning the Representation of Integers in Noninteger Bases

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Abstract. In this note it is shown how an integer x can be represented uniquely in a noninteger basis provided the "digits" of the representation are allowed to be nonintegers. It is then shown that the integer parts of these "digits" contain all the information necessary to recover x.

Let β be an integer greater than one and let x_0 be a nonnegative integer. Then x_0 has a unique base-β representation of the form

$$x_0 = d_0 + d_1\beta + d_2\beta^2 + \cdots + d_n\beta^n, \tag{1}$$

where the *digits* d_i are integers satisfying

$$0 \le d_i < \beta \qquad (i = 0, 1, \ldots, n). \tag{2}$$

When we relax the restriction that β be an integer and allow it to be a real number greater than one, we must also relax the restriction that the d_i be integers, in which case x_0 can be represented in infinitely many different ways in the form (1) where the digits satisfy (2). To obtain a unique representation, we must impose additional conditions. Perhaps the most natural is to demand that in addition to (2) the quantities x_i defined by

$$\begin{aligned} x_n &= d_n, \\ x_i &= d_i + \beta x_{i+1} \qquad (i = n-1, n-2, \ldots, 0) \end{aligned} \tag{3}$$

all be integers. In this case d_0 will be the unique nonnegative number less than β such that $x_0 - d_0$ is an integral multiple of β, and x_1 will then be $(x_0 - d_0)/\beta$. In general d_i will be the unique nonnegative number less than β such that $x_i - d_i$ is an integral multiple of β, and x_{i+1} will then be $(x_i - d_i)/\beta$. The naturalness comes from the fact that these statements characterize the integers d_i when β is an integer.*

As we noted above, when β is not an integer, the d_i in the expansion (1) are digits by courtesy only, since they are not integers. In fact they may be nonterminating decimals. Curiously enough, only the integer parts of the d_i are needed to determine x_0. Specifically, let

$$d'_i = \lfloor d_i \rfloor.$$

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*The referee has provided an equivalent characterization beginning "d_0 is the distance from x_0 to the largest multiple of β not greater than x_0, and x_1 is that multiple." Some may find this more natural.

Then the sequence x_0, $x_1, \ldots, x_n$ *is uniquely determined by the sequence* d'_0, $d'_1, \ldots, d'_n$.

To see this, first note that the requirement that x_n be an integer implies that $d_n = d'_n$. Now suppose that we have determined x_{i+1}, and set

$$x'_i = d'_i + \beta x_{i+1}.$$

Then $x_i - x'_i = d_i - d'_i < 1$. Since $x_i \geq x'_i$, it follows that $x_i = \lceil x'_i \rceil$.

There are two comments to be made about this result. First, if β were an integer with $2^t < \beta \leq 2^{t+1}$, then the digits of a base-β representation of a number would require $t+1$ bits for their binary representations. We have shown that for noninteger bases we can define integer digits d'_i representable by the same number of bits. Moreover, the integer x_0 can be evaluated by the recursion

$$\begin{aligned} x_n &= d'_n, \\ x_i &= \lceil d'_i + \beta x_{i+1} \rceil \qquad (i = n-1, n-2, \ldots, 0), \end{aligned}$$

which reduces to (3) when β is an integer.

The second comment concerns *subbinary representation*, where $1 < \beta < 2$. Here the digits d'_i are zeros or ones, and each zero digit causes the current x_i to increase by at least one. As β approaches one, the proportion of zero digits increases, until the representation reduces to a one followed by $x_0 - 1$ zeros, which may properly be called a base-one representation of x_0.

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