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Cosine Methods for Nonlinear Second-Order Hyperbolic Equations*

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Dedicated to Professor Eugene Isaacson on the occasion of his 70th birthday

Abstract. We construct and analyze efficient, high-order accurate methods for approximating the smooth solutions of a class of nonlinear, second-order hyperbolic equations. The methods are based on Galerkin type discretizations in space and on a class of fourth-order accurate two-step schemes in time generated by rational approximations to the cosine. Extrapolation from previous values in the coefficients of the nonlinear terms and use of preconditioned iterative techniques yield schemes whose implementation requires solving a number of linear systems at each time step with the same operator. L^2 optimal-order error estimates are proved.

1. Introduction. *The problem.* In this paper we shall study efficient, high-order accurate methods for approximating the solution of the following initial and boundary value problem: let Ω be a bounded domain in $\mathbf{R}^N$ ($N = 1,2,3$) with smooth boundary $\partial\Omega$ and let $0 < t^* < \infty$. We seek a real-valued function $u = u(x,t)$, $(x,t) \in \overline{\Omega} \times [0,t^*]$ satisfying

$$(1.1)\qquad \begin{aligned} u_{tt} = -L(t,u)u + f(t,u) \equiv \sum_{i,j=1}^{N} \partial_i(a_{ij}(x,t,u)\partial_j u) - a_0(x,t,u)u \\ + f(x,t,u) \quad \text{in } \Omega \times [0,t^*], \\ u(x,t) = 0 \qquad \text{on } \partial\Omega \times [0,t^*], \\ u(x,0) = u^0(x) \quad \text{in } \Omega, \\ u_t(x,0) = u_t^0(x) \quad \text{in } \Omega, \end{aligned}$$

where $a_{ij}, a_0, f, u^0, u_t^0$ are given functions. We shall discretize (1.1) in space by methods of Galerkin type and base the temporal discretization on a class of fourth-order accurate, two-step multiderivative schemes generated by rational approximations to the cosine, [3]. By extrapolating from previous values in the coefficients of the nonlinear terms we can implement the time-stepping schemes by solving only linear systems of equations at each time step. These systems may then be solved approximately by preconditioned iterative techniques, [12], [4], that require solving a number of linear systems with the same operator at every time step.

Galerkin type methods, coupled with two-step schemes of second-order accuracy in time, for the numerical solution of nonlinear problems similar to (1.1) have been analyzed in the past, cf., e.g., [10], [11], [14]; in [14] the linear systems at

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each time step are solved by preconditioned iterative techniques. High-order linear multistep methods were studied in [1] in the case of a semilinear problem. One of us, [2], has recently analyzed high-order time-stepping methods (generated by rational approximations to $\exp(ix)$) for (1.1) written in first-order system form. In this paper we shall discretize directly the second-order equation in (1.1). Our analysis relies in part on existing estimates in the case of the linear hyperbolic problem with time-dependent coefficients, [3], while some of the techniques of estimating nonlinear terms are adapted from the analogous techniques for parabolic problems due to Bramble and Sammon, [5].

For integral $s \geq 0$ and $p \in [1, \infty]$, let $W^{s,p} = W^{s,p}(\Omega)$ denote the usual Sobolev spaces of real functions on Ω with corresponding norm $\|\cdot\|_{s,p}$ and let $H^s = W^{s,2}$ with norm $\|\cdot\|_s$; $(\cdot,\cdot)$, resp. $\|\cdot\|$, will denote the inner product, resp. norm, on $L^2 = L^2(\Omega)$, while $|\cdot|_\infty$ will be the norm on $L^\infty = L^\infty(\Omega)$. As usual, $\overset{\circ}{H}{}^1$ will consist of those elements of H^1 that vanish on $\partial\Omega$ in the sense of trace. It is well known, cf., e.g., [6], [9], that the problem (1.1) has a unique solution, in general for small enough t^*, under appropriate smoothness and compatibility conditions on the data. Specifically, it is proved in [9] that if, for example, the coefficients a_{ij}, a_0, f are sufficiently smooth functions of their arguments for $(x, t, u) \in Q \equiv \overline{\Omega} \times \mathbf{R}_+ \times \mathbf{R}$, with (a_{ij}) symmetric and uniformly positive definite and a_0 nonnegative in Q, if the initial data are such that $u^0 \in H^m$, $u_t^0 \in H^{m-1}$ for some $m \geq [N/2] + 2$, and if appropriate compatibility conditions are satisfied at $t = 0$ (namely, if the functions u_j, $j = 0, 1, 2, \ldots,$—where $u_0 = u^0$, $u_1 = u_t^0$ and u_j, $j \geq 2$, denote $\partial_t^j u(\cdot, t)|_{t=0}$ as computed formally in terms of u_0 and u_1 by the differential equation in (1.1)—belong to $\overset{\circ}{H}{}^1$ for $0 \leq j \leq m - 1$), then, for some $t^* > 0$, there exists a unique solution u of (1.1) as a C^k map from $[0, t^*]$ into $H^{m-k}(\Omega)$ for $k = 0, 1, \ldots, m$. By Sobolev's theorem, the solution will be classical provided $m \geq [N/2] + 3$.

We shall assume therefore in the sequel that the data of (1.1) are smooth and compatible enough and t^* is sufficiently small so that a unique smooth solution u of (1.1) exists as above. As a consequence, we shall assume, for the purposes of the error analysis of our schemes, that, in addition to $u(x, t)$, temporal derivatives $\partial_t^j u(x, t)$ of high enough order also vanish for $x \in \partial\Omega$, $t > 0$. We remark that the error analysis will not require any artificial compatibility conditions on the nonhomogeneous term of the type, e.g., that $f(x, t, u) = 0$ for $x \in \partial\Omega$, $t > 0$.

To introduce some more notation, suppose that $u \in [m_1, m_2]$ for $(x, t) \in \overline{\Omega} \times [0, t^*]$. We shall assume that, for some fixed $\delta > 0$, a_{ij}, a_0 and f are defined and are smooth functions of their arguments (x, t, u) in $Q_\delta \equiv \overline{\Omega} \times [0, t^*] \times M_\delta$, where $M_\delta = [m_1 - \delta, m_2 + \delta]$. In particular, we shall repeatedly make use of the fact that the a_{ij}, a_0, f and some of their partial derivatives satisfy Lipschitz conditions with respect to the variable u in M_δ, uniformly with respect to $(x, t) \in \overline{\Omega} \times [0, t^*]$. We assume that (a_{ij}) is symmetric and uniformly positive definite and that a_0 is nonnegative in Q_δ.

Following the notation of [5], we let $Y = \{g \in W^{1,\infty} : g(x) \in M_\delta,\ x \in \overline{\Omega}\}$. For $t \in [0, t^*]$ and $g \in Y$, the operators $L(t, g)$ defined by (1.1) form a smooth family of selfadjoint elliptic operators on L^2 with domain $D_L = H^2 \cap \overset{\circ}{H}{}^1$. For such t and g, given $w \in L^2$, the boundary value problem $L(t, g)v = w$ in Ω, $v = 0$ on $\partial\Omega$, has a unique solution $v \in D_L$ which we represent as $v = T(t,g)w$ in terms of the solution operator $T(t,g)$: $L^2 \to D_L$ defined by $a(t,g)(Tw,\varphi) = (w,\varphi)$ $\forall \varphi \in \overset{\circ}{H}{}^1$, where, for $t \in [0,t^*]$, $g \in Y$,

$$a(t,g)(\varphi,\psi) = \int_\Omega \left[\sum_{i,j=1}^N a_{ij}(x,t,g)\partial_i\varphi\partial_j\psi + a_0(x,t,g)\varphi\psi\right] dx, \qquad \varphi,\psi \in \overset{\circ}{H}{}^1,$$

is a bilinear, symmetric and coercive form on $\overset{\circ}{H}{}^1 \times \overset{\circ}{H}{}^1$. If u is the solution of (1.1), we shall use the notation $L(t) = L(t,u(t))$, $T(t) = T(t,u(t))$ for $t \in [0,t^*]$ and regard $L(t)$, $T(t)$ as smooth families of bounded linear operators from $H^{m+2} \cap D_L$ into H^m, resp. H^m into $H^{m+2} \cap D_L$.

Quasi-Discrete Operators. For $0 < h < 1$, let S_h be a family of finite-dimensional subspaces of $W^{1,\infty}$ in which approximations to the solution of (1.1) will be sought. For $t \in [0,t^*]$ let $T_h(t)$: $L^2 \to S_h$ be a family of linear, bounded 'quasi-discrete' (in the sense that they depend on $u(t)$, the solution of (1.1)) operators, that approximate $T(t)$. Following, e.g., [4], [5], [2], we shall assume that S_h and T_h satisfy the following list of properties, that will be used in the sequel, usually without special reference. (Also, henceforth, c, c_i, etc. will denote, as is customary, positive generic constants, not necessarily the same in any two places, possibly depending on u, t^* and the data of (1.1), but not on discretization parameters such as h and the time step, or elements of S_h, the fully discrete approximations, etc.)

(i) $T_h(t)$ is a family of selfadjoint operators, positive semidefinite on L^2, positive definite on S_h uniformly in $t \in [0,t^*]$.

(ii) There exists an integer $r \geq 2$ and, for $j = 0,1,2,\ldots$, constants c_j such that for $2 \leq s \leq r$

$$\text{(a)} \quad \|(T^{(j)}(t) - T_h^{(j)}(t))f\| \leq c_j h^s \|f\|_{s-2},$$

for all $f \in H^{s-2}$. (In general, for a vector- or operator-valued function $u(t)$, we put $u^{(j)} = D_t^j u(t)$.) Moreover, there exists c such that

$$\text{(b)} \quad |(T(t) - T_h(t))f|_\infty \leq ch^r |\log(h)|^{\bar{r}} \|Tf\|_{r,\infty},$$

where $\bar{r} = 0$ if $r > 2$ and $0 < \bar{r} < \infty$ if $r = 2$, provided $Tf \in W^{r,\infty}$.

(iii) If $L_h(t) = T_h(t)^{-1}$ on S_h, $0 \leq t \leq t^*$, assume that there exist constants c_j, $j = 1,2,\ldots$, such that

$$|(L_h^{(j)}(t)\varphi,\varphi| \leq c_j(L_h(s)\varphi,\varphi) \quad \forall \varphi \in S_h,\ t,s \in [0,t^*].$$

(iv) Assume that there exists a constant c such that the following inverse assumptions hold on S_h (for a justification of (c) cf. Section 5):

(a) $(L_h(t)\varphi,\varphi) \leq ch^{-2}\|\varphi\|^2$ $\forall \varphi \in S_h$, $t \in [0,t^*]$.

(b) $|\varphi|_\infty \leq ch^{-N/2}\|\varphi\|$ $\forall \varphi \in S_h$.

(c) $|\varphi|_\infty \leq c\gamma(h)\|L_h(0)^{1/2}\varphi\|$ $\forall \varphi \in S_h$, where $0 \leq \gamma(h) \leq h^{-1/2}$ for h small enough.

(v) For $t \in [0,t^*]$, $g \in Y$, we postulate the existence of a symmetric bilinear form $a_h(t,g)(\cdot,\cdot)$ on $W^{1,\infty} \times W^{1,\infty}$, which is positive definite on S_h, and of a linear operator $L_h(t,g)$: $S_h \to S_h$ such that

$$\text{(a)} \quad L_h(t,u(t)) = L_h(t), \qquad t \in [0,t^*],$$

$$\text{(b)} \quad a_h(t,g)(\varphi,\psi) = (L_h(t,g)\varphi,\psi), \qquad \varphi,\psi \in S_h,\ t \in [0,t^*].$$

Moreover, assume that there exists c such that for $\varphi, \psi \in S_h$, $g, g_i \in Y$, $s, t \in [0, t^*]$:

(c) $|((L_h(t) - L_h(t,g))\varphi, \psi)| \le c|u(t) - g|_\infty \|L_h^{1/2}(t)\varphi\| \, \|L_h^{1/2}(t)\psi\|$,

(d) $|((L_h(t) - L_h(t,g))\varphi, \psi)| \le c\|u(t) - g\| \, \|\varphi\|_{1,\infty} \|L_h^{1/2}(t)\psi\|$,

(e) $|(a_h(t, g_1) - a_h(t, g_2) - a_h(s, g_3) + a_h(s, g_4))(\varphi, \psi)|$
$\le c[\|g_1 - g_2 - g_3 + g_4\|(1 + |g_3 - g_4|_\infty)$
$+ |g_1 - g_3|_\infty \|g_3 - g_4\| + |t - s| \, \|g_3 - g_4\|]\|\varphi\|_{1,\infty} \|L_h^{1/2}(t)\psi\|$.

An example of a pair S_h, $T_h(t)$ which satisfies the above properties (and from which this list of assumptions is motivated) is furnished by the *standard Galerkin method* in which $S_h \subset \overset{\circ}{H}{}^1 \cap W^{1,\infty}$ is endowed with the approximation property

$$\inf_{\chi \in S_h} (\|u - \chi\| + h\|u - \chi\|_1) \le ch^s \|u\|_s, \qquad 1 \le s \le r, \text{ for } u \in H^r \cap \overset{\circ}{H}{}^1,$$

where the $T_h(t)$: $L^2 \to S_h$ are defined for $f \in L^2$ by $a(t, u(t))(T_h(t)f, \chi) = (f, \chi)$ $\forall \chi \in S_h$ and where the bilinear form a_h coincides with a. For verification of (i)–(iv) in this case, cf., e.g., [2]–[4] and their references. For (iv.c), cf. Section 5. Properties (v.c,d,e) follow easily from the smoothness of the coefficients a_{ij}, a_0 in Q_δ and the definition of a_h.

A number of important inequalities now follow from the above list, cf. [3], [4]. We let in the sequel P: $L^2 \to S_h$ denote the L^2 projection operator onto S_h. Then there exist constants c_j, $j = 0, 1, 2, \ldots$, such that for $t, s \in [0, t^*]$, $\varphi, \psi \in S_h$:

$$\begin{aligned} &\|L_h^{(j)}(t)T_h(s)\|, \|T_h(s)L_h^{(j)}(t)P\| \le c_j, \\ &|(L_h^{(j)}(t)\varphi, \psi)| \le c_j \|L_h^{1/2}(s)\varphi\| \, \|L_h^{1/2}(t)\psi\|, \\ &\|L_h^{(j)}(t)\varphi\| \le c_j \|L_h(s)\varphi\|. \end{aligned} \tag{1.2}$$

Also, as a consequence of (ii.a), there exists c such that

$$\|v - Pv\| \le ch^s \|v\|_s \quad \text{if } 2 \le s \le r \text{ and } v \in H^s \cap D_L. \tag{1.3}$$

Moreover, we shall assume (for a justification, cf. Section 5) that for each $v \in L^\infty$, there exists a constant $c(v)$ such that

$$h\|Pv\|_{1,\infty} \le c(v). \tag{1.4}$$

If $u(t)$ is the solution of (1.1), we let $W(t) = P_I(t)u(t) = T_h(t)L(t)u(t)$ denote the *elliptic projection* of u. As a consequence of our assumptions (i)–(iv), the elliptic projection will satisfy, cf. [3],[4], the following properties, some of which are just restatements, for convenience in referencing, of previously listed ones: there exist constants c, c_i, c_{ij} such that for $t, t' \in [0, t^*]$

$$\|v - P_I(t)v\| \le ch^s \|v\|_s, \qquad 2 \le s \le r, \ v \in D_L \cap H^s, \tag{1.5}$$

$$\|u^{(m)}(t) - W^{(m)}(t)\| \le c_m h^s, \qquad 2 \le s \le r, \ m \ge 0, \tag{1.6}$$

$$\|L_h^{(i)}(t)W^{(j)}(t')\| \le c_{ij}, \qquad i, j \ge 0, \tag{1.7}$$

$$|u(t) - W(t)|_\infty \le ch^r |\log h|^{\bar{r}}, \qquad \bar{r} \text{ as in (ii.b)}. \tag{1.8}$$

We shall also need the property that for constants c_j

$$\|W^{(j)}(t)\|_{1,\infty} \le c_j, \qquad j = 0, 1, \ t \in [0, t^*], \tag{1.9}$$

which we shall justify under some additional assumptions in Section 5.

Full Discretizations. For the purpose of introducing the fully discrete approximations, we consider the 'quasi-discrete' problem, i.e., define w_h: $[0, t^*] \to S_h$ such that

$$w_h^{(2)}(t) + L_h(t) w_h(t) = Pf(t), \qquad 0 \le t \le t^*, \tag{1.10}$$

where $f(t) = f(t, u(t))$. As $w_h(t)$ will play no role in the analysis and the proofs, other than that of motivating the construction of the fully discrete schemes, we shall assume that supplementing (1.10) with initial conditions $w_h(0)$, $w_{h,t}(0)$ will produce a unique, sufficiently smooth solution $w_h(t)$, $0 \le t \le t^*$.

Our time-stepping procedures will be based on fourth-order accurate rational approximations $r(x)$ to $\cos(x)$, [3], of the form

$$r(x) = (1 + p_1 x^2 + p_2 x^4)/(1 + q_1 x^2 + q_2 x^4)$$

with $q_1, q_2 > 0$. We shall assume for accuracy and stability purposes that $p_1 = q_1 - 1/2$, $p_2 = q_2 - q_1/2 + 1/24$, and that the pair (q_1, q_2) belongs to the stability region $\overline{\mathscr{R}}$ of the $q_1, q_2 > 0$ quarterplane, [3]. Let $k > 0$ denote the time step, let $t_n = nk$, $n = 0, 1, 2, \ldots, J$, and assume that $t^* = Jk$. In the sequel we shall employ the following notation: $L_n = L_h(t_n)$, $L_n^{(j)} = L_h^{(j)}(t_n)$, $T_n = T_h(t_n)$, $T_n^{(j)} = T_h^{(j)}(t_n)$, $f^n = Pf(t_n)$, $f^{(j)n} = Pf^{(j)}(t_n) = Pf^{(j)}(t_n, u(t_n))$, $w^n = w_h(t_n)$, $w^{(j)n} = w_h^{(j)}(t_n)$. As in [3], approximating $\cosh(z) = \cos(iz)$ by $r(iz)$ in the formal relation $w^{n+1} + w^{n-1} = 2\cosh(kD_t) w^n$, $D_t = d/dt$, we have, for w_h smooth enough,

$$\begin{aligned}(I - q_1 k^2 D_t^2 + q_2 k^4 D_t^4)&(w^{n+1} + w^{n-1}) \\ &= 2(I - p_1 k^2 D_t^2 + p_2 k^4 D_t^4) w^n + O(k^6 w_h^{(6)}).\end{aligned}$$

Differentiating now (1.10), we obtain

$$\begin{aligned}w_h^{(4)}(t) = &- L_h(t)(-L_h(t) w_h(t) + Pf(t)) - L_h^{(2)}(t) w_h(t) - 2L_h^{(1)}(t) w_h^{(1)}(t) \\ &+ Pf^{(2)}(t).\end{aligned}$$

Substituting this in the above relation and using the notations $q(\tau) = 1 + q_1\tau + q_2\tau^2$, $p(\tau) = 1 + p_1\tau + p_2\tau^2$, $Q_n = q(k^2 L_n)$, $P_n = p(k^2 L_n)$, yields the following temporal discretization of (1.10):

$$\begin{aligned}Q_{n+1}&w^{n+1} - 2P_n w^n + Q_{n-1} w^{n-1} \\ &= k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) \\ &\quad + k^4(q_2 L_{n+1} f^{n+1} - 2p_2 L_n f^n + q_2 L_{n-1} f^{n-1}) \\ &\quad + q_2 k^4 (L_{n+1}^{(2)} w^{n+1} - 2L_n^{(2)} w^n + L_{n-1}^{(2)} w^{n-1}) + 2(q_2 - p_2) k^4 L_n^{(2)} w^n \\ &\quad + 2q_2 k^4 (L_{n+1}^{(1)} w^{(1)n+1} - 2L_n^{(1)} w^{(1)n} + L_{n-1}^{(1)} w^{(1)n-1}) \\ &\quad + 4(q_2 - p_2) k^4 L_n^{(1)} w^{(1)n} \\ &\quad - q_2 k^4 (f^{(2)n+1} - 2f^{(2)n} + f^{(2)n-1}) - 2(q_2 - p_2) k^4 f^{(2)n} + O(k^6).\end{aligned}$$

Since we are interested in fourth-order methods, we put $q_2 - p_2 = (q_1 - 1/12)/2$ and drop the (presumably of $O(k^6)$) second-order central differences in the right-hand side of the above. We also replace the derivative $w^{(1)n}$, using the relation

$w^{(1)n} = k^{-1}(w^n - w^{n-1}) + kw^{(2)n}/2 + O(k^2)$ and computing $w^{(2)n}$ by (1.10). The resulting relation yields that up to presumably $O(k^6)$ terms,

$$
\begin{aligned}
&Q_{n+1}w^{n+1} - 2P_n w^n + Q_{n-1}w^{n-1} \\
&\quad \cong k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) \\
&\quad + k^4(q_2 L_{n+1} f^{n+1} - 2p_2 L_n f^n + q_2 L_{n-1} f^{n-1}) \\
&\quad + (q_1 - 1/12)k^4\{L_n^{(2)} w^n + 2L_n^{(1)}[k^{-1}(w^n - w^{n-1}) \\
&\qquad + (k/2)(-L_n w^n + f^n)] - f^{(2)n}\}.
\end{aligned} \tag{1.11}
$$

Motivated by (1.11), we can now state the fully discrete scheme. We shall seek $U^n \in S_h$ approximating $u^n = u(t_n)$ for $0 \le n \le J$. To avoid solving nonlinear systems of equations at every time step, when called upon to evaluate the coefficients and the right-hand side at the advanced time level $n+1$, we shall substitute (as was done in the parabolic case in [5]) for U^{n+1} an approximation $\hat{U}^{n+1}$ to u^{n+1} obtained by suitable extrapolation from values of U^m, $m \le n$. The precise formulas for the $\hat{U}^{n+1}$ will be specified in Section 3. We shall also replace the derivatives $L_n^{(j)}$, $f^{(j)n}$ in (1.11) by appropriate difference quotients. To this end, we use the notations

$$
\begin{aligned}
&\delta^2 L_n(\hat{U}^{n+1}, U^n, U^{n-1}) \equiv k^{-2}(L_{n+1}(\hat{U}^{n+1}) - 2L_n(U^n) + L_{n-1}(U^{n-1})), \\
&\delta L_n(\hat{U}^{n+1}, U^{n-1}) \equiv (2k)^{-1}(L_{n+1}(\hat{U}^{n+1}) - L_{n-1}(U^{n-1})), \\
&\delta^2 f^n(\hat{U}^{n+1}, U^n, U^{n-1}) \equiv k^{-2}(f^{n+1}(\hat{U}^{n+1}) - 2f^n(U^n) + f^{n-1}(U^{n-1})),
\end{aligned} \tag{1.12}
$$

where, for $g^n \in Y$, $0 \le n \le J$, we put $L_n(g^n) = L_h(t_n, g^n)$ and $f^n(g^n) = Pf(t_n, g^n)$. Letting $\hat{A}_n = q(k^2 L_n(\hat{U}^n))$, $A_n = q(k^2 L_n(U^n))$, $B_n = p(k^2 L_n(U^n))$, we can finally state our fully discrete method, which we shall refer to as the *base scheme*:

$$
\begin{aligned}
&\hat{A}_{n+1}U^{n+1} - 2B_n U^n + A_{n-1}U^{n-1} = \Theta(\hat{U}^{n+1}, U^n, U^{n-1}) \\
&\quad \equiv k^2(q_1 f^{n+1}(\hat{U}^{n+1}) - 2p_1 f^n(U^n) + q_1 f^{n-1}(U^{n-1})) \\
&\quad + k^4(q_2 L_{n+1}(\hat{U}^{n+1}) f^{n+1}(\hat{U}^{n+1}) - 2p_2 L_n(U^n) f^n(U^n) \\
&\qquad + q_2 L_{n-1}(U^{n-1}) f^{n-1}(U^{n-1})) \\
&\quad + (q_1 - 1/12)k^4\{\delta^2 L_n(\hat{U}^{n+1}, U^n, U^{n-1})U^n + 2\delta L_n(\hat{U}^{n+1}, U^{n-1}) \\
&\qquad \cdot [k^{-1}(U^n - U^{n-1}) + (k/2)(-L_n(U^n)U^n + f^n(U^n))] \\
&\qquad - \delta^2 f^n(\hat{U}^{n+1}, U^n, U^{n-1})\}.
\end{aligned} \tag{1.13}
$$

We shall compute U^{n+1} for $1 \le n \le J-1$ from this scheme. In Section 3 we shall specify our starting procedure, i.e., the definitions of U^0, U^1 and the 'lagged' term $\hat{U}^{n+1}$, $1 \le n \le J-1$. In the same section we shall show that, under appropriate stability restrictions (in general that kh^{-1} remain arbitrary but bounded as $k, h \to 0$ and, for some choices of the parameters q_1, q_2, that kh^{-1} remain small), there exists a constant c such that

$$\max_{0\le n\le J}\|u^n - U^n\| \le c(k^4 + h^r),$$

i.e., that an optimal-order in space and time L^2 error estimate holds. However, solving for U^{n+1} by (1.13) necessitates solving linear systems with the operators $\hat{A}_{n+1}$ that change with each time step. Using preconditioned iterative techniques following [12], [4], [3], we show in Section 4 how to modify the base scheme so that the resulting fully discrete methods require solving $O(|\log(k)|)$ linear systems at each time step with the same matrix and preserve the stability and accuracy of the base scheme. These results are preceded by a series of technical lemmata and 'a priori' stability and convergence estimates, which we present in Section 2. The paper closes with an appendix (Section 5) in which we collect evidence of the validity of several technical inequalities that are assumed in the previous sections. The proofs of the main result of Section 2, of some results of Section 3, and all of Section 5 can be found in the Supplement to the paper in the supplements section of this issue in Sections S2, S3, S5, respectively.

2. Consistency and Preliminary Error Estimates. In this section we shall study the problem of existence of solutions and the consistency of the base scheme (1.13) and derive several preliminary error estimates and a priori stability results that will prepare the way for the main convergence theorem of Section 3. The proofs of many intermediate results can be found in detail in the Supplement to the paper in the supplements section of this issue.

We begin with a technical lemma that supplements the inequalities of the type (v.c, d) in Section 1.

LEMMA 2.1. *There exists a constant $c > 0$ such that for $g \in Y$, $t \in [0, t^*]$:*

$$\|(L_h(t) - L_h(t,g))\psi\| \le \begin{cases} ch^{-1}|u(t) - g|_\infty \|L_h^{1/2}(t)\psi\| \\ ch^{-1}\|u(t) - g\|\,\|\psi\|_{1,\infty} \end{cases} \quad \textit{for } \psi \in S_h, \tag{2.1}$$

$$\begin{aligned} &|((L_h^2(t) - L_h^2(t,g))\psi, \varphi)| \\ &\quad \le ch^{-1}|u(t) - g|_\infty(\|L_h^{1/2}(t)\psi\|\,\|L_h(t)\varphi\| + \|L_h(t)\psi\|\,\|L_h^{1/2}(t)\varphi\|) \\ &\qquad + ch^{-2}|u(t) - g|_\infty^2\|L_h^{1/2}(t)\psi\|\,\|L_h^{1/2}(t)\varphi\|, \quad \textit{for } \varphi, \psi \in S_h. \end{aligned} \tag{2.2}$$

Proof. The estimate (2.1) follows from (v.c,d) and (iv.a). Using, for $\varphi, \psi \in S_h$,

$$\begin{aligned} ((L_h^2(t) - L_h^2(t,g))\psi, \varphi) &= ((L_h(t) - L_h(t,g))\psi,\ L_h(t)\varphi) \\ &\quad + (L_h(t,g)\psi,\ (L_h(t) - L_h(t,g))\varphi) \end{aligned}$$

and noting that $\|L_h(t,g)\psi\| \le \|(L_h(t,g) - L_h(t))\psi\| + \|L_h(t)\psi\|$, we obtain (2.2) from (v.c), (iv.a) and (2.1). □

The next result concerns the invertibility of the linear operator $\hat{A}_{n+1}$ on S_h. In the sequel we denote $e^n = u^n - U^n$, $\hat{e}^n = u^n - \hat{U}^n$.

LEMMA 2.2. *Suppose that $1 \le n \le J-1$ and $\hat{U}^{n+1} \in S_h \cap Y$. Then there exists a constant c such that for $\varphi, \psi \in S_h$*

$$(2.3)\quad \begin{aligned} &|((Q_{n+1} - \hat{A}_{n+1})\psi, \varphi)| \\ &\quad \le cq_1 k^2 h^{-1} |\hat{e}^{n+1}|_\infty \|L_{n+1}^{1/2}\psi\|\,\|\varphi\| \\ &\quad + cq_2 k^4 h^{-1} |\hat{e}^{n+1}|_\infty (\|L_{n+1}^{1/2}\psi\|\,\|L_{n+1}\varphi\| + \|L_{n+1}\psi\|\,\|L_{n+1}^{1/2}\varphi\|) \\ &\quad + cq_2 k^4 h^{-2} |\hat{e}^{n+1}|_\infty^2 \|L_{n+1}^{1/2}\psi\|\,\|L_{n+1}^{1/2}\varphi\|. \end{aligned}$$

If in addition there exists $\alpha > 0$ such that $kh^{-1} \le \alpha$, and if $|\hat{e}^{n+1}|_\infty$ is sufficiently small (or if $|\hat{e}^{n+1}|_\infty \le ch$ and k is sufficiently small), then $\hat{A}_{n+1}$ is invertible on S_h, and U_{n+1}, defined by (1.13), *exists uniquely, given $U^n, U^{n-1}, \hat{U}^{n+1}$.*

Proof. Since

$$Q_{n+1} - \hat{A}_{n+1} = q_1 k^2 (L_{n+1} - L_{n+1}(\hat{U}^{n+1})) + q_2 k^4 (L_{n+1}^2 - L_{n+1}^2(\hat{U}^{n+1})),$$

(2.3) follows from (v.c), (iv.a), (2.2). Putting $\psi = \varphi$ in (2.3) and using the arithmetic-geometric mean (agm) inequality gives

$$\begin{aligned} |((Q_{n+1} - \hat{A}_{n+1})\varphi, \varphi)| &\le c(kh^{-1}|\hat{e}^{n+1}|_\infty + k^2 h^{-2} |\hat{e}^{n+1}|_\infty^2) \\ &\quad \cdot (\|\varphi\|^2 + k^2 \|L_{n+1}^{1/2}\varphi\|^2 + q_2 k^4 \|L_{n+1}\varphi\|^2). \end{aligned}$$

Letting $\tilde{Q}_{n+1} = I + k^2 L_{n+1} + q_2 k^4 (L_{n+1})^2$, one may easily see, cf. [4], that for positive constants c_i there holds $c_1(Q_{n+1}\varphi, \varphi) \le (\tilde{Q}_{n+1}\varphi, \varphi) \le c_2(Q_{n+1}\varphi, \varphi)$ for every $\varphi \in S_h$. Hence,

$$(2.4)\quad |((Q_{n+1} - \hat{A}_{n+1})\varphi, \varphi)| \le c(kh^{-1}|\hat{e}^{n+1}|_\infty + k^2 h^{-2}|\hat{e}^{n+1}|_\infty^2)(Q_{n+1}\varphi, \varphi)$$

for $\varphi \in S_h$, and the invertibility of $\hat{A}_{n+1}$ follows from that of $Q_{n+1} (q_1, q_2 > 0)$. □

Assuming that $1 \le n \le J-1$, that U^n, U^{n-1}, $\hat{U}^{n+1}$ exist in S_h and that the hypotheses of Lemma 2.2 hold, we let $E^n = U^n - W^n$, where $W^n = W(t_n) = P_I(t_n)u^n$. For $\varphi_j \in S_h$, $j = n-1, n, n+1$, we define

$$(2.5)\quad \mathbf{S}_n \varphi_n = (Q_{n+1} - \hat{A}_{n+1})\varphi_{n+1} - 2(P_n - B_n)\varphi_n + (Q_{n-1} - A_{n-1})\varphi_{n-1}$$

and obtain, using (1.13), the *error equation*

$$(2.6)\quad \begin{aligned} &Q_{n+1}E^{n+1} - 2P_n E^n + Q_{n-1}E^{n-1} \\ &\quad = \mathbf{S}_n E^n + \mathbf{S}_n W^n + \Theta(\hat{U}^{n+1}, U^n, U^{n-1}) \\ &\qquad - (Q_{n+1}W^{n+1} - 2P_n W^n + Q_{n-1}W^{n-1}). \end{aligned}$$

The next lemma is a *consistency* result for the scheme (1.13). (In the sequel we let $u^{(j)n} = u^{(j)}(t_n)$.)

LEMMA 2.3. *Let $1 \le n \le J-1$ and suppose that the solution u and the data of* (1.1) *are sufficiently smooth. Then*

$$(2.7)\quad \begin{aligned} &Q_{n+1}W^{n+1} - 2P_n W^n + Q_{n-1}W^{n-1} \\ &\quad = Y^n + k^2(q_1 f^{n+1} - 2p_1 f^n + q_1 f^{n-1}) \\ &\qquad + k^4(q_2 L_{n+1} f^{n+1} - 2p_2 L_n f^n + q_2 L_{n-1} f^{n-1}) \\ &\qquad + k^4(q_1 - 1/12)(PL^{(2)}(t_n)u^n + 2PL^{(1)}(t_n)u^{(1)n} - f^{(2)n}), \end{aligned}$$

where for some constant c

$$|(Y^n,\varphi)| \le ck^2(k^4+h^r)(\|\varphi\|+k^2\|L_n\varphi\|) \quad \forall\varphi\in S_h. \tag{2.8}$$

Proof. See Section S2 in the Supplement to the paper. □

Defining now, for $1\le n\le J-1$,

$$\begin{aligned}\Lambda(&\hat U^{n+1},U^n,U^{n-1})\\ &=\Theta(\hat U^{n+1},U^n,U^{n-1})-k^2(q_1f^{n+1}-2p_1f^n+q_1f^{n-1})\\ &\quad-k^4(q_2L_{n+1}f^{n+1}-2p_2L_nf^n+q_2L_{n-1}f^{n-1})\\ &\quad-k^4(q_1-1/12)(PL^{(2)}(t_n)u^n+2PL^{(1)}(t_n)u^{(1)n}-f^{(2)n}),\end{aligned} \tag{2.16}$$

we see that the error equation (2.6) may be written as

$$\begin{aligned}Q_{n+1}&E^{n+1}-2P_nE^n+Q_{n-1}E^{n-1}\\ &=\mathbf{S}_nE^n+\mathbf{S}_nW^n+\Lambda(\hat U^{n+1},U^n,U^{n-1})-Y^n,\end{aligned}$$

with Y^n as in (2.7)–(2.8). Taking the L^2 inner product of both sides of this equation with $E^{n+1}-E^{n-1}$, and using the symmetry of Q_n, P_n, we obtain

$$\begin{aligned}(Q_{n+1}&E^{n+1},E^{n+1})-(Q_{n-1}E^{n-1},E^{n-1})\\ &-2[(P_{n+1}E^{n+1},E^n)-(P_nE^n,E^{n-1})]\\ &=((Q_{n+1}-Q_{n-1})E^{n+1},E^{n-1})-2((P_{n+1}-P_n)E^{n+1},E^n)\\ &\quad+(\mathbf{S}_nE^n+\mathbf{S}_nW^n+\Lambda(\hat U^{n+1},U^n,U^{n-1})-Y^n,E^{n+1}-E^{n-1}).\end{aligned} \tag{2.17}$$

A basic error inequality is given in the following

LEMMA 2.4. *Suppose that* $1\le m\le l\le J-1$, *that* U^n, $m-1\le n\le l+1$ *and* $\hat U^{n+1}$, $m\le n\le l$, *exist uniquely in* S_h (*i.e, that the* $\hat A_{n+1}$ *are invertible for* $m\le n\le l$). *Then*

$$\begin{aligned}\eta^{(1)}_{l+1}\le \eta^{(1)}_m &+ck^2(k^4+h^r)^2((l-m+1)k)\\ &+ck\sum_{n=m}^{l}\{\|E^{n+1}-E^{n-1}\|^2\\ &\qquad+k^2(\|L_n^{1/2}E^{n+1}\|^2+\|L_n^{1/2}E^n\|^2+\|L_n^{1/2}E^{n-1}\|^2)\\ &\qquad+k^4(\|L_nE^{n+1}\|^2+\|L_nE^n\|^2+\|L_nE^{n-1}\|^2\\ &\qquad\qquad+\|L_n(E^{n+1}-E^{n-1})\|^2\}\\ &+\sum_{n=m}^{l}(\mathbf{S}_nE^n+\mathbf{S}_nW^n+\Lambda(\hat U^{n+1},U^n,U^{n-1}),E^{n+1}-E^{n-1}),\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}\eta^{(1)}_j\equiv\|E^j-E^{j-1}\|^2&+k^2((q_1-p_1)/2)\|L_j^{1/2}(E^j+E^{j-1})\|^2\\ &+k^2((q_1+p_1)/2)\|L_j^{1/2}(E^j-E^{j-1})\|^2\\ &+k^4((q_2-p_2)/2)\|L_j(E^j+E^{j-1})\|^2\\ &+k^4((q_2+p_2)/2)\|L_j(E^j-E^{j-1})\|^2.\end{aligned} \tag{2.19}$$

Proof. The proof follows by summing both sides of (2.17) from $n = m$ to $n = l$, proceeding as in the proof of Theorem 2.1 of [3]—noting that the analogs of (2.30) and (2.32) of [3] hold here too—and making use of the estimate, cf. (2.8):

$$\begin{aligned}\sum_{n=m}^{l}(Y^n, E^{n+1} - E^{n-1}) \\ \le c\sum_{n=m}^{l}(k^3(k^4+h^r)^2 + k\|E^{n+1}-E^{n-1}\|^2 + k^5\|L_n(E^{n+1}-E^{n-1})\|^2). \quad \square\end{aligned}$$

We must now estimate the last three sums in the right-hand side of (2.18). This is carried out in Section S2 of the Supplement to the paper. Specifically, in Lemma 2.5 in the Supplement, we estimate the term $\sum_n(\mathbf{S}_n E^n, E^{n+1} - E^{n-1})$ in a straightforward way, following estimates analogous to those that led to (2.3). The term $\sum_n(\mathbf{S}_n W^n, E^{n+1} - E^{n-1})$ is estimated piecemeal in Lemmata 2.6, 2.7 and 2.8 in the Supplement. (It turns out that further use of these estimates will be made in Section 3 in the cases $l \ge m+2$ and $l = m$. Lemmata 2.6–2.8 deal with the case $l \ge m+2$, while the term with $l = m$ is easily estimated in (2.40), cf. Section S2.) Finally, the term $\sum_n(\Lambda(\hat{U}^{n+1}, U^n, U^{n-1}), E^{n+1} - E^{n-1})$ is broken into five parts which are then estimated in Lemmata 2.9–2.13 in the Supplement and complete the a priori estimation of all terms in the right-hand side of (2.18). For convenience in later use we collect below, summarize and simplify the results of Lemmata 2.4–2.13, distinguishing between the cases $l \ge m+2$ and $l = m$.

PROPOSITION 2.1. *Suppose that $1 \le m, l \le J-1$ and $l \ge m+2$, that U^j, $m-1 \le j \le l$ exist in $S_h \cap Y$, that U^{l+1} exists in S_h, that $\hat{U}^j$, $m+1 \le j \le l+1$ exist in $S_h \cap Y$, that* (1.4) *and* (1.9) *hold and that there exists $\alpha > 0$ such that $kh^{-1} \le \alpha$. Then, with $\eta_m^{(j)}$, $j = 1,2,3$, defined by* (2.19), (2.24), (2.54) (*cf. Section* S2), *respectively, given $\varepsilon_1, \varepsilon_2 > 0$, there exists a constant $c(\varepsilon_1, \varepsilon_2) > 0$ such that*

$$\begin{aligned}\|E^{l+1} - E^l\|^2 &+ k^2((q_1-p_1)/2)\|L_{l+1}^{1/2}(E^{l+1}+E^l)\|^2 \\ &+ k^2((q_1+p_1)/2)\|L_{l+1}^{1/2}(E^{l+1}-E^l)\|^2 \\ &+ k^4((q_2-p_2)/2)\|L_{l+1}(E^{l+1}+E^l)\|^2 \\ &+ k^4((q_2+p_2)/2)\|L_{l+1}(E^{l+1}-E^l)\|^2 \le \sum_{j=1}^{7} F_j,\end{aligned} \tag{2.88}$$

where

$$\begin{aligned}F_1 &= \eta_m^{(1)} + \eta_m^{(2)} + \eta_m^{(3)}, \\ F_2 &= \varepsilon_1 k^2(\|L_{l+1}^{1/2}(E^{l+1}+E^l)\|^2 + \|L_{l+1}^{1/2}(E^{l+1}-E^l)\|^2) \\ &\quad + \varepsilon_2 k^4(\|L_{l+1}(E^{l+1}+E^l)\|^2 + \|L_{l+1}(E^{l+1}-E^l)\|^2) \\ &\quad + c(\varepsilon_1,\varepsilon_2)k^2\left[\sum_{j=1}^{l+1}\|\hat{e}^j\|^2(1+|\hat{e}^j|_\infty^2) + \sum_{j=l-2}^{l}\|e^j\|^2(1+|e^j|_\infty^2)\right],\end{aligned}$$

$$F_3 = ck^2(k^4 + h^r)^2(l - m + 1)k,$$

$$F_4 = ck \sum_{n=m-1}^{l} \{\|E^{n+1} - E^n\|^2 + k^2(\|L_{n+1}^{1/2}(E^{n+1} + E^n)\|^2 + \|L_{n+1}^{1/2}(E^{n+1} - E^n)\|^2) + k^4(\|L_{n+1}(E^{n+1} + E^n)\|^2 + \|L_{n+1}(E^{n+1} - E^n)\|^2)\},$$

$$\begin{aligned}F_5 = {} & ck \sum_{n=m+1}^{l-1} \{|e^{n-1}|_\infty^2 \|E^{n+1} - E^{n-1}\|^2 + |e^{n-2}|_\infty^2 \|E^n - E^{n-2}\|^2\} \\
& + ck \sum_{n=m}^{l} \{h^{-2}(|\hat{e}^{n+1}|_\infty^2 + |e^n|_\infty^2 + |e^{n-1}|_\infty^2) \\
& \quad \times (\|E^{n+1} - E^{n-1}\|^2 + \|E^n - E^{n-2}\|^2) \\
& \quad + h^{-1}(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)\|E^{n+1} - E^{n-1}\|^2 \\
& \quad + k^2h^{-1}(|\hat{e}^{n+1}|_\infty \|L_{n+1}^{1/2}E^{n+1}\|^2 + |e^n|_\infty \|L_n^{1/2}E^n\|^2 + |e^{n-1}|_\infty \|L_{n-1}^{1/2}E^{n-1}\|^2) \\
& \quad + k^2h^{-1}(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)\|L_{n+1}^{1/2}(E^{n+1} - E^n)\|^2 \\
& \quad + k^3h^{-2}(|\hat{e}^{n+1}|_\infty^2 \|L_{n+1}^{1/2}E^{n+1}\|^2 + |e^n|_\infty^2 \|L_n^{1/2}E^n\|^2 + |e^{n-1}|_\infty^2 \|L_{n-1}^{1/2}E^{n-1}\|^2) \\
& \quad + k^3h^{-2}(|\hat{e}^{n+1}|_\infty^2 + |e^n|_\infty^2 + |e^{n-1}|_\infty^2)\|L_{n+1}^{1/2}(E^{n+1} - E^{n-1})\|^2 \\
& \quad + k^4h^{-1}(|\hat{e}^{n+1}|_\infty \|L_{n+1}E^{n+1}\|^2 + |e^n|_\infty \|L_nE^n\|^2 + |e^{n-1}|_\infty \|L_{n-1}E^{n-1}\|^2) \\
& \quad + k^4h^{-1}(|\hat{e}^{n+1}|_\infty + |e^n|_\infty + |e^{n-1}|_\infty)\|L_{n+1}(E^{n+1} - E^{n-1})\|^2\} \\
& + ck \sum_{n=m+1}^{l-1} \left\{ k^2h^{-2} \left(|\hat{e}^{n+2}|_\infty^2 + |\hat{e}^n|_\infty^2 + \sum_{j=n-2}^{n+1} |e^j|_\infty^2 \right) \|L_n^{1/2}E^n\|^2 + k^2h^{2r}(|e^{n-1}|_\infty^2 + |e^{n-2}|_\infty^2) \right\},\end{aligned}$$

$$F_6 = ck \sum_{n=m}^{l} k^2(\|\hat{e}^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2),$$

$$F_7 = ck \sum_{n=m+1}^{l-1} \|\hat{e}^{n+2} - \hat{e}^n\|^2(1 + |\hat{e}^n|_\infty^2). \quad \square$$

We also examine for later use the case $l = m$, $1 \le m \le J - 1$. Assuming that for such m, U^j exist in $S_h \cap Y$ for $m - 1 \le j \le m$ and in S_h for $j = m + 1$, that $U^{m+1} \in S_h \cap Y$, that (1.4) and (1.9) hold and that there exists $\alpha > 0$ such that $kh^{-1} \le \alpha$, then, with $\eta_j^{(1)}$ defined by (2.19), we have that, given $\varepsilon_i > 0$, $1 \le i \le 4$, there exists a constant $c_\varepsilon \equiv c(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) > 0$ such that

$$\begin{aligned}
\eta_{m+1}^{(1)} \le \eta_m^{(1)} &+ \varepsilon_1 k^2 (\|L_{m+1}^{1/2} E^{m+1}\|^2 + \|L_{m+1}^{1/2} E^{m-1}\|^2) \\
&+ \varepsilon_2 k^4 (\|L_{m+1} E^{m+1}\|^2 + \|L_{m+1} E^{m-1}\|^2) \\
&+ \varepsilon_3 k^2 \|L_m^{1/2} E^{m+1}\|^2 + \varepsilon_4 k^2 \|L_m^{1/2} E^{m-1}\|^2 \\
&+ c_\varepsilon k^2 \left[\|\hat{e}^{m+1}\|^2 (1 + |\hat{e}^{m+1}|_\infty^2) + \sum_{j=m-1}^{m} \|e^j\|^2 (1 + |e^j|_\infty^2) \right] \\
&+ ck^3 (k^4 + h^r)^2 + ck \|E^{m+1} - E^{m-1}\|^2 \\
&+ ck^3 \left(\sum_{j=m-1}^{m+1} \|L_j^{1/2} E^j\|^2 + \|L_m^{1/2} (E^{m+1} - E^m)\|^2 \right. \\
&\qquad \left. + \|L_m^{1/2} (E^m - E^{m-1})\|^2 \right) \\
&+ ck^5 \left(\sum_{j=m-1}^{m+1} \|L_j E^j\|^2 + \|L_m (E^{m+1} - E^{m-1})\|^2 \right) \\
&+ ckh^{-1} (|\hat{e}^{m+1}|_\infty + |e^m|_\infty + |e^{m-1}|_\infty) \|E^{m+1} - E^{m-1}\|^2 \\
&+ ckh^{-2} (|\hat{e}^{m+1}|_\infty^2 + |e^m|_\infty^2 + |e^{m-1}|_\infty^2) \\
&\times (\|E^{m+1} - E^m\|^2 + \|E^m - E^{m-1}\|^2) \\
&+ ck^3 h^{-1} (|\hat{e}^{m+1}|_\infty \|L_{m+1}^{1/2} E^{m+1}\|^2 + |e^m|_\infty \|L_m^{1/2} E^m\|^2 \\
&\qquad + |e^{m-1}|_\infty^2 \|L_{m-1}^{1/2} E^{m-1}\|^2) \\
&+ ck^3 h^{-1} (|\hat{e}^{m+1}|_\infty + |e^m|_\infty + |e^{m-1}|_\infty) \|L_m^{1/2} (E^{m+1} - E^{m-1})\|^2 \\
&+ ck^4 h^{-2} (|\hat{e}^{m+1}|_\infty^2 \|L_{m+1}^{1/2} E^{m+1}\|^2 + |e^m|_\infty^2 \|L_m^{1/2} E^m\|^2 \\
&\qquad + |e^{m-1}|_\infty^2 \|L_{m-1}^{1/2} E^{m-1}\|^2) \\
&+ ck^4 h^{-2} (|\hat{e}^{m+1}|_\infty^2 + |e^m|_\infty^2 + |e^{m-1}|_\infty^2) \|L_{m+1}^{1/2} (E^{m+1} - E^{m-1})\|^2 \\
&+ ck^5 h^{-1} (|\hat{e}^{m+1}|_\infty \|L_{m+1} E^{m+1}\|^2 + |e^m|_\infty \|L_m E^m\|^2 \\
&\qquad + |e^{m-1}|_\infty \|L_{m-1} E^{m-1}\|^2) \\
&+ ck^5 h^{-1} (|\hat{e}^{m+1}|_\infty + |e^m|_\infty + |e^{m-1}|_\infty) \|L_m (E^{m+1} - E^{m-1})\|^2 \\
&+ ck^3 (\|\hat{e}^{m+1}\|^2 + \|e^m\|^2 + \|e^{m-1}\|^2).
\end{aligned} \tag{2.89}$$

3. Starting and Convergence of the Scheme. In this section we shall complete the base scheme (1.13) by specifying U^0, U^1, and the formulas for computing $\hat{U}^{n+1}$. We shall then prove, in Theorem 3.1, an optimal-order L^2-error estimate for the base scheme. The starting will be done in two phases: first we specify U^0 and compute U^1 using a single-step method; we also prove some associated error estimates. The values $U^j, j \ge 2$, will be computed using the base scheme. It turns out that it is necessary to analyze the error of the approximation U^j, $2 \le j \le 5$ (and compute the associated $\hat{U}^j$) in a special way. Finally, we specify $\hat{U}^j$ for $j > 5$ and prove the main stability-convergence result. The proofs and statements of many intermediate results appear in the Supplement to the paper.

Computing U^0, U^1. We shall take

$$U^0 = W^0 = T_0 L(0) u^0. \tag{3.1}$$

To define U^1, let $S_h^2 = S_h \times S_h$ and, adopting the notation of [3, Section 3] or [2], introduce the inner product $((\Phi, \Psi))_n = (\varphi_1, \psi_1) + (T_n \varphi_2, \psi_2)$ for $\Phi = (\varphi_1, \varphi_2)^T$, $\Psi = (\psi_1, \psi_2)^T \in S_h^2$, and the associated norm $|||\Phi|||_n = ((\Phi, \Phi))_n^{1/2}$. Let $\tilde{r}(z)$ be the (2,2)-Padé approximant to e^z, i.e., let

$$\tilde{r}(z) = (1 + z/2 + z^2/12)/(1 - z/2 + z^2/12) \equiv \tilde{p}(z)/\tilde{q}(z). \tag{3.2}$$

Defining

$$\mathbf{L}_m = \mathbf{L}_h(t_m) \equiv \begin{pmatrix} 0 & I \\ -L_m & 0 \end{pmatrix},$$

$$\mathbf{L}_m(g) = \mathbf{L}_h(t_m, g) \equiv \begin{pmatrix} 0 & I \\ -L_m(g) & 0 \end{pmatrix}, \qquad g \in Y,$$

and $\mathbf{U}^0 \in S_h^2$ as

$$\mathbf{U}^0 \equiv (U_1^0, U_2^0)^T = (W^0, W^{(1)0})^T \equiv \mathbf{W}^0 \tag{3.3}$$

(so that $U^0 = U_1^0 = W^0$), compute for $j = 1, 2, 3$, $\hat{U}_1^j \in S_h$ by

$$\hat{U}_1^j = U^0 + P[jku^{(1)0} + (jk)^2 u^{(2)0}/2! + (jk)^3 u^{(3)0}/3!]. \tag{3.4}$$

It is assumed that in (3.3), (3.4), $u^{(2)0}$, $u^{(3)0}$ and $W^{(1)0} = (T_h(t)L(t)u(t))^{(1)}|_{t=0}$ will be evaluated using the differential equation in (1.1) at $t = 0$. As U^1 we shall then take

$$U^1 = U_1^1, \tag{3.5}$$

where $\mathbf{U}^1 = (U_1^1, U_2^1)^T \in S_h^2$ is the solution of the linear system

$$\mathbf{A}_1 \mathbf{U}^1 = \mathbf{B}_0 \mathbf{U}^0 + \mathbf{F}^0 \tag{3.6}$$

with

$$\begin{aligned} \mathbf{A}_1 &= \tilde{q}(k\mathbf{L}_1(U_1^1)) \\ &\quad + (k^2/12)[(6k)^{-1}(-\mathbf{L}_3(\hat{U}_1^3) + 6\mathbf{L}_2(\hat{U}_1^2) - 3\mathbf{L}_1(\hat{U}_1^1) - 2\mathbf{L}_0(U^0))], \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathbf{B}_0 &= \tilde{p}(k\mathbf{L}_0(U^0)) \\ &\quad + (k^2/12)[(6k)^{-1}(2\mathbf{L}_3(\hat{U}_1^3) - 9\mathbf{L}_2(\hat{U}_1^2) + 18\mathbf{L}_1(\hat{U}_1^1) - 11\mathbf{L}_0(U^0))], \end{aligned} \tag{3.8}$$

$$\begin{aligned} \mathbf{F}^0 = (&k^2(f^0 - f^1(\hat{U}_1^1))/12, \\ &kf^0/2 + (k^2/12)[2f^3(\hat{U}_1^3) - 9f^2(\hat{U}_1^2) + 18f^1(\hat{U}_1^1) - 11f^0]/6k \\ &+ kf^1(\hat{U}_1^1)/2 \\ &- (k^2/12)[-f^3(\hat{U}_1^3) + 6f_2(\hat{U}_1^2) - 3f^1(\hat{U}_1^1) - 2f^0(U^0)]/6k)^T. \end{aligned} \tag{3.9}$$

For the proof of convergence of the overall scheme we shall need error estimates for U^1 in a special norm. For this purpose we state and prove some preliminary results in the Lemmata 3.1 and 3.2 of the Supplement. These results lead to Proposition 3.1 and (3.29) (see Supplement), which summarize the error analysis at the time levels t_j, $j = 0, 1$.

Computing $U^j, \hat{U}^j$, $2 \le j \le 5$. We then compute (and estimate the errors of) a few steps ($2 \le j \le 5$) of the numerical solution U^j using the cosine base scheme (1.13). To do this, we must also provide the necessary $\hat{U}^j$, $2 \le j \le 5$. It turns out that the error analysis must be done in a special way for these first few steps. We start with the preparatory Lemma 3.3, the heart of the step-by-step estimation argument, albeit good only for a few time steps. Its statement and proof can be found in the Supplement.

Then we define in an inductive fashion $\hat{U}^{j+1}$ for $j = 1, \dots, 4$ as follows:

$$\hat{U}^2 = 8U^1 - 7U^0 - 6kPu^{(1)0} - 2k^2Pu^{(2)0}, \tag{3.38.2}$$

$$\hat{U}^3 = (9/2)U^2 - 9U^1 + (11/2)U^0 + 3kPu^{(1)0}, \tag{3.38.3}$$

$$\hat{U}^4 = 4U^3 - 6U^2 + 4U^1 - U^0, \tag{3.38.4}$$

$$\hat{U}^5 = 4U^4 - 6U^3 + 4U^2 - U^1. \tag{3.38.5}$$

In these formulas, the U^j, $2 \le j \le 4$, are computed successively by (1.13), once the required U^i, $i < j$ and $\hat{U}^j$ have been computed.

For the motivation behind this special choice of $\hat{U}^{j+1}$ for $1 \le j \le 4$ and the relevant error estimation we refer the reader to the Supplement. Here, for purposes of easy reference, summarizing the results of Proposition 3.1, Lemma 3.3 and the subsequent discussion in the Supplement, we state:

PROPOSITION 3.2. *Suppose that there exists* $\alpha > 0$ *such that* $kh^{-1} \le \alpha$, *that* k, h *are sufficiently small and assume the stability conditions on* (q_1, q_2) *of Lemma* 3.3. *Suppose also that* (1.4), (1.7), (1.9) *hold and let* U^0, $\mathbf{U}^0$, $\hat{U}_1^j$, $1 \le j \le 3$, *be given by* (3.1), (3.3), (3.4). *Then* $\mathbf{U}^1$, *the solution of* (3.6), *exists uniquely. Define* U^1 *by* (3.5). *Then*

for $j = 1, \dots, 4$:

define $\hat{U}^{j+1}$ *by* $(3.38.j+1)$,

then U^{j+1}, *the solution of* (1.13) *for* $n = j$, *exists uniquely.*

Moreover, $U^j \in S_h \cap Y$, $0 \le j \le 5$, $\hat{U}^j \in S_h \cap Y$, $2 \le j \le 5$. *If* $E^j = U^j - W^j$ $(E^0 = 0)$, *if* $E_{j,j-1}$ *is defined by* (3.31) *and if* $e^j = u^j - U^j$, $\hat{e}^j = u^j - \hat{U}^j$ *as usual, we have*

$$\begin{array}{lll} \text{(a)} & E_{j,j-1} \le c_j k^2 (k^4 + h^r)^2, & 1 \le j \le 5, \\ \text{(b)} & \|E^j\| \le c_j k (k^4 + h^r), & 0 \le j \le 5, \\ \text{(c)} & \|e^j\| \le c_j (k^4 + h^r), & 0 \le j \le 5, \\ \text{(d)} & |e^j|_\infty \le h, & 0 \le j \le 5, \\ \text{(e)} & \|\hat{e}^{j+1}\| \le \hat{c}_j (k^4 + h^r), & 1 \le j \le 4, \\ \text{(f)} & |\hat{e}^{j+1}|_\infty \le h, & 1 \le j \le 4. \end{array} \tag{3.39}$$

Stability and Convergence of the Base Scheme. We now proceed to the central result of this section. Having already defined and estimated U^n, $0 \le n \le 5$, and $\hat{U}^{n+1}$, $1 \le n \le 4$, we shall let, for $5 \le n \le J-1$, provided of course that the U^j, $j \le n$ exist,

$$\hat{U}^{n+1} = \sum_{j=1}^{4} \alpha_j U^{n+1-j} \equiv 4U^n - 6U^{n-1} + 4U^{n-2} - U^{n-3} \tag{3.40}$$

and compute U^{n+1} as the solution of (1.13).

THEOREM 3.1. *Assume all hypotheses and definitions of Proposition* 3.2. *Then, with* $\hat{U}^{n+1}$ *defined by* (3.38.$n+1$) *for* $1 \le n \le 4$ *and by* (3.40) *for* $5 \le n \le J-1$, *the* U^n, $2 \le n \le J$, *exist uniquely as solutions of* (1.13). *Let* $E^n = U^n - W^n$ *and let* $E_{j,j-1}$ *be given for* $j \ge 1$ *by*

$$E_{j,j-1} = \|E^j - E^{j-1}\|^2 + k^2\|L_j^{1/2}(E^j - E^{j-1})\|^2 + k^2\|L_j^{1/2}(E^j + E^{j-1})\|^2 + k^4\|L_j(E^j - E^{j-1})\|^2 + k^4\|L_j(E^j + E^{j-1})\|^2. \tag{3.41}$$

Then there exists a positive c, independent of h and k, such that

$$\max_{0\le n\le J}\left(\|E^n\| + \sum_{j=1}^{n}(E_{j,j-1})^{1/2}\right) \le c(k^4 + h^r), \tag{3.42}$$

$$\max_{0\le n\le J}\|u^n - U^n\| \le c(k^4 + h^r). \tag{3.43}$$

Proof (by induction). Let l be an integer such that $5 \le l \le J-1$. We make the following induction hypothesis on l:

$$\begin{aligned}
&\text{(a)}\quad U^n, 0 \le n \le l \text{ exist (as solutions of (1.13) for } n \ge 2) \text{ in } S_h \cap Y,\\
&\text{(b)}\quad \|E^n\| + \textstyle\sum_{j=1}^{n}(E_{j,j-1})^{1/2} \le \sigma e^{\sigma t_n}(k^4 + h^r), \qquad 0 \le n \le l,\\
&\text{(c)}\quad |e^n|_\infty \le h, \qquad 0 \le n \le l,\\
&\text{(d)}\quad \hat{U}^{n+1}, 1 \le n \le l, \text{ belong to } S_h \cap Y,\\
&\text{(e)}\quad |\hat{e}^{n+1}|_\infty \le h, \qquad 1 \le n \le l.
\end{aligned} \tag{3.44}$$

(In (3.44.b), σ is a finite positive constant, independent of k, n, h or l, whose value will be specified in the proof.) Obviously, the hypothesis holds for $l = 5$, cf. (3.39). Also, if k is sufficiently small, (2.4) shows that $\hat{A}_{l+1}$ is invertible, i.e., that U^{l+1}, the solution of (1.13) for $n = l$, exists uniquely in S_h. We now turn to Proposition 2.1 which we shall use for $m = 3$. All its hypotheses are fulfilled and therefore, for any $\varepsilon_1, \varepsilon_2 > 0$, there exists a constant $c(\varepsilon_1, \varepsilon_2) > 0$ such that (2.88) holds for $m = 3$ and our current l ($\ge m + 2 = 5$), or any other l' such that $5 \le l' \le l$. As a preliminary note we remark that the induction hypothesis (3.44.b) gives

$$\|e^n\| \le \|E^n\| + \|u^n - W^n\| \le \sigma e^{\sigma t_n}(k^4 + h^r) + ch^r, \qquad 0 \le n \le l. \tag{3.45}$$

Consequently, in view of (3.44.d), (3.40), we have, for $5 \le n \le l$,

$$\begin{aligned}
\|\hat{e}^{n+1}\| &\le \left\|\sum_{j=1}^{4}\alpha_j e^{n+1-j}\right\| + \left\|u^{n+1} - \sum_{j=1}^{4}\alpha_j u^{n+1-j}\right\|\\
&\le c(k^4 + h^r)\sum_{j=1}^{4}(\sigma e^{\sigma t_{n+1-j}}) + c(k^4 + h^r).
\end{aligned}$$

Combining with (3.39.e), we have

$$\|\hat{e}^{n+1}\| \le c(k^4 + h^r)\left(\sum_{j=1}^{4}\sigma e^{\sigma t_{n+1-j}}\right) + c(k^4 + h^r), \qquad 1 \le n \le l. \tag{3.46}$$

We now embark upon estimating the terms F_i of the right-hand side of (2.88). We immediately conclude by (3.39a, c-f) that

$$F_1 = \eta_3^{(1)} + \eta_3^{(2)} + \eta_3^{(3)} \leq ck^2(k^4 + h^r)^2. \tag{3.47}$$

Now, using the L^∞ bounds (3.44.c,e), we shall estimate for the time being

$$\begin{aligned} F_2 &\leq \varepsilon_1 k^2(\|L_{l+1}^{1/2}(E^{l+1} + E^l)\|^2 + \|L_{l+1}^{1/2}(E^{l+1} - E^l)\|^2) \\ &\quad + \varepsilon_2 k^4(\|L_{l+1}(E^{l+1} + E^l)\|^2 + \|L_{l+1}(E^{l+1} - E^l)\|^2) \\ &\quad + c(\varepsilon_1, \varepsilon_2)k^2 \left(\sum_{j=1}^{l+1} \|\hat{e}^j\|^2 + \sum_{j=l-2}^{l} \|e^j\|^2\right). \end{aligned} \tag{3.48}$$

We also immediately note that

$$F_3 \leq ck^2(k^4 + h^r)^2, \tag{3.49}$$

$$F_4 \leq ck \sum_{n=2}^{l} E_{n+1,n}. \tag{3.50}$$

Using (3.44.c,e), it is straightforward to see that

$$F_5 \leq ck \sum_{n=2}^{l} E_{n+1,n} + ck^2 h^{2r}. \tag{3.51}$$

Then, using (3.44.b) and (3.45), (3.46), we obtain

$$F_6 \leq ck^2(k^4 + h^r)^2 + ck^3(k^4 + h^r)^2 \sigma^2 e^{2\sigma t_3}(e^{2\sigma k(l-1)} - 1)/(e^{2\sigma k} - 1). \tag{3.52}$$

For the purpose of estimating F_7, note that by (3.38.4,5) and (3.40) we have for $4 \leq n \leq l-1$

$$\begin{aligned} \|\hat{e}^{n+2} - \hat{e}^n\| &\leq \left\| \left(u^{n+2} - \sum_{j=1}^{4} \alpha_j u^{n+2-j}\right) - \left(u^n - \sum_{j=1}^{4} \alpha_j u^{n-j}\right) \right\| \\ &\quad + \left\| \sum_{j=1}^{4} \alpha_j [(u^{n+2-j} - W^{n+2-j}) - (u^{n-j} - W^{n-j})] \right\| \\ &\quad + \left\| \sum_{j=1}^{4} \alpha_j [(U^{n+2-j} - W^{n+2-j}) - (U^{n-j} - W^{n-j})] \right\| \\ &\leq ck^5 + ckh^r + c\sum_{j=1}^{4} \|E^{n+2-j} - E^{n-j}\|. \end{aligned}$$

Hence, using (3.44.c) and (3.39.a), we obtain

$$F_7 \leq ck^2(k^4 + h^r)^2 + ck \sum_{n=2}^{l-1} \|E^{n+1} - E^n\|^2. \tag{3.53}$$

Collecting terms, we see that from (3.47)–(3.53) and (2.88) there follows that, with $\eta_j^{(1)}$ defined by (2.19),

$$
\begin{aligned}
\eta_{l+1}^{(1)} \leq\ & ck^2(k^4+h^r)^2 + ck\sum_{n=2}^{l} E_{n+1,n} \\
& + \varepsilon_1 k^2 (\|L_{l+1}^{1/2}(E^{l+1}+E^l)\|^2 + \|L_{l+1}^{1/2}(E^{l+1}-E^l)\|^2) \\
& + \varepsilon_2 k^4 (\|L_{l+1}(E^{l+1}+E^l)\|^2 + \|L_{l+1}(E^{l+1}-E^l)\|^2) \\
& + c(\varepsilon_1,\varepsilon_2)k^2 \left(\sum_{j=l}^{l+1}\|\hat{e}^j\|^2 + \sum_{j=l-2}^{l}\|e^j\|^2\right) \\
& + ck^3(k^4+h^r)^2\sigma^2 e^{2\sigma t_3}(e^{2\sigma k(l-1)}-1)/(e^{2\sigma k}-1).
\end{aligned} \tag{3.54}
$$

(Let us remark again that, e.g., (3.54) holds if we replace l by any integer l' such that $5 \leq l' \leq l$.) At this stage, the stability assumptions on q_1, q_2 yield—basically as in the proof of Theorem 2.1 of [3]—that it is possible, by taking k and $\varepsilon_1, \varepsilon_2$ sufficiently small, to hide the third and fourth term in the right-hand side of the above in analogous terms of the left-hand side, which may be subsequently bounded below by a positive constant times $E_{l+1,l}$. Hence we obtain for k sufficiently small

$$
\begin{aligned}
E_{l+1,l} \leq\ & ck^2(k^4+h^r)^2 + ck^2\left(\sum_{j=1}^{l+1}\|\hat{e}^j\|^2 + \sum_{j=l-2}^{l}\|e^j\|^2\right) \\
& + ck^3(k^4+h^r)^2\sigma^2 e^{2\sigma t_3}(e^{2\sigma k(l-1)}-1)/(e^{2\sigma k}-1) \\
& + ck\sum_{n=2}^{l-1} E_{n+1,n}.
\end{aligned} \tag{3.55}
$$

Inserting now the assumed (by (3.45) and (3.46)) bounds for $\|\hat{e}^j\|$, $l \leq j \leq l+1$, $\|e^j\|$, $l-2 \leq j \leq l$, we see, in view of (3.39.a), that for all l', $0 \leq l' \leq l$, there holds

$$
E_{l'+1,l'} \leq ck^2(k^4+h^r)^2 A_{l'} + ck\sum_{n=0}^{l'-1} E_{n+1,n}, \tag{3.56}
$$

where

$$
A_{l'} \equiv 1 + \sigma^2 e^{2\sigma t_{l'}} + k\sigma^2 e^{2\sigma t_3}(e^{2\sigma k(l'-1)}-1)/(e^{2\sigma k}-1).
$$

By Gronwall's lemma we conclude therefore, since for $x \geq 0$, $x(e^x-1)^{-1} \leq 1$, that

$$
(E_{n+1,n})^{1/2} \leq ck(k^4+h^r)(1+\sigma e^{\sigma t_n} + \sqrt{\sigma}e^{\sigma t_{n+2}}), \qquad 0 \leq n \leq l, \tag{3.57}
$$

where c is independent of σ. We shall eventually choose $\sigma \geq 1$; hence

$$
\|E^{n+1}-E^n\| + (E_{n+1,n})^{1/2} \leq ck(k^4+h^r)\sigma e^{\sigma t_{n+2}}, \qquad 0 \leq n \leq l.
$$

Since $E^0 = 0$, summation yields

$$
\|E^{l+1}\| + \sum_{n=0}^{l}(E_{n+1,n})^{1/2} \leq (k^4+h^r)(c_* e^{2\sigma k})e^{\sigma t_{l+1}}, \tag{3.58}
$$

where the positive constant c_* is independent of σ; we assume $c_* > 1$. Now—with 20/20 hindsight—choosing $\sigma = 2c_*$ and picking k small enough so that $e^{4c_* k} \leq 2$, gives $c_* e^{2\sigma k} \le \sigma$, i.e., that in the above

$$\|E^{l+1}\| + \sum_{j=1}^{l} (E_{j,j-1})^{1/2} \le \sigma e^{\sigma t_{l+1}} (k^4 + h^r), \tag{3.59}$$

which is (3.44.b) for $n = l + 1$. With this choice of σ, (3.57) implies

$$\|L_{l+1}^{1/2} E^{l+1}\| \le c(k^4 + h^r), \tag{3.60}$$

i.e., in view of (iv.c), that $|E^{l+1}|_\infty \le ch^{3/2}$. Hence, $|e^{l+1}|_\infty \le ch^{3/2} \le h$ for h sufficiently small. This is (3.44.c) for $n = l+1$; the fact that $U^{l+1} \in Y$ also follows.

Finally, if $l + 1 = J$, we are done. If $l + 1 < J$, define $\hat{U}^{l+2} = \sum_{j=1}^{4} \alpha_j U^{l+2-j}$ and obtain, by (3.57), (3.60) for h sufficiently small, that

$$\begin{aligned} |\hat{e}^{l+2}|_\infty &\le \left| u^{l+2} - \sum_{j=1}^{4} \alpha_j u^{l+2-j} \right|_\infty + \left| \sum_{j=1}^{4} \alpha_j (u^{l+2-j} - W^{l+2-j}) \right|_\infty \\ &\quad + \left| \sum_{j=1}^{4} \alpha_j E^{l+2-j} \right|_\infty \\ &\le c(k^2 + h^{3/2} + \gamma(h)(k^4 + h^r)) \le h, \end{aligned}$$

which establishes (3.44.d,e) for $n = l + 1$. The inductive step is complete; (3.42) and (3.43) follow from (3.44.b). □

4. Preconditioned Iterative Methods. The implementation of the base scheme (1.13) requires, at each time step n, the solution of a linear system with operator $\hat{A}_{n+1}$, which changes from step to step. Following [12], [4], [3], we shall use preconditioned iterative techniques with suitable starting values to approximate U^{n+1} in a stable and accurate way by solving a number of linear systems per step with an operator that does not change with n. Most of the required estimates are similar to those of Section 3 and follow in general lines the analogous estimates in [3]. Hence we shall just state here the relevant algorithms and results without proofs.

We shall denote by V^n, $n \ge 0$, the new fully discrete approximations to be computed, to distinguish them from U^n, the solutions of the base scheme (1.13). To establish notation, following [4], let H be a finite-dimensional Hilbert space equipped with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H = (\cdot,\cdot)_H^{1/2}$. To approximate the solution $\bar{x} \in H$ of a linear system $A\bar{x} = b$, $b \in H$, where A is a selfadjoint, positive definite operator on H, we suppose that there exists a positive definite, selfadjoint, easily invertible operator ${}^P A$ (the preconditioner) and constants $0 < \lambda_0 \le \lambda_1$, such that

$$\lambda_0 ({}^P A z, z)_H \le (Az, z)_H \le \lambda_1 ({}^P A z, z)_H, \qquad z \in H. \tag{4.1}$$

Then, there are iterative methods, for solving the system $A\bar{x} = b$, which, given an initial guess $x^{(0)} \in H$, generate a sequence $x^{(j)}$, $j \ge 1$, of approximations to $\bar{x}$ in such a way that calculating $x^{(j+1)}$, given $x^{(i)}$, $0 \le i \le j$, only requires multiplying A with vectors, solving systems with operator ${}^P A$ and computing inner products and linear combinations of vectors. Moreover, there is a smooth decreasing function

$\sigma\colon (0,1] \to [0,1)$ with $\sigma(1) = 0$ and a constant c such that $\|{}^P A^{1/2}(\bar{x} - x^{(j)})\|_H \leq c[\sigma(\lambda_0/\lambda_1)]^j \|{}^P A^{1/2}(\bar{x} - x^{(0)})\|_H$. In our applications we shall perform at each step n, $1 \leq n \leq J$, j_n iterations, sufficiently many so as to achieve, with $x = x^{(j_n)}$,

$$\|{}^P A^{1/2}(\bar{x} - x)\|_H \leq \beta_n \|{}^P A^{1/2}(\bar{x} - x^{(0)})\|_H,$$

where $\beta_n > 0$ are small preassigned tolerances. We shall always take $\beta_n = O(k^\nu)$, $\nu \geq 1$, so that, as a consequence of the geometric convergence of the iterative method, $j_n = O(|\log(k)|)$.

We follow the structure and notation of Section 3. As a first step we seek $V^j \cong u^j$, $j = 0, 1$. We let $\mathbf{V}^0 = \mathbf{U}^0$, $\hat{V}_1^j = \hat{U}_1^j$, $1 \leq j \leq 3$, where $\mathbf{U}^0$, $\hat{U}_1^j$ are given by (3.3), (3.4), respectively. Suppose that $\overline{\mathbf{V}}^1 \in S_h^2$ is the exact solution of

$$\mathbf{A}_1 \overline{\mathbf{V}}^1 = \mathbf{B}_0 \mathbf{V}^0 + \mathbf{F}^0,$$

i.e., let $\overline{\mathbf{V}}^1 = \mathbf{U}^1$, cf. (3.6). We now let $H = S_h^2$, $(\cdot,\cdot)_H$ be the $L^2 \times L^2$ inner product on H, $\mathbf{A}_1^*$ be the associated adjoint of $\mathbf{A}_1$ and $\mathbf{T}_0$ be the operator $\operatorname{diag}(I, T_0)$ on S_h^2. $\mathbf{T}_0$ is a selfadjoint positive definite operator on H, but $\mathbf{A}_1$ is not. For our purposes we regard $\overline{\mathbf{V}}^1$ as the exact solution of the problem

$$(\mathbf{A}_1^* \mathbf{T}_0 \mathbf{A}_1)\overline{\mathbf{V}}^1 = \mathbf{A}_1^* \mathbf{T}_0(\mathbf{B}_0 \mathbf{V}^0 + \mathbf{F}^0),$$

which will be the system on H to be solved by iterative techniques. As preconditioner we use, with $\beta > 0$, the operator

$${}^P\mathbf{A} = \operatorname{diag}((I + \beta k^2 L_0)^2, (I + \beta k^2 L_0) T_0 (I + \beta k^2 L_0))$$

(it satisfies (4.1)) and compute, by a preconditioned iterative method satisfying our stated general properties, $\mathbf{V}^1 = [V_1^1, V_2^1]^T$ as an approximation to $\overline{\mathbf{V}}^1$ satisfying

$$\|{}^P\mathbf{A}^{1/2}(\overline{\mathbf{V}}^1 - \mathbf{V}^1)\|_H \leq \beta_1 \|{}^P\mathbf{A}^{1/2}(\overline{\mathbf{V}}^1 - \mathbf{V}^{(0)1})\|_H,$$

where we take $\beta_1 = \min(\gamma, k^4)$ for some constant $0 < \gamma < 1$ and where $\mathbf{V}^{(0)1} = \mathbf{V}^0$. We set $V^1 = V_1^1$.

For the rest of this section we let $H = S_h$ and $(\cdot,\cdot)_H$ be the L^2 inner product on S_h. We compute first V^j, $2 \leq j \leq 5$, (and the needed extrapolated values $\hat{V}^j$, $2 \leq j \leq 5$) as approximations to the exact solutions V^j, $2 \leq j \leq 5$, of the cosine scheme, cf. (1.13),

$$\hat{A}_{n+1} \overline{V}^{n+1} - 2B_n V^n + A_{n-1} V^{n-1} = \Theta(\hat{V}^{n+1}, V^n, V^{n-1}), \qquad n \geq 1,$$

where, although we use the same notation $\hat{A}_{n+1}$, A_n, B_n as before, we mean of course that $\hat{A}_{n+1} = q(k^2 L_{n+1}(\hat{V}^{n+1}))$, $B_n = p(k^2 L_n(V^n))$, $A_n = q(k^2 L_n(V^n))$ etc. The operator $\hat{A}_{n+1}$ will now play the role of A. As preconditioner we shall choose the time-independent operator

$${}^P Q = (I + \beta k^2 L_0)^2, \qquad \beta > 0,$$

for which (4.1) is satisfied, cf. [3]. The approximations V^{j+1}, $1 \leq j \leq 4$, to $\overline{V}^{j+1}$ are then computed so that

$$\|{}^P Q^{1/2}(\overline{V}^{n+1} - V^{n+1})\| \leq \beta_{n+1} \|{}^P Q^{1/2}(\overline{V}^{n+1} - V^{(0)n+1})\|$$

holds for $n = j$, $1 \leq j \leq 4$. We take $\beta_{n+1} = \min(\gamma, k^4)$ and $V^{(0)n+1} = V^n$; the $\hat{V}^j$, $2 \leq j \leq 5$, are given by the formulas (3.38.j), replacing U^j by V^j. We continue for $n \geq 5$ by computing $\hat{V}^{n+1}$ by (3.40) with $U^j = V^j$, and U^{n+1} as the approximation to the solution $\overline{V}^{n+1}$ of (4.8), so that (4.10) is satisfied, where now $\beta_{n+1} = \min(\gamma, k)$ and $V^{(0)n+1} = 5V^n - 10V^{n-1} + 10V^{n-2} - 5V^{n-3} + V^{n-4}$. It may be proved, under the assumptions of Theorem 3.1, that all intermediate approximations exist uniquely; moreover, there exists a constant $c > 0$ such that $\|V^n - u^n\| \leq c(k^4 + h^r)$, i.e., that V^n asymptotically satisfies the same type of L^2 optimal-order error estimate as does U^n.

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