

On Four-Dimensional Terminal Quotient Singularities*

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Abstract. We report on an investigation of four-dimensional terminal cyclic quotient singularities which are not Gorenstein. (For simplicity, we focus on quotients by cyclic groups of prime order.) An enumeration, using a computer, of all such singularities for primes < 1600 led us to conjecture a structure theorem for these singularities (which is rather more complicated than the known structure theorem in dimension three). We discuss this conjecture and our evidence for it; we also discuss properties of the anticanonical and antibicanonical linear systems of these singularities.

Introduction. The recent successes in understanding the birational geometry of algebraic varieties of dimension greater than two have focused attention on a class of singularities (called *terminal* singularities) which appear on the birational models which the theory selects. In dimension three, the structure of these terminal singularities is known in some detail: The terminal quotient singularities were classified, in what is now called the “terminal lemma”, by several groups of people working independently (cf. [1], [2], [5], [8]), and all other three-dimensional terminal singularities were subsequently classified by work of the first author [6] and of Kollár and Shepherd-Barron [4]. Both of these classifications were based on key results of Reid [10], [11] who reduced the problem to an analysis of quotients of smooth points and double points by cyclic group actions. (A detailed account of the classification can be found in [12].)

A consequence of this classification, apparently first observed by Reid [12], is that for any three-dimensional terminal singularity T , the general element of the anticanonical linear system $|-K_T|$ has only canonical singularities. This in turn implies that if we form the double cover of T branched on the general antibicanonical divisor $D \in |-2K_T|$, that double cover will also have only canonical singularities. At first glance, this second property appears to have no advantages over the first, but looking at things in these terms proved decisive in a slightly more global context: Kawamata [3] showed that if T is an “extremal neighborhood” of a rational curve C on a threefold, then the existence of such a divisor D globally on T is sufficient to conclude the existence of a certain kind of birational modification (a “directed flip”) centered on C . The first author [7] showed that such divisors always exist, completing the proof of the Minimal Model Theorem for threefolds. (See [7] for more details.)

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All of this is by way of explaining why we chose to concentrate on properties of $|-K_T|$ and $|-2K_T|$ when we began to study terminal singularities in dimension four. The detailed structure of four-dimensional terminal singularities is still unknown; even for quotient singularities, the number-theoretic methods used in the various proofs of the terminal lemma appear to give no information in higher dimension. We restrict our attention here to non-Gorenstein four-dimensional quotient singularities which are quotients by groups of prime order. (The four-dimensional Gorenstein cases were completely classified some time ago by the second author and G. Stevens [8].) This class of singularities is a natural place to begin an investigation: Among other pleasant features, computations with cyclic quotient singularities can be made readily, by using the techniques of toric geometry. In particular, to decide whether a given quotient singularity is terminal requires only a finite computation (of a distinctly number-theoretic character).

We easily found (by hand) examples for which the singularities of the general anticanonical divisor are worse than canonical. (An infinite number of examples is given in Example 2.5 below.) To find examples for which $|-2K_T|$ is badly behaved was more difficult, and we turned to a machine computation. We did find such examples, but contrary to the expectations we formed after finding the first few, we have to date in a systematic search found exactly six examples with index less than 1600, and all of these have index between 83 and 109. Our examples and these computations are described in Section 2.

In order for $|-2K_T|$ to be well behaved, the terminal singularity T must pass one of two computational tests (corresponding to Corollary 2.8(ii) and Lemma 2.14 below). The second test is computationally very expensive, so we resorted to it as seldom as possible. And we were surprised to observe that once the index passed 420, exactly 20 singularities for each index required the use of this second test. We have arrived at an explanation for this phenomenon in the form of a conjectural “four-dimensional terminal lemma”, which we discuss in Section 1. Among other things, this approach allows us to handle the 20 cases we encountered experimentally in a systematic fashion (cf. Table 2.11). But in addition, the truth of the conjecture would imply that our six examples of terminal quotient singularities of prime index for which $|-2K_T|$ is badly behaved are the only ones.

The results of this paper are related to our computer calculations in several ways. First, Theorem 1.3, which describes a large class of terminal quotient singularities, could not have been formulated without the results of our computations: We discovered most of these singularities by examining the output of our programs. Second, the proofs of Theorems 1.3 and 2.12 are computer-aided. The computer assistance is rather trivial in the case of Theorem 1.3 (the required calculation could be done on a programmable hand-held calculator), but the proof of Theorem 2.12 requires the solution of a large number of linear programming problems. And finally, the calculations we made inspired our conjectural classification of these singularities, and have provided evidence for that conjecture.

1. A Conjectural Classification. In this section we wish to propose a conjectural classification of those four-dimensional terminal singularities which are quotients by cyclic groups of prime order. More precisely, we will write down a list of $\mathbf{Z}/p\mathbf{Z}$ -quotient singularities some of whose entries depend on one or two parameters

in $(\mathbf{Z}/p\mathbf{Z})^*$ —and verify that all the singularities on our list are terminal. Our conjecture is that these are exactly the isolated terminal $\mathbf{Z}/p\mathbf{Z}$ -quotient singularities of index p if $p \geq 421$.

First some notation. If m and k are integers, we write $\langle m \rangle_p$ or simply $\langle m \rangle$ for the unique element of $\{0, 1, 2, \dots, p - 1\}$ which is congruent to m modulo p and $m^{(k)}$ for $\langle mk \rangle$. Fix once and for all a generator σ of $\mathbf{Z}/p\mathbf{Z}$. If σ acts on \mathbf{C}^4 by the recipe

$$(w, x, y, z) \rightarrow (\zeta^a w, \zeta^b x, \zeta^c y, \zeta^d z) \quad \text{with } \zeta = e^{2\pi i/p},$$

then we will denote the corresponding $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity by $[a, b, c, d]_p$ and call a, b, c , and d the *weights* of the singularity. The singularity is isolated exactly when each of the weights is relatively prime to p .

Given a $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity $[a, b, c, d]_p$, we let

$$s = \langle a + b + c + d \rangle.$$

When we wish to emphasize the value of s , we will sometimes write $[s|a, b, c, d]_p$ for $[a, b, c, d]_p$. More generally, we let

$$s_k = a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)}.$$

(Note that while $s_k \equiv s^{(k)} \pmod{p}$, s_k need only lie between 0 and $4p$.) Reid ([10, Theorem 3.1]; cf. also [12, Theorem 4.6]) has given a criterion which characterizes terminal quotient singularities: The singularity $[a, b, c, d]_p$ is terminal if and only if

$$(1.1) \quad p < s_k \quad \text{for all } k \in \{1, 2, \dots, p - 1\}.$$

A $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity can be alternatively described by means of a quintuple $Q = (a, b, c, d, e)$ of integers such that $a + b + c + d + e = 0$ and $p > M_Q := \max\{|a|, |b|, |c|, |d|, |e|\}$. Such a quintuple gives rise to the singularity $[a, b, c, d]_p$ with $s \equiv -e \pmod{p}$. In these terms, the condition that $[a, b, c, d]_p$ be terminal can be expressed as

$$p + e^{(k)} < a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)}$$

for all $k \in \{1, 2, \dots, p - 1\}$. Since the expression on the right is a multiple of p , this condition is equivalent to

$$(1.2) \quad 2p \leq a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)}$$

for all $k \in \{1, 2, \dots, p - 1\}$. We say that the quintuple $Q = (a, b, c, d, e)$ is *p-terminal* when $p > M_Q$ and condition (1.2) is satisfied.

For computational purposes, a slight reformulation of conditions (1.1) and (1.2) is often convenient. We call a subset \mathcal{S} of $\{1, 2, \dots, p - 1\}$ a *CM-type*, if \mathcal{S} contains exactly one element from each of the pairs $(k, p - k)$. If the singularity $[a, b, c, d]_p$ is isolated, then condition (1.1) is equivalent to

$$p < s_k < 3p \quad \text{for all } k \text{ in some CM-type } \mathcal{S}.$$

(This follows from the fact that $m^{(k)} + m^{(p-k)} = p$ for any integer m relatively prime to p). Similarly, when each of a, b, c, d , and e is nonzero (and $p > M_Q$), condition (1.2) may be replaced by

$$2p \leq a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)} \leq 3p \quad \text{for all } k \in \mathcal{S}.$$

Given a p -terminal quintuple $Q = (a, b, c, d, e)$, since the “ p -terminal” condition is symmetric, each of the five $\mathbf{Z}/p\mathbf{Z}$ -quotient singularities $[a, b, c, d]_p$, $[a, b, c, e]_p$, $[a, b, d, e]_p$, $[a, c, d, e]_p$, and $[b, c, d, e]_p$ is terminal; we call these the *associated p -singularities* of Q . A quintuple $Q = (a, b, c, d, e)$ is called *stable* if it is p -terminal for all sufficiently large p . We say that a terminal singularity $[a, b, c, d]_p$ is *stable**** if it is associated with a stable quintuple $Q = (a, b, c, d, e)$ with $p > M_Q$; we call $[a, b, c, d]_p$ *sporadic* if it is not stable.

Every terminal singularity has an *index*; in the case of a quotient singularity \mathbf{C}^4/G , the index is the smallest natural number n with the property that the group G acts trivially on $(K_{\mathbf{C}^4})^{\otimes n}$. For a $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity $[a, b, c, d]_p$, the index is 1 if $s = 0$ (in this case, the singularity is Gorenstein), and the index is p if s is relatively prime to p .

If the entries in a quintuple Q are all nonzero and $p > M_Q$, then each associated p -singularity has index p . If one of the entries in Q is zero, then one of the associated p -singularities is isolated and Gorenstein, while the others are nonisolated. Now the classification of nonisolated terminal quotient singularities reduces to a problem in lower dimension, and is related (via the quintuples Q) to the classification of four-dimensional Gorenstein terminal quotient singularities. A complete classification in both of these cases is known [8]: Phrased in terms of quintuples, the classification says that each quintuple corresponding to Gorenstein and to nonisolated terminal quotient singularities must be equivalent mod p to one of the form $(\alpha, -\alpha, \beta, -\beta, 0)$.

For the remainder of this paper, we restrict our attention to isolated singularities of prime index. We will implicitly assume that each quotient singularity we consider is isolated, and that the order of the quotient group coincides with the index, thus, the phrase “terminal quotient singularity of index p ” serves as an abbreviation for “isolated terminal $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity of index p ”.

An extensive computer study of terminal quotient singularities of prime index led us to the discovery of some large classes of stable quintuples.

THEOREM 1.3. *Let Q be a quintuple of integers summing to zero, and let p be a prime number. Suppose that either*

- (a) $Q = (\alpha, -\alpha, \beta, \gamma, -\beta - \gamma)$ with $0 < |\alpha|, |\beta|, |\gamma| < p/2$, and $\beta + \gamma \neq 0$, or
- (b) $Q = (\alpha, -2\alpha, \beta, -2\beta, \alpha + \beta)$ with $0 < |\alpha|, |\beta| < p/2$, and $\alpha + \beta \neq 0$, or
- (c) Q is one of the 29 quintuples listed in Table 1.9 and $p > M_Q$.

Then Q is p -terminal.

It follows immediately that each of the quintuples described in the theorem is stable, and that each of their associated p -singularities is terminal and stable (when p satisfies the given restrictions). We call the 29 quintuples of case (c) *exceptional stable quintuples* because they do not fit into infinite families like those of cases (a) and (b). These two infinite families are in fact characterized by certain linear relations among a, b, c, d , and e which will be important in Section 2. (Specifically, we have $a + b = c + d + e = 0$ in case (a) and $2a + b = 2c + d = a + c - e = 0$ in case (b).)

Before proving this theorem, we wish to state the conjecture about terminal quotient singularities suggested by our computer calculations. To do so, we first

***This is unrelated to the notions of stability arising in geometric invariant theory.

note that the group $(\mathbf{Z}/p\mathbf{Z})^*$ acts on the set of $\mathbf{Z}/p\mathbf{Z}$ -quotient singularities by

$$[s|a, b, c, d]_p \rightarrow [s^{(k)}|a^{(k)}, b^{(k)}, c^{(k)}, d^{(k)}]_p,$$

and the symmetric group S^4 acts by permuting the weights. Both these actions preserve the subset of isolated terminal singularities of index p .

CONJECTURE 1.4 (four-dimensional terminal lemma). *Fix $p \geq 421$. Up to the actions of $(\mathbf{Z}/p\mathbf{Z})^*$ and S^4 , each isolated four-dimensional terminal $\mathbf{Z}/p\mathbf{Z}$ -quotient singularity of index p is associated with one of the p -terminal quintuples given in Theorem 1.3.*

Note that the descriptions of our infinite families (cases (a) and (b) in Theorem 1.3) are somewhat redundant when we consider the actions by $(\mathbf{Z}/p\mathbf{Z})^*$ and S^4 : By a further action of $(\mathbf{Z}/p\mathbf{Z})^*$, for example, we could have assumed that $\alpha = 1$. However, it will be more convenient to work with the larger families of quintuples as we have described them.

Our conjecture says in part that every terminal quotient singularity of index p is stable when $p \geq 421$. However, for most $p < 421$, there definitely exist sporadic terminal quotient singularities of index p ; these will be discussed further at the end of this section. Some of the sporadic terminal singularities will play an important role in Section 2, where we will also give some explicit examples.

We now turn to the proof of Theorem 1.3; the proof is a case analysis. Observe that $p > M_Q$ in each case, so that we only need to verify condition (1.2).

Case (a). This is the case analogous to the one occurring in dimension 3, where the terminal lemma asserts that two of the weights of any terminal quotient singularity must sum to the index p (or equivalently, that any p -terminal quadruple is equivalent mod p to one of the form $(\alpha, -\alpha, \beta, -\beta)$). In our case, since $\alpha^{(k)} + (-\alpha)^{(k)} = p$ and $\beta^{(k)} + \gamma^{(k)} + (-\beta - \gamma)^{(k)} = p$ or $2p$ for all $k \in \{1, 2, \dots, p - 1\}$, the p -terminal condition (1.2) is immediate.

The remaining cases have no analogues in dimension less than 4.

Case (b). If we act on the quintuple $(\alpha, -2\alpha, \beta, -2\beta, \alpha + \beta)$ by an element of $(\mathbf{Z}/p\mathbf{Z})^*$ then, possibly after modifying some of the entries of the resulting quintuple by multiples of p , we get another quintuple of the same form. Thus, it suffices to show that

$$2p \leq \langle \alpha \rangle + \langle -2\alpha \rangle + \langle \beta \rangle + \langle -2\beta \rangle + \langle \alpha + \beta \rangle.$$

Write $\langle -2\alpha \rangle = (j + 1)p - 2\langle \alpha \rangle$, $\langle -2\beta \rangle = (j + 1)p - 2\langle \beta \rangle$, and $\langle \alpha + \beta \rangle = (k - 1)p + \langle \alpha \rangle + \langle \beta \rangle$, so that $0 \leq i, j, k \leq 2$. Then

$$\langle \alpha \rangle + \langle -2\alpha \rangle + \langle \beta \rangle + \langle -2\beta \rangle + \langle \alpha + \beta \rangle = (i + j + k + 1)p.$$

But if $i = j = k = 0$, then $\langle \alpha \rangle < p/2$, $\langle \beta \rangle < p/2$, and $\langle \alpha \rangle + \langle \beta \rangle > p$, a contradiction. Thus, $i + j + k + 1 \geq 2$, verifying the condition.

Case (c). *Stable Quintuples.* Let $Q = (a, b, c, d, e)$ be a quintuple of integers summing to zero, let l_Q be the least common multiple of $\{|a|, |b|, |c|, |d|, |e|\}$ and M_Q be the maximum of this last set. To state our main result about stable quintuples, we need a definition. Let

$$R_Q(x) = \lim_{y \rightarrow x^+} ([ay] + [by] + [cy] + [dy] + [ey]),$$

with $[z]$ denoting the greatest integer less than or equal to z .

PROPOSITION 1.5. *The following are equivalent:*

- (a) $Q = (a, b, c, d, e)$ is stable;
- (b) $-3 \leq R_Q(i/l_Q) \leq -2$, for $i = 1, 2, \dots, l_Q - 1$.

Proof. For each integer m dividing l_Q , the function $y \mapsto [my]$ is constant on the open intervals $(i/l_Q, i + 1/l_Q)$. Thus, the function

$$x \mapsto \lim_{y \rightarrow x^+} [my]$$

is constant on the half-open intervals $[i/l_Q, i + 1/l_Q)$. (The limit is needed when m is negative.) In particular, the function R_Q is constant on such intervals, so that

$$(1.6) \quad R_Q(x) = R_Q([xl_Q]/l_Q).$$

Therefore, the proposition will follow if we show

Claim 1.7. *If $p > M_Q$ and $0 < k < p$, then $a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)} = -R_Q(k/p)p$.*

To see the claim, note that $m^{(k)}/p + [mk/p] = mk/p$ for any nonzero integer m ; moreover, since $p > M_Q$, mk/p is never of the form i/l_Q . Thus,

$$\begin{aligned} R_Q(k/p) + (a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)})/p \\ &= [ak/p] + [bk/p] + [ck/p] + [dk/p] + [ek/p] \\ &\quad + (a^{(k)} + b^{(k)} + c^{(k)} + d^{(k)} + e^{(k)})/p \\ &= ak/p + bk/p + ck/p + dk/p + ek/p \\ &= 0. \end{aligned}$$

This proves the claim, and hence the proposition. \square

Claim (1.7) has a nice

COROLLARY 1.8. *If Q is p -terminal for any prime $p > l_Q$, then Q is stable. Moreover, in this case Q is p -terminal for every $p > M_Q$.*

Proof. The terminal condition for a given prime p implies, by the claim, that all $R_Q(k/p)$'s equal -2 or -3 . But if $p > l_Q$, then as k runs from 1 to $p - 1$, $i = [kl_Q/p]$ runs through the entire set $\{0, 1, 2, \dots, l_Q - 1\}$. In view of (1.6), this yields condition (b) of the proposition. But now Q is stable, and the claim implies that Q is p -terminal for every $p > M_Q$. \square

Proposition 1.5 provides a simple computational method for testing[†] whether a given quintuple Q is stable. Using it, the reader can easily repeat (or have his computer repeat) our verification that each of the quintuples listed in Table 1.9 below is in fact stable. This then implies by Corollary 1.8 that each of these quintuples is p -terminal for every $p > M_Q$. We have thus completed the proof of Theorem 1.3.

Table 1.9 lists the 29 exceptional stable quintuples. The entries in the second column of the table are coefficients of certain linear relations among a, b, c, d , and e which we will need in Section 2. (For example, in the first line of the table, the

[†]This test is easily implemented on programmable calculators and computers of almost any size; a Pascal program which we wrote for this purpose is available from the third author upon request.

entries 02100, 11002, and 20122 indicate that the quintuple satisfies the relations $2b + c = 0$, $a + b + 2e = 0$, and $2a + c + 2d + 2e = 0$, respectively.)

TABLE 1.9

<i>Stable Quintuple</i>	<i>Linear Relations</i>
(9, 1, -2, -3, -5)	02100, 11002, 20122
(9, 2, -1, -4, -6)	01200, 02010, 20212
(12, 3, -4, -5, -6)	02001, 10002, 12220
(12, 2, -3, -4, -7)	02010, 11002, 20212
(9, 4, -2, -3, -8)	01200, 02001, 20221
(12, 1, -2, -3, -8)	02100, 12021, 20122
(12, 3, -1, -6, -8)	02010, 10020, 12202
(15, 4, -5, -6, -8)	02001, 20221
(12, 2, -1, -4, -9)	01200, 02010, 20212
(10, 6, -2, -5, -9)	02120, 10020, 12202
(15, 1, -2, -5, -9)	02100, 20122
(12, 5, -3, -4, -10)	02001, 02210, 20221
(15, 2, -3, -4, -10)	02010, 20212
(6, 4, 3, -1, -12)	02221, 20001
(7, 5, 3, -1, -14)	02221, 20001
(9, 7, 1, -3, -14)	02001, 20221
(15, 7, -3, -5, -14)	02001, 20221
(8, 5, 3, -1, -15)	02211, 20011
(10, 6, 1, -2, -15)	00210, 22012
(12, 5, 2, -4, -15)	00210, 22012
(9, 6, 4, -1, -18)	02221, 20001
(9, 6, 5, -2, -18)	02221, 20001
(12, 9, 1, -4, -18)	02001, 20221
(10, 7, 4, -1, -20)	02221, 20001
(10, 8, 3, -1, -20)	02221, 20001
(10, 9, 4, -3, -20)	02221, 20001
(12, 10, 1, -3, -20)	02001, 20221
(12, 8, 5, -1, -24)	02221, 20001
(15, 10, 6, -1, -30)	02221, 20001

We conclude this section with a few comments on Conjecture 1.4 and our evidence for it. The existence of sporadic singularities shows that it is possible for a quintuple Q to be p -terminal for some $p > M_Q$, and yet not be stable. To say that Q is not stable means, by Proposition 1.5, that $I_Q := \{i < l_Q \mid R_Q(i/l_Q) < -3 \text{ or } R_Q(i/l_Q) > -2\}$ is nonempty. On the other hand, by Claim 1.7, Q can be p -terminal only if for all $k < p$, $R_Q(k/p)$ is either -2 or -3 . In other words, the set $J_{p,l} := \{[kl/p] \mid k < p\}$ must be disjoint from I_Q . We say that p detects an unstable quintuple Q if

$$I_Q \cap J_{p,l} \neq \emptyset.$$

Corollary 1.8 shows that any $p > l_Q$ detects an unstable Q . This result is somewhat unsatisfactory since if we act on Q by an element of $(\mathbf{Z}/p\mathbf{Z})^*$ we will certainly change its least common multiple l without affecting its p -terminality.

Question 1.10. Is there a bound N_Q defined in terms of M_Q , so that every $p > N_Q$ detects an unstable Q ?

This question has a positive answer if Conjecture 1.4 is true.

PROPOSITION 1.11. *Let $Q = (a_1, \dots, a_5)$ be a quintuple of integers, let $p > 6M_Q$ and suppose that for some $k \in (\mathbf{Z}/p\mathbf{Z})^*$ and some quintuple $Q' = (a'_1, \dots, a'_5)$ listed in Theorem 1.3 we have $Q \equiv kQ' \pmod{p}$. Then:*

- (i) *If Q' satisfies the conditions in cases (a) or (b) of Theorem 1.3, then so does Q .*
- (ii) *If Q' is one of the 29 exceptional stable quintuples of Table 1.9, then Q is an integral multiple of Q' .*

It follows that if Conjecture 1.4 is true, then every stable quintuple is an integral multiple of one of the ones listed in Theorem 1.3, and every $p > 6M_Q$ detects an unstable quintuple.

Proof. Regarding Q' as an element of the vector space \mathbf{Q}^5 , consider the annihilator $\text{Ann}(Q') \subset (\mathbf{Q}^5)^*$. If $v = (\alpha'_1, \dots, \alpha'_5)$ is in this annihilator and satisfies

$$(1.12) \quad \alpha'_i \in \mathbf{Z} \quad \text{for all } i, \text{ and } \sum |\alpha'_i| \leq 6,$$

then we claim that $v \in \text{Ann}(Q)$. For

$$|v \cdot Q| \leq \sum |\alpha'_i a_i| \leq \sum |\alpha'_i| M_Q \leq 6M_Q,$$

while $v \cdot Q = kv \cdot Q' = 0 \pmod{p}$. Since $p > 6M_Q$, $v \cdot Q = 0$.

If Q' comes from case (a) of Theorem 1.3, then $\text{Ann}(Q')$ contains $\{(1, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$. Thus, these same vectors lie in $\text{Ann}(Q)$; but since $p > 2M_Q$, Q itself then satisfies the conditions of (1.3)(a).

A similar argument, using the set $\{(2, 1, 0, 0, 0), (0, 0, 2, 1, 0), (1, 0, 1, 0, -1)\}$, gives the result when Q' comes from case (b).

If Q' is one of the 29 exceptional stable quintuples of Table 1.9, it suffices to show that $\text{Ann}(Q) = \text{Ann}(Q')$, for then Q will be an integral multiple of Q' . For this purpose, we only need to show that $\text{Ann}(Q')$ has a basis consisting of vectors satisfying (1.12). Suppose a'_i and a'_j are two entries in Q' such that a'_j divides a'_i , and $m_{ij} := a'_i/a'_j$ has absolute value between 2 and 5. Then $\text{Ann}(Q')$ contains a vector whose nonzero entries are -1 and m_{ij} , and since $|-1| + |m_{ij}| \leq 6$, this vector satisfies (1.12). Now if there are three pairs of entries of Q' with this property, which collectively involve at least four of the entries of Q' , then the three corresponding vectors in $\text{Ann}(Q')$ together with the vector $(1, 1, 1, 1, 1)$ form a basis of $\text{Ann}(Q')$ with the required property. As is easily verified, there do exist three such pairs for 27 of the 29 exceptional stable quintuples; in the remaining cases, we simply write down an appropriate basis for $\text{Ann}(Q')$. Namely, when $Q' = (15, 4, -5, -6, -8)$ we take as a basis $\{(0, 2, 0, 0, 1), (1, 0, 3, 0, 0), (0, 3, 0, 2, 0), (1, 1, 1, 1, 1)\}$ while when $Q' = (10, 8, 3, -1, -20)$ we take as a basis $\{(2, 0, 0, 0, 1), (0, 0, 1, 3, 0), (0, 1, -2, 2, 0), (1, 1, 1, 1, 1)\}$. \square

We may define a stable $(n + 1)$ -tuple Q by analogy as an $(n + 1)$ -tuple of integers summing to 0, all $n + 1$ of whose associated n -dimensional $(\mathbf{Z}/p\mathbf{Z})$ -quotient singularities are terminal for all large primes p . Letting M_Q be the maximum of

the absolute values of the weights of Q , l_Q be their least common multiple and

$$R_Q(x) = \lim_{y \rightarrow x^+} \sum_i [a_i y],$$

a straightforward modification of the arguments above then shows that when all entries of Q are nonzero, Q is a stable $(n + 1)$ -tuple if and only if $-(n - 1) \leq R_Q(i/l_Q) \leq -2$ for $i = 1, 2, \dots, l_Q - 1$. Moreover, any time a proper subset of the weights of Q sums to 0, these inequalities automatically hold. Hence, isolated terminal quotient singularities of index p in any dimension for which a subset of the weights sums to zero mod p are stable. When $n = 3$, the terminal lemma asserts that *all* three-dimensional terminal quotient singularities are associated with such quadruples and hence that *every prime larger than M_Q detects an unstable quadruple*. (In particular, the answer to the analogue of Question 1.10 is positive.) Conversely, the italicized assertion would imply the terminal lemma.

It is easy to see that any stable quadruple has the form $Q = (\alpha, -\alpha, \beta, -\beta)$. For the condition for a quadruple to be stable is simply that R_Q be identically -2 . But if a is a weight of largest absolute value, then for $x = 1/a$ the values $R_Q(x - \varepsilon)$ and $R_Q(x + \varepsilon)$ will differ unless there is also a weight $a' = a$. Hence, the terminal lemma may also be restated as: “For all p , any three-dimensional terminal quotient singularity of index p is stable”. Since Conjecture 1.4 would have as a consequence the statement: “For all $p \geq 421$, any four-dimensional terminal quotient singularity of index p is stable”, it is tempting to ask

Question 1.13. Does there exist, for each n , a lower bound $p_0(n)$ such that every n -dimensional terminal quotient singularity of index $p \geq p_0(n)$ is stable?

We have, however, no evidence that the answer is yes, except in dimension ≤ 4 .

Let us conclude with a few remarks on our evidence for Conjecture 1.4. In view of Theorem 1.3 and Proposition 1.11, the force of the conjecture is to assert that

- (i) there are no sporadic terminal quotient singularities of prime index $p \geq 421$, and
- (ii) there are no stable quintuples other than integral multiples of those listed in (1.3).

Our evidence for both these assertions is computational. In the course of an investigation, discussed in Section 2, as to which terminal quotient singularities have *good anticanonical covers*, we systematically tabulated all the terminal quotient singularities of prime index at most 1600. All the singularities found are either associated with one of the stable quintuples of Theorem 1.3, or are sporadic with index $p < 421$. A summary of our tabulation of the sporadic singularities appears in Table 1.14, where for each prime $p < 421$ we give the number S_p of sporadic terminal quotient singularities of index p up to the actions of $(\mathbf{Z}/p\mathbf{Z})^*$ and \mathbf{S}^4 . (Note that the correspondence between p -terminal quintuples and sporadic terminal singularities is more complicated than one might initially suspect; for example, when $p = 61$, the five associated p -singularities of the p -terminal quintuple $(1, -3, 9, -27, 20)$ are all equivalent under the actions of $(\mathbf{Z}/61\mathbf{Z})^*$ and \mathbf{S}^4 .)

Could there possibly be more sporadic terminal singularities? As can be seen by inspecting the table, the number of sporadic terminal quotient singularities grows to a maximum of 300 when $p = 83$, and then tapers off to zero. Moreover, for

$421 \leq p < 1600$, the only terminal quotient singularities found are those predicted by Conjecture 1.4.

Could there possibly be more stable quintuples? By Corollary 1.8 and Proposition 1.11, if Q is a stable quintuple which is not an integral multiple of one of those in (1.3), and if $p > \max\{l_Q, 6M_Q\}$, then the associated p -singularities of Q are terminal but are not associated with any quintuple on our list (1.3). Thus, our computations imply that any new stable quintuple must have l_Q at least 1600 or M_Q at least 267. On the other hand, the largest value of l_Q for the 29 quintuples Q in Table 1.9 is 210 for $Q = (7, 5, 3, -1, -14)$ and $Q = (15, 7, -3, -5, -14)$, and the largest value of M_Q is 30.

Our faith in the conjecture reflects our belief that these phenomena represent a general pattern for large primes. We find it hard to imagine that new sporadic singularities or stable quintuples will appear after such a long period of predictability.

TABLE 1.14

p	S_p	p	S_p	p	S_p	p	S_p
2	0	73	220	179	105	283	10
3	0	79	275	181	65	293	25
5	0	83	300	191	40	307	0
7	0	89	275	193	60	311	5
11	0	97	230	197	65	313	5
13	0	101	201	199	55	317	5
17	9	103	255	211	20	331	5
19	13	107	270	223	35	337	0
23	28	109	220	227	45	347	5
29	39	113	200	229	30	349	10
31	30	127	120	233	45	353	5
37	50	131	145	239	15	359	0
41	76	137	140	241	30	367	0
43	110	139	185	251	25	373	0
47	100	149	130	257	15	379	0
53	195	151	95	263	35	383	0
59	260	157	55	269	10	389	0
61	186	163	85	271	20	397	5
67	205	167	90	277	0	409	0
71	250	173	75	281	0	419	5

2. Anticanonical Divisors and Antibicanonical Covers. In this section we indicate how the techniques of toric geometry can be used to decide whether a given terminal quotient singularity has either of the two important properties described in the introduction: whether its general anticanonical divisor has only canonical singularities, and whether the double cover branched on the general antibicanonical divisor of the quotient singularity has only canonical singularities. We will not completely settle the question for anticanonical divisors, contenting ourselves with giving an infinite number of examples for each alternative (see Examples 2.4 and 2.5). We will be more concerned with whether each terminal quotient singularity

has a double cover branched on an antibicanonical divisor with only canonical singularities. (We call this a *good antibicanonical cover*.) Our principal results are that *all the terminal quotient singularities associated with the stable quintuples of Theorem 1.3 do have such covers* (cf. Proposition 2.10) and that up to the actions of \mathbf{S}^4 and $(\mathbf{Z}/p\mathbf{Z})^*$, *there exist exactly six sporadic terminal quotient singularities of prime index less than 1600 which do not possess such covers* (cf. Theorem 2.12 and Table 2.15). These singularities all have indices between 83 and 109, and we conjecture that they are the only such singularities: This would follow from Conjecture 1.4.

We begin by reviewing the notion of a hyperquotient singularity. Let y_0, \dots, y_n denote coordinates on \mathbf{C}^{n+1} , let a generator σ of $\mathbf{Z}/r\mathbf{Z}$ act on \mathbf{C}^{n+1} by $(y_0, \dots, y_n) \rightarrow (\zeta^{a_0}y_0, \dots, \zeta^{a_n}y_n)$, where $\zeta = e^{2\pi i/r}$, and let $f \in \mathbf{C}[y_0, \dots, y_n]$ be a $\mathbf{Z}/r\mathbf{Z}$ -semi-invariant polynomial. Then the singularity of $(f = 0)/(\mathbf{Z}/r\mathbf{Z})$ at the origin is called a *hyperquotient singularity*. Let $\overline{M} \cong \mathbf{Z}^{n+1}$ be the lattice of (rational) monomials on \mathbf{C}^{n+1} , let \mathbf{y}^m denote the monomial corresponding to $m \in \overline{M}$, let \overline{N} be the dual lattice, let N be the overlattice of \overline{N} defined by $N = \overline{N} + \mathbf{Z} \cdot (a_0/r, \dots, a_n/r)$, and let $M \subset \overline{M}$ be the dual sublattice. Let N^+ be the intersection of N with the nonnegative quadrant in $N_{\mathbf{R}} := N \otimes \mathbf{R}$.

For $m \in \overline{M}$, say that the monomial \mathbf{y}^m *appears in* f (written $\mathbf{y}^m \in f$) if its coefficient in f is nonzero. Then for any $\alpha \in N$, we define

$$\alpha(f) = \min\{\alpha(\mathbf{y}^m) \mid \mathbf{y}^m \in f\}.$$

The methods of toric geometry lead one to:

THE HYPERQUOTIENT CRITERION 2.1 ([6, Theorem 2 and Corollary 2.1]; [12, Theorem 4.6]). *If the hyperquotient singularity $(f = 0)/(\mathbf{Z}/r\mathbf{Z})$ is canonical, then $\alpha(y_0 \cdots y_n) \geq \alpha(f) + 1$ for every primitive (nonzero) vector $\alpha \in N^+$. Conversely, if $\alpha(y_0 \cdots y_n) \geq \alpha(f) + 1$ for every primitive nonzero $\alpha \in N^+$ and if the coefficients of the monomials appearing in f are sufficiently general, then the hyperquotient singularity $(f = 0)/(\mathbf{Z}/r\mathbf{Z})$ is canonical. \square*

If we define the *Newton polyhedron* of f to be the lattice polyhedron in $M_{\mathbf{R}}$ defined by

$$\text{Newton}(f) = \{u \in M_{\mathbf{R}} \mid \alpha(u) \geq \alpha(f) \text{ for every } \alpha \in N^+\},$$

then the hyperquotient criterion (2.1) depends only on the polyhedron $\text{Newton}(f)$ and not on f itself. The exact translation of (2.1) to a condition on the Newton polyhedron is a bit complicated (see Reid [12, appendix to Section 4] for details), but (2.1) certainly implies that the vector $(1, 1, \dots, 1)$ lies in the interior of $\text{Newton}(f)$.

We will apply the hyperquotient criterion to four-dimensional quotient singularities in two ways. Let $\mathcal{M} \cong \mathbf{Z}^4$ denote the set of (rational) monomials in w, x, y, z ; we will represent a monomial $m \in \mathcal{M}$ either by a quadruple $m = (m_1, m_2, m_3, m_4)$ or by the symbol $\mathbf{x}^m := w^{m_1}x^{m_2}y^{m_3}z^{m_4}$. \mathcal{M} has a natural partial ordering defined by $m' \leq m$ if and only if $m'_i \leq m_i$ for each $i = 1, 2, 3, 4$. Let \mathcal{N} denote the dual lattice of \mathcal{M} , and let \mathcal{N}^+ be its intersection with the positive quadrant. Given a terminal quotient singularity $T = [s|a, b, c, d]_p$, we define the weight function $w_T: \mathcal{M} \rightarrow \mathbf{Z}/p\mathbf{Z}$ by

$$w_T(m) = \langle am_1 + bm_2 + cm_3 + dm_4 \rangle_p$$

and let $\mathcal{M}_k = \mathcal{M}_k(T)$ denote the fiber of this map over the residue k ; we let \mathcal{M}^+ and \mathcal{M}_k^+ denote the subsets of \mathcal{M} and \mathcal{M}_k , respectively, which consist of holomorphic monomials. \mathcal{M}_k^+ is a basis of the ζ^k -eigenspace of the action of our generator $\sigma \in \mathbf{Z}/p\mathbf{Z}$ on the polynomial ring $\mathbf{C}[w, x, y, z]$, and this eigenspace is naturally a module over the ring R_T of invariants of the action. As such, \mathcal{M}_k contains a unique minimal R_T -module basis $\mathcal{L}_k = \mathcal{L}_k(T)$ characterized by

$$(2.2) \quad \mathcal{L}_k = \{m \in \mathcal{M}_k^+ \mid \text{if } m' \in \mathcal{M}_k^+ \text{ and } m' \leq m \text{ then } m' = m\}.$$

An easy residue calculation shows that each element of the linear system $|-rK_T|$ is represented by a polynomial f with eigenvalue ζ^{rs} . Thus, if we define $\alpha(\mathcal{L}_k) = \min\{\alpha(m) \mid m \in \mathcal{L}_k\} = \min\{\alpha(m) \mid m \in \mathcal{M}_k^+\}$ for $\alpha \in \mathcal{N}^+$, and

$$\text{Newton}(|-rK_T|) = \{u \in M_{\mathbf{R}} \mid \alpha(u) \geq \alpha(\mathcal{L}_{rs}) \text{ for every } \alpha \in \mathcal{N}^+\},$$

then $\text{Newton}(f) \subset \text{Newton}(|-rK_T|)$, with equality whenever each monomial in \mathcal{L}_{rs} appears in f .

We define some special elements α_k of $\text{Hom}(\mathcal{M}, \mathbf{Q})$ as follows:

$$\alpha_k(m) = m_1\{ak/p\} + m_2\{bk/p\} + m_3\{ck/p\} + m_4\{dk/p\},$$

where $\{x\} := x - [x]$ denotes the fractional part.

PROPOSITION 2.3. *The general anticanonical divisor of T has a canonical singularity if and only if*

- (a) $\mathbf{1} := (1, 1, 1)$ is in the interior of $\text{Newton}(|-K_T|)$, and
- (b) for each $k \in (\mathbf{Z}/p\mathbf{Z})^*$, there is some monomial $m \in \mathcal{M}_s^+$ such that $\alpha_k(wxyz) \geq \alpha_k(\mathbf{x}^m) + 1$.

Proof. We apply the hyperquotient criterion with $(y_0, \dots, y_n) = (w, x, y, z)$ and $(a_0, \dots, a_n) = (a, b, c, d)$; the semigroup N^+ is then generated by \mathcal{N}^+ and $\{\alpha_k\}$. Thus, if $f = 0$ defines an anticanonical divisor with canonical singularities, for each k there is some $m \in \mathcal{M}_s$ such that $\mathbf{x}^m \in f$ and $\alpha_k(wxyz) \geq \alpha_k(\mathbf{x}^m) + 1$. Moreover, for each $\alpha \in \mathcal{N}^+$ there is some $\mathbf{x}^m \in f$ with $\alpha(\mathbf{1}) = \alpha(wxyz) > \alpha(\mathbf{x}^m)$. In particular, $\mathbf{1}$ is in the interior of $\text{Newton}(f)$, and hence in the interior of $\text{Newton}(|-K_T|)$.

Conversely, if $\text{Newton}(|-K_T|)$ and \mathcal{M}_s^+ satisfy the conditions in the proposition, let f be a general linear combination of the monomials in \mathcal{L}_s ; we must show that $\alpha(wxyz) \geq \alpha(f) + 1$ for any nonzero primitive $\alpha \in N^+$. Any nonzero $\alpha \in N^+$ can be written as $\alpha = \alpha' + \alpha''$, where $\alpha' \in \mathcal{N}^+$ and $\alpha'' \in \{0, \alpha_k\}$, such that at least one of α' and α'' is nonzero; we then have $\alpha(f) \geq \alpha'(f) + \alpha''(f)$. If $\alpha'' \neq 0$, then $\alpha'(wxyz) \geq \alpha'(f)$ and $\alpha''(wxyz) \geq \alpha''(f) + 1$, and the assertion follows in this case. On the other hand, if $\alpha'' = 0$, then since $\mathbf{1}$ is in the interior of $\text{Newton}(|-K_T|) = \text{Newton}(f)$, we see that $\alpha(wxyz) = \alpha'(wxyz) > \alpha'(f) = \alpha(f)$. Since both sides of this inequality are integers, we conclude that $\alpha(wxyz) \geq \alpha(f) + 1$ in this case as well. \square

Example 2.4. The terminal singularities associated to the stable quintuples in case (a) of Theorem 1.3 satisfy the conditions of the proposition. For in that case, after reordering the variables either yz or z is in the set \mathcal{M}_s^+ . For either of these monomials, α_k has the stated property; moreover, if either is in the set \mathcal{M}_s^+ , then

1 is in the interior of $\text{Newton}(| - K_T |)$ (cf. the proofs of Corollaries 2.7 and 2.8(i) below).

Example 2.5. Many of the terminal singularities associated with the stable quintuples in case (b) of Theorem 1.3 do *not* have an anticanonical divisor with only canonical singularities. For example, if we take the quintuple $(-1, 2, \beta, -2\beta, \beta - 1)$ with $\beta \geq 3$ and its associated p -singularity $[2, \beta, -2\beta, \beta - 1]_p$, and if we suppose that $p > 2\beta + 2$, then $\alpha_1(wxyz) = 1 + (1/p)$. By the proposition, if there were such an anticanonical divisor, there would be some monomial $m \in \mathcal{M}_1^+$ such that $\alpha_1(x^m) = 1/p$. But since each of the numbers $\{2/p\}, \{\beta/p\}, \{-2\beta/p\} = 1 - (2\beta/p)$ and $\{(\beta - 1)/p\}$ is $\geq 2/p$, no nonnegative integral linear combination of them can equal $1/p$.

We now consider antibicanonical covers of T . That is, we select a divisor D from the linear system $| - 2K_T |$ and form the double cover of T branched on D . If D is defined by an equation $f = 0$, then we can describe that double cover as follows: We introduce a new variable v and let $(y_0, \dots, y_4) = (v, w, x, y, z)$ and $(a_0, \dots, a_4) = (s, a, b, c, d)$ determine an action of $\mathbf{Z}/p\mathbf{Z}$ on \mathbf{C}^5 as in the discussion of the hyperquotient criterion. The double cover is then the quotient of the hypersurface $v^2 = f(w, x, y, z)$ by the given $\mathbf{Z}/p\mathbf{Z}$ -action.

PROPOSITION 2.6. *Let T be a terminal quotient singularity. The general antibicanonical cover of T has a canonical singularity if and only if $\mathbf{2} := (2, 2, 2, 2)$ is in the interior of $\text{Newton}(| - 2K_T |)$.*

Proof. We again apply the hyperquotient criterion: In this case, the semigroup N^+ is generated by \mathcal{N}^+ (the elements of which have value zero on the new variable $v = y_0$) together with the elements β_k and γ defined by $\beta_k(m_0, \dots, m_4) = \sum m_i \{a_i k/p\}$ and $\gamma(m_0, \dots, m_4) = m_0$.

Suppose that the hyperquotient singularity $(v^2 - f = 0)/(\mathbf{Z}/p\mathbf{Z})$ is canonical. For any $\alpha \in \mathcal{N}^+$, let $\delta = \alpha - [-\alpha(f)/2]\gamma \in N^+$; then $\delta(v^2 - f) = \alpha(f)$, while $\delta(vwxyz) = \alpha(wxyz) - [-\alpha(f)/2]$. Thus, since $\delta(vwxyz) \geq \delta(v^2 - f) + 1$, we see that $\alpha(\mathbf{2}) = 2\alpha(wxyz) \geq 2\alpha(f) + 2[-\alpha(f)/2] + 2 \geq \alpha(f) + 1$. In particular, $\mathbf{2}$ is in the interior of $\text{Newton}(f)$, and hence in the interior of $\text{Newton}(| - 2K_T |)$.

Conversely, if $\mathbf{2}$ is in the interior of $\text{Newton}(| - 2K_T |)$, we let f be a general linear combination of the monomials in \mathcal{L}_{2s} . Write any nonzero element of N^+ in the form $\alpha = \alpha' + \alpha'' + \alpha'''$, where $\alpha' \in \mathcal{N}^+$, $\alpha'' \in \{0, \beta_k\}$ and $\alpha''' = n\gamma$, such that at least one of $\alpha', \alpha'',$ and α''' is nonzero. We make the following claims, which clearly suffice to show that $\alpha(vwxyz) \geq \alpha(v^2 - f) + 1$ (and thus complete the proof of the proposition):

- (i) $\beta_k(vwxyz) \geq \beta_k(v^2 - f) + 1$;
- (ii) If $\alpha' \neq 0$, then $(\alpha' + n\gamma)(vwxyz) \geq (\alpha' + n\gamma)(v^2 - f) + 1$;
- (iii) If $n \geq 1$, then $n\gamma(vwxyz) \geq n\gamma(v^2 - f) + 1$.

To prove claim (i), since T is terminal, $s_k \geq s^{(k)} + p$. Moreover, since v^2 appears in $v^2 - f$, we have $\beta_k(v^2 - f) \leq \beta_k(v^2) = 2s^{(k)}/p$. Thus

$$\beta_k(vwxyz) = (s_k + s^{(k)})/p \geq (2s^{(k)}/p) + 1 \geq \beta_k(v^2 - f) + 1,$$

proving (i). To prove claim (ii), since all terms in the inequality are integers, it suffices to show that $(\alpha' + n\gamma)(vwxyz) \geq (\alpha' + n\gamma)(v^2 - f) + 1/2$. Now $\mathbf{2}$ is

in the interior of $\text{Newton}(| - 2K_T|) = \text{Newton}(f)$, so that for $\alpha' \neq 0$ we have $2\alpha'(wxyz) \geq \alpha'(f) + 1$ (since all are integers). Thus,

$$\begin{aligned} (\alpha' + n\gamma)(vwxyz) &= \alpha'(wxyz) + n \geq (\alpha'(f) + 1)/2 + n \\ &\geq \min(\alpha'(f), 2n) + 1/2 = (\alpha' + n\gamma)(v^2 - f) + 1/2. \end{aligned}$$

Finally, to prove (iii), note that $\gamma(v^2 - f) = 0$ so that $n\gamma(vwxyz) = n \geq 1 = \gamma(v^2 - f) + 1$. \square

Note that $\mathbf{2} \in \mathcal{M}_{2s}$ and hence that $\mathbf{2}$ always lies in $\text{Newton}(| - 2K_T|)$. The content of the proposition is that a good antibicanonical cover can exist only if $\mathbf{2}$ does not lie on the boundary of $\text{Newton}(| - 2K_T|)$. Let $\overline{\mathcal{L}}_k$ denote the convex hull of \mathcal{L}_k in \mathbf{R}^4 , and let us call a vector n in \mathbf{R}^4 *small* if each coordinate is between 0 and 2 and n is neither the zero vector nor the vector $\mathbf{2}$.

COROLLARY 2.7. *The singularity T has a good antibicanonical cover if and only if there is a small vector n in $\overline{\mathcal{L}}_{2s}$.*

Proof. If $\mathbf{2}$ is in the interior of $\text{Newton}(| - 2K_T|)$, then $n = (1 - \varepsilon)\mathbf{2} \in \overline{\mathcal{L}}_{2s}$ for an appropriate positive $\varepsilon < 1$. Conversely, given $n = (i, j, k, l) \in \overline{\mathcal{L}}_{2s}$, we may suppose without loss of generality that $i < 2$. Define $i' \in \mathbf{Z}$ by the conditions $0 < i' < p$ and $i'a \equiv 2s \pmod{p}$. Then $n' = (i', 0, 0, 0) \in \overline{\mathcal{L}}_{2s}$, and, for small positive ε , $n'' = (1 - \varepsilon)n + \varepsilon n'$ is in $\overline{\mathcal{L}}_{2s}$ and has all coordinates strictly less than 2. Therefore, $\mathbf{2}$ is in the interior of $\text{Newton}(| - 2K_T|)$. \square

Hence, we have

COROLLARY 2.8. (i) *If \mathcal{M}_0^+ contains a small vector, then T has a good antibicanonical cover.*

(ii) *If T is associated with a quintuple Q , and if there is a linear relation among the entries of Q with coefficients in $\{0, 1, 2\}$ such that the coefficient of the entry of Q which is not a weight of T is 0, then T has a good antibicanonical cover.*

(iii) *The infinite families of terminal quotient singularities associated with the stable quintuples of cases (a) and (b) of Theorem 1.3 have good antibicanonical covers.*

Proof. (i) Suppose that $n \in \mathcal{M}_0^+$ is a small invariant monomial, and choose n to be maximal among small invariant monomials under the natural partial order on \mathcal{M} . Then, since $\mathbf{2} \in \mathcal{M}_{2s}$, $n' := \mathbf{2} - n$ is a small vector in \mathcal{L}_{2s} .

(ii) By symmetry, we may assume that $T = [a, b, c, d]_p$ and $Q = (a, b, c, d, e)$. If the linear relation is given by $ia + jb + kc + ld + 0e = 0$, then $n = (i, j, k, l)$ is a small invariant monomial for T .

(iii) The required linear relations are $a + b = c + d + e = 0$ in case (a) and $2a + b = 2c + d = 0$ in case (b). \square

The associated p -singularities of the exceptional stable quintuples Q of case (c) of Theorem 1.3 also have good antibicanonical covers. For some of these associated p -singularities, the criterion in Corollary 2.8(ii) above can be used to show this, but for others we need a different method.

COROLLARY 2.9. *Let $Q = (a, b, c, d, e)$ be a quintuple with $a + b + c + d + e = 0$ and $e < 0$, and for a monomial $m = (i, j, k, l)$ define $\tilde{w}(m) := ia + jb + kc + ld$.*

Suppose that there is a small vector n in the closed convex hull of the set

$$\mathcal{F} := \{m \in \mathcal{M}^+ \mid \tilde{w}(m) = -2e\}.$$

Then the associated p -singularity $T_p = [a, b, c, d]_p$ has a good anticanonical cover for every $p > M_Q$.

Proof. If $\tilde{w}(m) = -2e$, then $w_{T_p}(m) \equiv \tilde{w}(m) \equiv 2s \pmod{p}$. Thus, $\mathcal{F} \subset \mathcal{M}_{2s}^+$, so for an appropriate positive $\varepsilon < 1$, $(1 - \varepsilon)n$ will be a small vector in $\overline{\mathcal{L}}_{2s}$. The corollary now follows from Corollary 2.7. \square

In Table 1.9, for each exceptional stable quintuple Q we listed the coefficients for certain linear relations among the entries of Q with coefficients in $\{0, 1, 2\}$. For 10 of the quintuples, we listed three relations, and in each of these cases it is easily verified that each entry of Q has coefficient 0 in at least one of those relations; thus, all 5 associated p -singularities have good anticanonical covers by Corollary 2.8(ii). For 18 of the remaining 19 quintuples, there is exactly one entry in Q which has nonzero coefficient in every relation (and for the quintuple $(8, 5, 3, -1, -15)$ both -1 and -15 have nonzero coefficients in every relation). We thus have 20 cases remaining to be checked, and in each of these the relevant entry of Q is negative. Corollary 2.9 holds for each of these cases, as we verify in Table 2.11. This table lists those cases along with a small vector n and a convex linear combination of elements of \mathcal{F} which equals n . Hence, we get

PROPOSITION 2.10. *All the terminal quotient singularities associated with the stable quintuples of Theorem 1.3 have good anticanonical covers. \square*

TABLE 2.11

$[s a, b, c, d]$	small vector n	convex linear combination
[8 15, 4, -5, -6]	$(3/2, 2, 1/2, 2)$	$= 1/2 (0, 4, 0, 0) + 1/2 (3, 0, 1, 4)$
[2 15, 1, -5, -9]	$(3/2, 2, 1/2, 2)$	$= 1/2 (0, 4, 0, 0) + 1/2 (3, 0, 1, 4)$
[4 15, 2, -3, -10]	$(1, 2, 2, 1/2)$	$= 1/2 (0, 4, 0, 0) + 1/2 (2, 0, 4, 1)$
[12 6, 4, 3, -1]	$(2, 3/2, 2, 0)$	$= 1/2 (2, 0, 4, 0) + 1/2 (2, 3, 0, 0)$
[14 7, 5, 3, -1]	$(2, 2, 3/2, 1/2)$	$= 1/2 (3, 1, 1, 1) + 1/2 (1, 3, 2, 0)$
[14 9, 7, 1, -3]	$(3/2, 2, 1/2, 0)$	$= 1/2 (3, 0, 1, 0) + 1/2 (0, 4, 0, 0)$
[14 15, 7, -3, -5]	$(3/2, 2, 2, 1/2)$	$= 1/2 (0, 4, 0, 0) + 1/2 (3, 0, 4, 1)$
[1 8, 5, 3, -15]	$(2, 1/2, 2, 3/2)$	$= 1/2 (4, 0, 0, 2) + 1/2 (0, 1, 4, 1)$
[15 8, 5, 3, -1]	$(2, 2, 3/2, 1/2)$	$= 1/2 (3, 0, 2, 0) + 1/2 (1, 4, 1, 1)$
[2 10, 6, 1, -15]	$(1/2, 2, 2, 1)$	$= 1/2 (1, 4, 0, 2) + 1/2 (0, 0, 4, 0)$
[4 12, 5, 2, -15]	$(2, 1/2, 2, 3/2)$	$= 1/2 (4, 1, 0, 3) + 1/2 (0, 0, 4, 0)$
[18 9, 6, 4, -1]	$(2, 2, 3/2, 0)$	$= 1/2 (2, 3, 0, 0) + 1/2 (2, 1, 3, 0)$
[18 9, 6, 5, -2]	$(2, 3/2, 2, 1/2)$	$= 1/2 (1, 2, 3, 0) + 1/2 (3, 1, 1, 1)$
[18 12, 9, 1, -4]	$(3/2, 2, 0, 0)$	$= 1/2 (3, 0, 0, 0) + 1/2 (0, 4, 0, 0)$
[20 10, 7, 4, -1]	$(2, 2, 3/2, 0)$	$= 1/2 (4, 0, 0, 0) + 1/2 (0, 4, 3, 0)$
[20 10, 8, 3, -1]	$(2, 2, 3/2, 1/2)$	$= 1/2 (1, 3, 2, 0) + 1/2 (3, 1, 1, 1)$
[20 10, 9, 4, -3]	$(2, 2, 1/2, 0)$	$= 1/2 (4, 0, 0, 0) + 1/2 (0, 4, 1, 0)$
[20 12, 10, 1, -3]	$(3/2, 2, 2, 0)$	$= 1/2 (0, 4, 0, 0) + 1/2 (3, 0, 4, 0)$
[24 12, 8, 5, -1]	$(11/6, 2, 2, 0)$	$= 1/2 (1, 2, 4, 0) + 1/3 (2, 3, 0, 0) + 1/6(4, 0, 0, 0)$
[30 15, 10, 6, -1]	$(2, 2, 5/3, 0)$	$= 1/2 (4, 0, 0, 0) + 1/3(0, 6, 0, 0) + 1/6(0, 0, 10, 0)$

There remains the question of whether the sporadic terminal singularities which exist for $p < 421$ have good antibicanonical covers. The answer is nearly always yes.

THEOREM 2.12. *Every terminal quotient singularity of index $p < 421$ has a good antibicanonical cover with the exception of those equivalent under the actions of \mathbf{S}^4 or $(\mathbf{Z}/p\mathbf{Z})^*$ to one of the six singularities in Table 2.15.*

Combining this with Proposition 2.10 yields the

COROLLARY 2.13. *If Conjecture 1.4 holds, then every four-dimensional terminal quotient singularity of prime index, except those occurring in Table 2.15, has a good antibicanonical cover. \square*

Our approach to Theorem 2.12 is a computational one based on a restatement of Corollary 2.7. Fix a terminal quotient singularity T . Let Λ denote the set of real vectors $\lambda = (\lambda_m)$ indexed by the elements $m = (i_m, j_m, k_m, l_m)$ of \mathcal{L}_{2s} for which

- (i) $\lambda_m \geq 0$ for all m , and
- (ii) $i_\lambda := \sum_{m \in \mathcal{L}_{2s}} \lambda_m i_m \leq 2$, and likewise for the analogously defined sums j_λ, k_λ , and l_λ ,

and let

$$\mu_\lambda = \sum_{m \in \mathcal{L}_{2s}} \lambda_m \quad \text{and} \quad \mu_T = \max_{\lambda \in \Lambda} \{\mu_\lambda\}.$$

LEMMA 2.14. *The singularity T has a good antibicanonical cover if and only if $\mu_T > 1$.*

Proof. If $\mathbf{2}$ lies in the interior of $\text{Newton}(|-2K_T|)$, then, by Corollary 2.7, there is a vector $n \in \overline{\mathcal{L}_{2s}}$ with all coordinates strictly smaller than two. This n can be written as a linear combination $\lambda = (\lambda_m)$ in Λ for which $\mu_\lambda = 1$ and hence $\mu_T > 1$. Conversely, if $\mu_T > 1$ and the λ which realizes this value corresponds to the vector n , then $(1/\mu_T)n$ is a small vector in $\overline{\mathcal{L}_{2s}}$. \square

This lemma essentially corresponds to dualizing the linear programming problem posed by Corollary 2.7. Moreover, the calculation of μ_T as defined above is a standard exercise in linear programming. All that is required as input is an enumeration of the generating set \mathcal{L}_{2s} using the definition given in (2.2). We have carried out this calculation for all the terminal quotient singularities T of prime index less than 421 on the Macintosh and Ridge computers of the Columbia University mathematics department. Our linear programs were solved^{††} using the Pascal subroutines which are described in Section 10.8 and listed on pp. 743–746 of [9]. Theorem 2.12 is the result of these computations.

The six terminal quotient singularities which do *not* have good antibicanonical covers are described in Table 2.15. Since $\mathbf{2}$ is always in \mathcal{L}_{2s} , these T have $\mu_T = 1$, so the output of our programs is not especially interesting. Instead, we give the reader data from which it is straightforward to verify that $\mathbf{2}$ does not lie in the interior of $\text{Newton}(|-2K_T|)$. More precisely, we give for each T a complete list of the monomials in \mathcal{L}_{2s} and the equation of a hyperplane $H := H_T$ in \mathbf{R}^4 containing

^{††}A listing of the complete Pascal program we used is available from the authors upon request (directed to the third author).

2 and such that every other element of \mathcal{L}_{2s} either lies on H or lies on the positive side of H . (Those which lie *on* H are prefixed by an * in the listing of \mathcal{L}_{2s} .)

Note that, in view of Proposition 2.10, any terminal quotient singularity of prime index < 1600 which does not have a good antibicanonical cover must be sporadic. Hence, the six singularities of Table 2.15 are examples of sporadic four-dimensional terminal quotient singularities. From them one can easily construct examples of p -terminal quintuples which are not stable: for example, the unstable quintuple $(-1, 3, 14, 23, -39)$ which is p -terminal for $p = 83$.

TABLE 2.15

$p: [s a, b, c, d]$ (monomials in \mathcal{L}_{2s})	<i>hyperplane</i> H_T)*
83: [1 3, 14, 23, 44]	$24i+29j+18k+20l=182$	
(0, 0, 0, 34) (0, 0, 3, 23) (0, 0, 6, 12) *(0, 0, 9, 1) (0, 0, 47, 0)		
(0, 1, 2, 10) (0, 3, 1, 8) (0, 5, 0, 6) (0, 8, 2, 4) (0, 10, 1, 2)		
(0, 12, 0, 0) *(1, 0, 1, 7) (1, 0, 18, 0) *(1, 2, 0, 5) (1, 5, 2, 3)		
(1, 7, 1, 1) (2, 0, 0, 15) *(2, 0, 3, 4) *(2, 2, 2, 2) *(2, 4, 1, 0)		
*(3, 0, 5, 1) (3, 11, 0, 2) (4, 0, 14, 0) (4, 1, 0, 7) (4, 8, 0, 1)		
(5, 1, 2, 4) (5, 3, 1, 2) (5, 5, 0, 0) *(6, 0, 1, 1) (7, 0, 0, 9)		
(7, 0, 10, 0) (8, 4, 0, 2) (9, 1, 0, 1) (10, 0, 6, 0) (12, 0, 0, 3)		
(13, 0, 2, 0) (16, 1, 1, 0) (19, 2, 0, 0) (56, 0, 0, 0))		
103: [1 4,57,59,87]	$8i+7j+11k+9l=70$	
((0, 0, 0, 90) (0, 0, 1, 10) (0, 0, 14, 0) (0, 1, 5, 9) (0, 2, 0, 7)		
(0, 3, 4, 6) (0, 4, 8, 5) (0, 5, 3, 3) (0, 6, 7, 2) (0, 7, 2, 0)		
(0, 12, 0, 4) (0, 22, 0, 1) (0, 94, 0, 0) (1, 0, 0, 13) *(1, 0, 4, 2)		
(1, 1, 8, 1) (1, 2, 12, 0) (1, 5, 2, 6) (1, 7, 1, 3) (1, 9, 0, 0)		
(2, 0, 3, 5) *(2, 2, 2, 2) (2, 3, 6, 1) (2, 4, 10, 0) (2, 7, 0, 6)		
(3, 0, 2, 8) (3, 2, 1, 5) *(3, 4, 0, 2) (3, 5, 4, 1) (3, 6, 8, 0)		
(4, 0, 5, 0) (5, 2, 3, 0) (6, 4, 1, 0) (12, 1, 0, 0) (52, 0, 0, 0))		
103: [1 9,15,22,58]	$18i+30j+44k+19l=210$	
((0, 0, 0, 32) *(0, 0, 3, 6) (0, 0, 15, 5) (0, 0, 27, 4) (0, 0, 39, 3)		
(0, 0, 51, 2) (0, 0, 63, 1) (0, 0, 75, 0) (0, 1, 2, 15) (0, 2, 1, 24)		
*(0, 7, 0, 0) *(1, 0, 2, 8) (1, 1, 1, 17) (1, 2, 0, 26) *(1, 2, 3, 0)		
*(2, 0, 1, 10) (2, 0, 6, 1) (2, 0, 18, 0) (2, 1, 0, 19) *(2, 2, 2, 2)		
*(3, 0, 0, 12) (3, 0, 5, 3) *(3, 2, 1, 4) (4, 0, 4, 5) *(4, 2, 0, 6)		
*(5, 4, 0, 0) (6, 0, 7, 0) *(10, 1, 0, 0) (46, 0, 0, 0))		
107:[1 19,22,31,36]	$2i+2j+2k+3l=18$	
(* (0, 0, 0, 6) (0, 0, 15, 2) (0, 0, 76, 0) (0, 1, 12, 1) (0, 2, 9, 0)		
(0, 5, 8, 2) (0, 6, 5, 1) *(0, 7, 2, 0) (0, 10, 1, 2) (0, 13, 0, 4)		
(0, 23, 1, 0) (0, 26, 0, 2) (0, 39, 0, 0) (1, 0, 4, 5) (1, 0, 19, 1)		
*(1, 1, 1, 4) (1, 1, 16, 0) (2, 0		

$p: [s a, b, c, d]$ (monomials in \mathcal{L}_s)	<i>hyperplane</i> H_T
109: [1 4,35,89,91]	$40i+23j+18k+38l=238$
((0, 0, 0, 12) *(0, 0, 9, 2) (0, 0, 98, 0) (0, 1, 18, 0) (0, 2, 8, 7)	
(0, 3, 7, 4) *(0, 4, 6, 1) (0, 6, 5, 6) (0, 7, 4, 3) *(0, 8, 3, 0)	
(0, 10, 2, 5) (0, 11, 1, 2) (0, 13, 0, 7) (0, 23, 2, 0) (0, 26, 0, 2)	
(0, 38, 1, 0) (0, 53, 0, 0) (1, 0, 2, 10) *(1, 0, 11, 0) (1, 1, 1, 7)	
*(1, 2, 0, 4) (2, 0, 4, 8) (2, 1, 3, 5) *(2, 2, 2, 2) (3, 0, 6, 6)	
(3, 1, 5, 3) *(3, 2, 4, 0) (4, 0, 8, 4) (4, 1, 7, 1) *(5, 0, 0, 1)	
(19, 1, 0, 0) (55, 0, 0, 0))	
109: [1 14,19,30,47]	$14i+19j+30k+47l=220$
((0, 0, 0, 7) (0, 0, 26, 2) (0, 0, 53, 1) (0, 0, 80, 0) *(0, 1, 2, 3)	
*(0, 10, 1, 0) (0, 46, 0, 0) (1, 0, 11, 2) (1, 0, 38, 1) (1, 0, 65, 0)	
(2, 0, 23, 1) (2, 0, 50, 0) (2, 1, 0, 6) *(2, 2, 2, 2) (3, 0, 8, 1)	
(3, 0, 35, 0) (4, 0, 20, 0) (4, 2, 0, 5) *(4, 3, 2, 1) *(5, 0, 5, 0)	
(6, 3, 0, 4) *(6, 4, 2, 0) (7, 0, 3, 3) (8, 4, 0, 3) *(9, 0, 0, 2)	
*(11, 1, 0, 1) *(13, 2, 0, 0) (18, 0, 1, 1) (20, 1, 1, 0) (27, 0, 2, 0)	
(78, 0, 0, 0))	

*Monomials lying on H_T are marked with an *. Those above H_T are unmarked.

Added in Proof. As of June 1, 1988 the conjecture has been numerically verified for all primes < 2600 .

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