

# Nonconforming Finite Elements for the Stokes Problem\*

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*Dedicated to Professor Eugene Isaacson on the occasion of his 70th birthday*

**Abstract.** A new stability result is obtained for the approximation of the stationary Stokes problem by nonconforming piecewise cubic approximations to the velocities and a discontinuous piecewise quadratic approximation to the pressure. The basic result is that for most reasonable meshes, these elements form a stable pair without the addition of quartic bubble functions (which had previously been added to insure stability).

**1. Introduction.** In the finite element approximation of the velocity-pressure formulation of the stationary Stokes equations using triangular finite elements, several approaches appear in the literature. One might classify these into conforming schemes which use a continuous piecewise polynomial approximation to the velocity and a discontinuous piecewise polynomial approximation to the pressure, conforming schemes which use continuous piecewise polynomial approximations for both velocity and pressure, macroelement schemes in which the pressure elements (usually discontinuous) are defined on a coarser mesh than the velocity elements, and nonconforming schemes in which the velocities are only continuous at appropriate Gauss points on the triangle edges, and discontinuous pressures are used.

In an early paper on the finite element approximation of the Stokes problem by the first author and P.-A. Raviart [4], several combinations of conforming and nonconforming velocity elements and discontinuous pressure elements were analyzed. Specifically, in the case of conforming velocity elements, it was shown that piecewise constant pressures could be paired with piecewise quadratic velocities (an idea suggested by Fortin [5]) to give a (suboptimal)  $O(h)$  energy norm convergence rate. When discontinuous piecewise linear pressures are used, a corresponding velocity space of continuous piecewise quadratics augmented by cubic bubble functions improves the convergence rate to  $O(h^2)$ . In the case of discontinuous quadratic pressures, a corresponding velocity space of continuous piecewise cubics augmented by two quartic bubbles for each component of velocity gives an energy norm convergence rate of  $O(h^3)$ . In later work of Bernardi and Raugel [1], following another idea of Fortin [6], it was shown that pairing piecewise constant pressures with a velocity space consisting of continuous piecewise linear functions augmented by one (vector) quadratic per edge also yield an  $O(h)$  energy norm convergence rate. In the case of higher-order elements, Scott and Vogelius [10] proved that except for some exceptional meshes, the use of discontinuous piecewise polynomials of degree

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$n$  ( $n \geq 3$ ) for pressure and continuous piecewise polynomials of degree  $n + 1$  for velocity (without adding any bubble functions) give methods with optimal-order convergence rates.

Since the pairing of constant pressures with continuous piecewise linear velocities is not convergent (for most meshes, the set of such velocities with zero divergence is  $\{0\}$ ) and the substitution of quadratic velocities leads to a suboptimal convergence rate, Crouzeix and Raviart also studied the use of nonconforming velocity spaces. They showed that constant pressures paired with nonconforming piecewise linear velocities give an optimal  $O(h)$  energy norm convergence rate. The case of nonconforming quadratic velocities was not considered in their paper, but was considered in a later paper of Fortin [7], where it was proved that nonconforming quadratic velocities may be paired with discontinuous linear pressures to produce an  $O(h^2)$  method. Crouzeix and Raviart also considered the case of nonconforming cubic velocities and discontinuous quadratic pressures, but obtained an  $O(h^3)$  convergence rate only by augmenting each component of the velocity space by two quartic bubble functions (as in the conforming case). The purpose of this paper is to improve this last result by showing that for most commonly used meshes, the quartic bubble functions are not needed to give a convergent method of optimal order. In light of the work of Scott and Vogelius (mentioned above) that conforming velocity elements of degree  $\geq 4$  already give optimal-order methods, the three lowest-order cases of nonconforming elements appear to be the only ones of interest.

An outline of this paper is as follows. In Section 2 we define the notation to be used and recall some of the theory of the finite element approximation of the stationary Stokes equations. In the now standard approach to this problem, the essential feature of the analysis is the verification of an appropriate form of the inf-sup condition. The verification of such a condition for the choice of nonconforming cubic velocities and discontinuous quadratic pressures is done in the remaining sections for various types of meshes.

**2. Notation and Preliminaries.** For  $\Omega$  a polygonal domain in  $\mathbb{R}^2$ , we thus consider the approximation of the stationary Stokes problem: Find  $\mathbf{u} = (u_1, u_2)$  and  $p$  satisfying

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\mathbf{u}$  is the fluid velocity,  $p$  is the pressure,  $\mathbf{f}$  are the body forces per unit mass, and  $\nu$  is the viscosity. The variational formulation of this problem is:

Find  $\mathbf{u} = (u_1, u_2) \in (H_0^1(\Omega))^2$  and  $p \in L^2(\Omega)/\mathbb{R}$  satisfying

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) && \text{for all } \mathbf{v} \in (H_0^1(\Omega))^2, \\ (\operatorname{div} \mathbf{u}, q) &= 0 && \text{for all } q \in L^2(\Omega)/\mathbb{R}, \end{aligned}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \nu \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

and  $(\cdot, \cdot)$  denotes the  $L^2(\Omega)$  inner product.

The finite element methods we consider may be described abstractly as follows. We let  $\tau_h$  denote a triangulation of  $\Omega$  by triangles  $T$  of diameter  $\leq h$ . We then denote by  $\mathbf{W}_h$  a finite-dimensional approximation of  $(H_0^1(\Omega))^2$ . Since we are considering nonconforming methods,  $\mathbf{W}_h \not\subseteq (H_0^1(\Omega))^2$ , but  $\mathbf{v}_h|_T \in H^1(T)$  for all  $\mathbf{v}_h \in \mathbf{W}_h$  and  $T \in \tau_h$ . We assume that

$$\|\mathbf{v}_h\|_{1,h} = \left( \sum_{T \in \tau_h} |\mathbf{v}_h|_{1,T}^2 \right)^{1/2}$$

is a norm on  $\mathbf{W}_h$ . Let  $Q_h$  denote a finite-dimensional subspace of  $L^2(\Omega)/\mathbb{R}$ . The approximation scheme is then: Find  $\mathbf{u}_h \in \mathbf{W}_h$ ,  $p_h \in Q_h$  satisfying

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}_h \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{W}_h, \\ (\operatorname{div}_h \mathbf{u}_h, q) &= 0 \quad \text{for all } q \in Q_h, \end{aligned}$$

where

$$a_h(\mathbf{u}, \mathbf{v}) = \nu \sum_{T \in \tau_h} \sum_{i,j=1}^2 \int_T \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx$$

and  $\operatorname{div}_h \mathbf{v}$  is the  $L^2(\Omega)$  function whose restriction to each triangle  $T \in \tau_h$  is given by  $(\operatorname{div} \mathbf{v})|_T$ .

The analysis of this type of method is well understood. If the space  $\mathbf{W}_h$  were conforming, the general theory of saddle-point problems developed by Babuška and Brezzi could be applied directly. In that case the only difficulty in the analysis is the verification of the inf-sup condition

$$(2.1) \quad \inf_{0 \neq q_h \in Q_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{W}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1 \|q_h\|_0} \geq \gamma.$$

When (2.1) holds, one obtains the quasi-optimal error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C \inf(\|\mathbf{u} - \mathbf{v}_h\|_1 + \|p - q_h\|_0),$$

where the inf is taken over all  $\mathbf{v}_h \in \mathbf{W}_h$  and  $q_h \in Q_h$ . For nonconforming methods, a straightforward modification of this result leads to the conclusion that if

$$(2.2) \quad \inf_{0 \neq q_h \in Q_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{W}_h} \frac{(\operatorname{div}_h \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,h} \|q_h\|_0} \geq \gamma,$$

then one obtains the error estimate

$$(2.3) \quad \begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_0 \\ &\leq C \left[ \inf(\|\mathbf{u} - \mathbf{v}_h\|_{1,h} + \|p - q_h\|_0) + \sup \frac{\sum_T \int_{\partial T} (\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p \mathbf{n}) \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{1,h}} \right], \end{aligned}$$

where the inf is taken over all  $\mathbf{v}_h \in \mathbf{W}_h$  and  $q_h \in Q_h$  and the sup is taken over all  $\mathbf{w}_h \in \mathbf{W}_h$ .

Since [4] predated the work of Brezzi [3], the analysis in the former paper does not proceed by giving a direct verification of condition (2.2). However, using the interpolants constructed in [4], condition (2.2) can be easily verified on each triangle for the elements considered, with the global result following immediately. This local verification of condition (2.2) depends on the fact that appropriate bubble

functions have been added to the basic nonconforming spaces. In order to verify (2.2) without adding these bubble functions, we instead seek to verify (2.2) first on a patch of elements. This idea was previously used by Boland and Nicolaides [2] and Stenberg [11]. The basic technique is described in Girault and Raviart [8, pp. 129–132] for the case of conforming elements. In our case, we first define a collection of subdomains  $\Omega_r \subset \Omega$ ,  $r = 1, 2, \dots, R$ , such that  $\bar{\Omega}_r$  is a union of triangles  $T_{0,r}, T_{1,r}, \dots, T_{M_r,r} \in \tau_h$  and

$$\bar{\Omega} = \bigcup_{r=1}^R \bar{\Omega}_r.$$

We assume that there exists a constant  $L$  independent of  $h$  such that for all  $T \in \tau_h$  the number of  $r$  such that  $\Omega_r \cap T \neq \emptyset$  is bounded by  $L$ . We next define

$$\mathbf{V}_h(\Omega_r) = \{ \mathbf{u}_h : \mathbf{u}_h|_{T_{k,r}} \in (P_3)^2, k = 0, 1, \dots, M_r, \mathbf{u}_h = 0 \text{ in } \Omega/\Omega_r, \\ \mathbf{u}_h \text{ continuous at the Gauss points on all triangle sides} \},$$

$$H_h(\Omega_r) = \left\{ q \in L^2(\Omega_r) : q|_{T_{k,r}} \in P_2 \text{ and } \int_{T_{k,r}} q \, dx = 0, \right. \\ \left. k = 0, 1, \dots, M_r, q = 0 \text{ in } \Omega/\Omega_r \right\}.$$

The weak local verification of condition (2.2) on the patch of elements  $\Omega_r$  will consist of establishing the validity of the following hypothesis:

*Hypothesis H0.* There exists a positive constant  $\gamma^*$ , independent of  $h$  and  $r$ , such that

$$(2.4) \quad \inf_{0 \neq q_h \in H_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div}_h \mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,h,\Omega_r} \|q_h\|_{0,\Omega_r}} \geq \gamma^*.$$

We shall do this by showing the validity of

*Hypothesis H1.* For all  $q_h \in H_h(\Omega_r)$ , there exists  $\mathbf{u}_h \in \mathbf{V}_h(\Omega_r)$  such that

$$(2.5) \quad \operatorname{div}_h \mathbf{u}_h = q_h \quad \text{and} \quad \|\mathbf{u}_h\|_{1,h,\Omega_r} \leq \|q_h\|_{0,\Omega_r} / \gamma^*.$$

As in the papers mentioned above (e.g. see Theorem 1.12 of [8]), we can establish the following result. We include a proof for the sake of completeness and also since we do not assume, as in [2] and [8], that the domains  $\Omega_r$  are disjoint.

**THEOREM 2.1.** *Let*

$$\mathbf{W}_h = \{ \mathbf{u}_h : \mathbf{u}_h|_T \in (P_3)^2, T \in \tau_h, \mathbf{u}_h \text{ continuous at the Gauss points} \\ \text{on all triangle sides, } \mathbf{u}_h = 0 \text{ at the Gauss points on } \partial\Omega \},$$

$$Q_h = \left\{ q \in L^2(\Omega), q|_T \in P_2, T \in \tau_h, \int_{\Omega} q \, dx = 0 \right\}.$$

*Then, if hypothesis H1 is satisfied, condition (2.2) holds with a constant  $\gamma$  independent of  $h$ .*

*Proof.* For  $q_h \in Q_h$ , one can construct by standard techniques a function  $\mathbf{w}_h \in \mathbf{W}_h$  satisfying for all  $T \in \tau_h$  the conditions

$$\int_T \operatorname{div}_h \mathbf{w}_h \, dx = \int_T q_h \, dx, \quad \|\mathbf{w}_h\|_{1,h,\Omega} \leq \|q_h\|_{0,\Omega} / \gamma_1,$$

with  $\gamma_1$  independent of  $h$ . Setting

$$(2.6) \quad p_h = q_h - \operatorname{div}_h \mathbf{w}_h,$$

we may write

$$p_h = \sum_{r=1}^R p_{h,r},$$

where

$$p_{h,r} \in H_h(\Omega_r) \quad \text{and} \quad p_{h,r} \cdot p_{h,r'} = 0 \quad \text{for } r \neq r'.$$

Hence we have

$$\|p_h\|_{0,\Omega}^2 = \sum_{r=1}^R \|p_{h,r}\|_{0,\Omega_r}^2.$$

Using (2.5), we may write

$$p_{h,r} = \operatorname{div}_h \mathbf{u}_{h,r} \quad \text{with} \quad \|\mathbf{u}_{h,r}\|_{1,h,\Omega_r} \leq \|p_{h,r}\|_{0,\Omega_r} / \gamma^*.$$

Then we set

$$(2.7) \quad \mathbf{u}_h = \sum_{r=1}^R \mathbf{u}_{h,r} \quad \text{and} \quad \mathbf{v}_h = \mathbf{u}_h + \mathbf{w}_h,$$

so that we have

$$(2.8) \quad \operatorname{div}_h \mathbf{v}_h = q_h$$

and

$$\begin{aligned} \|\mathbf{u}_h\|_{1,h,\Omega}^2 &= \sum_{T \in \tau_h} \|\mathbf{u}_h\|_{1,T}^2 \leq L \sum_{T \in \tau_h} \sum_{r=1}^R \|\mathbf{u}_{h,r}\|_{1,T}^2 \\ &\leq L \sum_{r=1}^R \|\mathbf{u}_{h,r}\|_{1,h,\Omega_r}^2 \leq L \|p_h\|_{0,\Omega}^2 / \gamma^{*2}. \end{aligned}$$

From (2.6) it follows that

$$\begin{aligned} \|p_h\|_{0,\Omega} &\leq \|q_h\|_{0,\Omega} + \|\operatorname{div}_h \mathbf{w}_h\|_{0,\Omega} \leq \|q_h\|_{0,\Omega} + \sqrt{2} \|\mathbf{w}_h\|_{1,h,\Omega} \\ &\leq (1 + \sqrt{2}/\gamma_1) \|q_h\|_{0,\Omega}. \end{aligned}$$

Using (2.6) and (2.7), we then obtain

$$\|\mathbf{v}_h\|_{1,h,\Omega} \leq \|\mathbf{u}_h\|_{1,h,\Omega} + \|\mathbf{w}_h\|_{1,h,\Omega} \leq \sqrt{L}(\gamma_1 + \sqrt{2}) \|q_h\|_{0,\Omega} / (\gamma^* \gamma_1) + \|q_h\|_{0,\Omega} / \gamma_1.$$

Finally,

$$\|\mathbf{v}_h\|_{1,h,\Omega} \leq \|q_h\|_{0,\Omega} / \gamma,$$

where

$$\gamma = \gamma^* \gamma_1 / [\sqrt{L}(\gamma_1 + \sqrt{2}) + \gamma^*].$$

Together with (2.8), this implies (2.2).

It then follows directly from (2.3) and standard estimates for nonconforming methods (see [4] and [9]) that if  $\mathbf{u} \in (H^4(\Omega))^2$  and  $p \in H^3(\Omega)$ , then

$$(2.9) \quad \|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\|_0 \leq Ch^3 (\|\mathbf{u}\|_4 + \|p\|_3).$$

Thus, to establish Theorem 2.1 and the error estimate (2.9), we need only show that Hypothesis H1 is valid. Before doing so in the next sections, we first state some basic facts that will facilitate the forthcoming analysis. We recall that the barycentric coordinates  $\lambda_i = \lambda_i(\mathbf{x})$ ,  $1 \leq i \leq 3$ , of any point  $\mathbf{x} \in \mathbb{R}^2$  with respect to the points  $A_1, A_2, A_3$  are the unique solutions of the linear system

$$(2.10) \quad \sum_{j=1}^3 A_j \lambda_j = \mathbf{x},$$

$$(2.11) \quad \sum_{j=1}^3 \lambda_j = 1.$$

Denoting by  $\overrightarrow{A_1 A_2}$  the vector from  $A_1$  to  $A_2$ , we have

$$(2.12) \quad \mathbf{x} = A_1 + \lambda_2 \overrightarrow{A_1 A_2} + \lambda_3 \overrightarrow{A_1 A_3}.$$

Hence,

$$(2.13) \quad I = \nabla \mathbf{x}^t = \nabla \lambda_2 (\overrightarrow{A_1 A_2})^t + \nabla \lambda_3 (\overrightarrow{A_1 A_3})^t,$$

and for any vector  $\mathbf{u}$ ,

$$(2.14) \quad \mathbf{u} = (\mathbf{u} \cdot \nabla \lambda_2) \overrightarrow{A_1 A_2} + (\mathbf{u} \cdot \nabla \lambda_3) \overrightarrow{A_1 A_3}.$$

By applying (2.13) to the vector  $\nabla \lambda_2$ , it easily follows that

$$(2.15) \quad \nabla \lambda_2 \cdot \overrightarrow{A_1 A_2} = 1 \quad \text{and} \quad \nabla \lambda_2 \cdot \overrightarrow{A_1 A_3} = 0.$$

**3. Verification of Hypothesis H1—Mesh I.** The verification of condition (2.5) will be done for three choices of the subdomain  $\Omega_r$ . In the first of these,  $\Omega_r = T_0 \cup T_1 \cup T_2 \cup T_3$ , where the four triangles  $T_0, T_1, T_2, T_3$  are aligned in the configuration depicted in Figure 3.1.

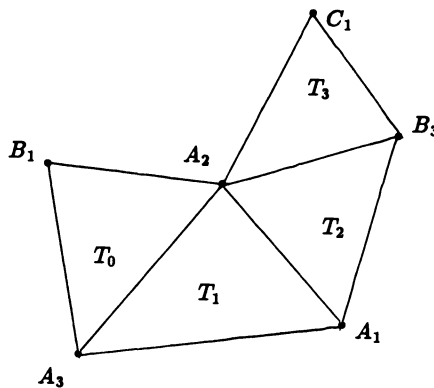


FIGURE 3.1

We denote by

- $(\mu_1, \mu_2, \mu_3)$  the barycentric coordinates of  $B_3$  with respect to  $A_1, A_2, A_3$ ,
- $(\nu_1, \nu_2, \nu_3)$  the barycentric coordinates of  $B_1$  with respect to  $A_1, A_2, A_3$ ,
- $(\tau_1, \tau_2, \tau_3)$  the barycentric coordinates of  $C_1$  with respect to  $A_1, A_2, B_3$ .

The verification of Hypothesis H1 will depend on a geometrical condition on the mesh patch  $\Omega_r$  given in terms of the barycentric coordinates defined above. Following the verification, we describe some common situations which are and are not covered. To simplify the proof of our result, it is convenient to first define the following subspace of  $\mathbf{V}_h(\Omega_r)$ . Let

$$\hat{\mathbf{V}}_h(\Omega_r) = \left\{ \mathbf{u}_h \in \mathbf{V}_h(\Omega_r) : \int_s \mathbf{u}_h d\sigma = 0 \text{ for all sides } s \text{ of } \Omega_r \right\}.$$

The verification of condition (2.5) is contained in the following proposition.

**PROPOSITION 3.1.** *If  $D \equiv -\nu_2\tau_2\mu_1^2 - (\nu_1 - 1)\tau_2\mu_2\mu_3 + (\tau_1 - 1)\nu_2\mu_2 \neq 0$ , then the operator  $\text{div}_h$  from  $\hat{\mathbf{V}}_h$  into  $H_h$  is one-to-one and for all  $\mathbf{u}_h \in \hat{\mathbf{V}}_h$*

$$\|\mathbf{u}_h\|_{1,h,\Omega_r} \leq \frac{C(\theta)}{|D|} \|\text{div}_h \mathbf{u}_h\|_{0,\Omega_r},$$

where  $C$  is a constant which depends only on the smallest angle  $\theta$  of the triangles  $T_0, T_1, T_2$ , and  $T_3$ .

*Proof.* First we remark that  $\mathbf{u}_h \in \hat{\mathbf{V}}_h$  implies  $\int_{T_k} \text{div}_h \mathbf{u}_h dx = 0$ , so that  $\text{div}_h(\hat{\mathbf{V}}_h) \subseteq H_h$ . Since  $\dim \hat{\mathbf{V}}_h = \dim H_h = 20$ ,  $\text{div}_h$  is one-to-one if and only if  $\text{Ker}(\text{div}_h) = \{0\}$ .

In the triangle  $T_k$ , we may write

$$\begin{aligned} \mathbf{u}_h &= \sum_{i=1}^3 \mathbf{u}_i^{(k)} \lambda_i^3 + \sum_{i \neq j} \mathbf{u}_{ij}^{(k)} \lambda_i^2 \lambda_j + \mathbf{u}_{123}^{(k)} \lambda_1 \lambda_2 \lambda_3, \\ q &= \sum_{i=1}^3 q_i^{(k)} \lambda_i^2 + \sum_{i < j} q_{ij}^{(k)} \lambda_i \lambda_j, \end{aligned}$$

where  $\lambda_i = \lambda_i^{(k)}$  are the barycentric coordinates in the triangle  $T_k$ . We shall omit the superscript  $(k)$  when there is no ambiguity.

We now turn to the lengthy process of expressing the continuity of  $\mathbf{u}_h$  at the Gauss points and the satisfaction of the equations  $\text{div}_h \mathbf{u}_h = q$  on each triangle  $T$  and  $\int_s \mathbf{u}_h d\sigma = 0$  for each triangle side  $s$ , in terms of the degrees of freedom of  $\mathbf{u}_h$  and  $q$ . On the side  $\overrightarrow{A_2A_3}$  or  $\overrightarrow{A_2B_3}$ , the condition  $\int_s \mathbf{u}_h d\sigma = 0$  becomes

$$(3.1) \quad 3(\mathbf{u}_2 + \mathbf{u}_3) + (\mathbf{u}_{23} + \mathbf{u}_{32}) = 0.$$

Similarly, we have on  $\overrightarrow{A_2B_1}$ ,  $\overrightarrow{A_2A_1}$ ,  $\overrightarrow{A_2C_1}$

$$(3.2) \quad 3(\mathbf{u}_2 + \mathbf{u}_1) + (\mathbf{u}_{21} + \mathbf{u}_{12}) = 0.$$

The continuity at the Gauss points on  $\overrightarrow{A_2A_3}$  (barycentric coordinates  $(0, 1/2, 1/2)$ ,  $(0, \theta, 1 - \theta)$ ,  $(0, 1 - \theta, \theta)$  with  $\theta(1 - \theta) = 1/10$ ) gives

$$(3.3) \quad (\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_{23} + \mathbf{u}_{32})^{(0)} = (\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_{23} + \mathbf{u}_{32})^{(1)},$$

$$(3.4) \quad \begin{aligned} & (\theta^3 \mathbf{u}_2 + (1 - \theta)^3 \mathbf{u}_3 + \theta(1 - \theta)[\theta \mathbf{u}_{23} + (1 - \theta) \mathbf{u}_{32}])^{(0)} \\ & = (\theta^3 \mathbf{u}_2 + (1 - \theta)^3 \mathbf{u}_3 + \theta(1 - \theta)[\theta \mathbf{u}_{23} + (1 - \theta) \mathbf{u}_{32}])^{(1)}, \end{aligned}$$

$$(3.5) \quad \begin{aligned} & ([1 - \theta]^3 \mathbf{u}_2 + \theta^3 \mathbf{u}_3 + \theta(1 - \theta)[(1 - \theta) \mathbf{u}_{23} + \theta \mathbf{u}_{32}])^{(0)} \\ & = ([1 - \theta]^3 \mathbf{u}_2 + \theta^3 \mathbf{u}_3 + \theta(1 - \theta)[(1 - \theta) \mathbf{u}_{23} + \theta \mathbf{u}_{32}])^{(1)}. \end{aligned}$$

Since  $\theta(1 - \theta) = 1/10$  implies  $\theta^2 = \theta - 1/10$  and  $\theta^3 = (9\theta - 1)/10$ , we have

$$\begin{aligned} & \theta^3 \mathbf{u}_2 + (1 - \theta)^3 \mathbf{u}_3 + \theta(1 - \theta)[\theta \mathbf{u}_{23} + (1 - \theta) \mathbf{u}_{32}] \\ & = \{(9\theta - 1) \mathbf{u}_2 + (8 - 9\theta) \mathbf{u}_3 + [\theta \mathbf{u}_{23} + (1 - \theta) \mathbf{u}_{32}]\} / 10 \\ & = \{(1 - \theta)[3(\mathbf{u}_2 + \mathbf{u}_3) + \mathbf{u}_{23} + \mathbf{u}_{32}] + (2\theta - 1)(9\mathbf{u}_2 + \mathbf{u}_{23}) \\ & \quad + (5 - 6\theta)(\mathbf{u}_2 + \mathbf{u}_3)\} / 10. \end{aligned}$$

Using an analogous rewriting of Eq. (3.5) and taking into account Eq. (3.1), we have the continuity at the Gauss points on  $\overrightarrow{A_2 A_3}$  and  $\overrightarrow{A_2 B_3}$  if and only if

$$(3.6) \quad (\mathbf{u}_2 + \mathbf{u}_3)^{(k)} = (\mathbf{u}_2 + \mathbf{u}_3)^{(k+1)}, \quad k = 0, 2,$$

$$(3.7) \quad (9\mathbf{u}_2 + \mathbf{u}_{23})^{(k)} = (9\mathbf{u}_2 + \mathbf{u}_{23})^{(k+1)}, \quad k = 0, 2.$$

Similarly, continuity at the Gauss points on  $\overrightarrow{A_2 A_1}$  becomes

$$(3.8) \quad (\mathbf{u}_1 + \mathbf{u}_2)^{(1)} = (\mathbf{u}_1 + \mathbf{u}_2)^{(2)},$$

$$(3.9) \quad (9\mathbf{u}_1 + \mathbf{u}_{12})^{(1)} = (9\mathbf{u}_1 + \mathbf{u}_{12})^{(2)}.$$

The continuity with 0 at the Gauss points of  $\overrightarrow{A_1 A_3}$ ,  $\overrightarrow{A_1 B_3}$ ,  $\overrightarrow{B_1 A_3}$ ,  $\overrightarrow{C_1 B_3}$  (which implies  $\int_s \mathbf{u}_h d\sigma = 0$ ) may be written

$$(3.10) \quad \mathbf{u}_1 + \mathbf{u}_3 = 0,$$

$$(3.11) \quad 9\mathbf{u}_1 + \mathbf{u}_{13} = 0,$$

$$(3.12) \quad 9\mathbf{u}_3 + \mathbf{u}_{31} = 0.$$

Noticing that

$$\int_T q dx = 0 \Leftrightarrow 2(q_1 + q_2 + q_3) + q_{12} + q_{23} + q_{13} = 0,$$

the equation  $\operatorname{div}_h \mathbf{u}_h = q$  on  $T$  is equivalent to

$$(3.13) \quad 3\mathbf{u}_1 \cdot \nabla \lambda_1 + \mathbf{u}_{12} \cdot \nabla \lambda_2 + \mathbf{u}_{13} \cdot \nabla \lambda_3 = q_1,$$

$$(3.14) \quad 3\mathbf{u}_2 \cdot \nabla \lambda_2 + \mathbf{u}_{23} \cdot \nabla \lambda_3 + \mathbf{u}_{21} \cdot \nabla \lambda_1 = q_2,$$

$$(3.15) \quad 3\mathbf{u}_3 \cdot \nabla \lambda_3 + \mathbf{u}_{31} \cdot \nabla \lambda_1 + \mathbf{u}_{32} \cdot \nabla \lambda_2 = q_3,$$

$$(3.16) \quad 2\mathbf{u}_{23} \cdot \nabla \lambda_2 + 2\mathbf{u}_{32} \cdot \nabla \lambda_3 + \mathbf{u}_{123} \cdot \nabla \lambda_1 = q_{23},$$

$$(3.17) \quad 2\mathbf{u}_{31} \cdot \nabla \lambda_3 + 2\mathbf{u}_{13} \cdot \nabla \lambda_1 + \mathbf{u}_{123} \cdot \nabla \lambda_2 = q_{13}.$$

Taking into account (3.2), continuity with 0 at the Gauss points of  $\overrightarrow{A_2 B_1}$  and  $\overrightarrow{A_2 C_1}$  is equivalent to

$$(3.18) \quad (\mathbf{u}_1 + \mathbf{u}_2)^{(k)} = 0, \quad k = 0, 3,$$

$$(3.19) \quad (9\mathbf{u}_1 + \mathbf{u}_{12})^{(k)} = 0, \quad k = 0, 3.$$



Using all of these equations, we now study  $\text{Ker}(\text{div}_h)$ . To do so, we set  $q = 0$ . In triangle  $T_0$  we obtain from (3.10), (3.11), (3.12), (3.2), (3.18), and (3.19) that

$$\mathbf{u}_2 = \mathbf{u}_3 = -\mathbf{u}_1, \quad \mathbf{u}_{31} = \mathbf{u}_{21} = -\mathbf{u}_{13} = -\mathbf{u}_{12} = 9\mathbf{u}_1.$$

We deduce from (3.13) that

$$(3.20) \quad 12\mathbf{u}_1 \cdot \nabla \lambda_1 = 0,$$

from (3.14) that

$$\mathbf{u}_{23} \cdot \nabla \lambda_3 = 3\mathbf{u}_1 \cdot \nabla \lambda_2,$$

and from (3.15) that

$$\mathbf{u}_{32} \cdot \nabla \lambda_2 = 3\mathbf{u}_1 \cdot \nabla \lambda_3.$$

Therefore,

$$(9\mathbf{u}_2 + \mathbf{u}_{23}) \cdot \nabla \lambda_3 = -9\mathbf{u}_1 \cdot \nabla \lambda_3 + 3\mathbf{u}_1 \cdot \nabla \lambda_2 = 12\mathbf{u}_1 \cdot \nabla \lambda_2,$$

and using (3.1),

$$(9\mathbf{u}_2 + \mathbf{u}_{23}) \cdot \nabla \lambda_2 = (6\mathbf{u}_2 - 3\mathbf{u}_3 - \mathbf{u}_{32}) \cdot \nabla \lambda_2 = 0.$$

Let

$$(3.21) \quad \alpha = (\mathbf{u}_1 \cdot \nabla \lambda_2)^{(0)}.$$

Then

$$(\mathbf{u}_2 + \mathbf{u}_3)^{(0)} = 2\alpha \overrightarrow{A_2 A_3},$$

$$(9\mathbf{u}_2 + \mathbf{u}_{23})^{(0)} = 12\alpha \overrightarrow{B_1 A_3}.$$

In triangle  $T_1$  we use (3.6) and (3.7) with  $k = 0$  to obtain

$$(3.22) \quad (\mathbf{u}_2 + \mathbf{u}_3)^{(1)} = 2\alpha \overrightarrow{A_2 A_3},$$

$$(3.23) \quad (9\mathbf{u}_2 + \mathbf{u}_{23})^{(1)} = 12\alpha \overrightarrow{B_1 A_3},$$

and using (3.1),

$$(9\mathbf{u}_3 + \mathbf{u}_{32})^{(1)} = -12\alpha \overrightarrow{B_1 A_2}.$$

Next, we use (3.15) and (3.12) to get

$$12(\mathbf{u}_3 \cdot \nabla \lambda_3 - \alpha \overrightarrow{B_1 A_2} \cdot \nabla \lambda_2) = 0,$$

i.e.,

$$(3.24) \quad \mathbf{u}_3 \cdot \nabla \lambda_3 = \alpha(1 - \nu_2).$$

Let

$$(3.25) \quad \beta = (\mathbf{u}_3 \cdot \nabla \lambda_1)^{(1)}.$$

Then

$$\mathbf{u}_3 = \beta \overrightarrow{A_2 A_1} + \alpha(1 - \nu_2) \overrightarrow{A_2 A_3},$$

and from (3.10),

$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_2 - \mathbf{u}_3 = 2\alpha \overrightarrow{A_2 A_3} - 2\mathbf{u}_3 = -2(\beta \overrightarrow{A_2 A_1} - \alpha \nu_2 \overrightarrow{A_2 A_3}),$$

i.e.,

$$(3.26) \quad (\mathbf{u}_1 + \mathbf{u}_2)^{(1)} = -2(\beta \overrightarrow{A_2 A_1} - \alpha \nu_2 \overrightarrow{A_2 A_3}).$$

From (3.13) and (3.11) we obtain

$$(3.27) \quad \begin{aligned} (9\mathbf{u}_1 + \mathbf{u}_{12}) \cdot \nabla \lambda_2 &= 9\mathbf{u}_1 \cdot \nabla \lambda_2 - 3\mathbf{u}_1 \cdot \nabla \lambda_1 - \mathbf{u}_{13} \cdot \nabla \lambda_3 \\ &= -12\mathbf{u}_1 \cdot \nabla \lambda_1 = 12\mathbf{u}_3 \cdot \nabla \lambda_1 = 12\beta. \end{aligned}$$

From (3.2), (3.10), (3.14), and (3.22),

$$\begin{aligned} (9\mathbf{u}_1 + \mathbf{u}_{12}) \cdot \nabla \lambda_1 &= (6\mathbf{u}_1 - 3\mathbf{u}_2 - \mathbf{u}_{21}) \cdot \nabla \lambda_1 \\ &= (6\mathbf{u}_1 - 3\mathbf{u}_2) \cdot \nabla \lambda_1 + 3\mathbf{u}_2 \cdot \nabla \lambda_2 + \mathbf{u}_{23} \cdot \nabla \lambda_3 \\ &= (6\mathbf{u}_1 - 3\mathbf{u}_2) \cdot \nabla \lambda_1 + 3\mathbf{u}_2 \cdot \nabla \lambda_2 - 9\mathbf{u}_2 \cdot \nabla \lambda_3 + 12\alpha \overrightarrow{B_1 A_3} \cdot \nabla \lambda_3 \\ &= 6\mathbf{u}_1 \cdot \nabla \lambda_1 + 6\mathbf{u}_2 \cdot (\nabla \lambda_2 - \nabla \lambda_3) + 12\alpha(1 - \nu_3) \\ &= -6\mathbf{u}_3 \cdot \nabla \lambda_1 + 12\alpha \overrightarrow{A_2 A_3} \cdot (\nabla \lambda_2 - \nabla \lambda_3) - 6\mathbf{u}_3 \cdot (\nabla \lambda_2 - \nabla \lambda_3) \\ &\quad + 12\alpha(1 - \nu_3) \\ &= 12\mathbf{u}_3 \cdot \nabla \lambda_3 - 12\alpha(1 + \nu_3) = 12\alpha(\nu_1 - 1). \end{aligned}$$

With (3.27) that gives

$$(3.28) \quad (9\mathbf{u}_1 + \mathbf{u}_{12})^{(1)} = 12[\alpha(\nu_1 - 1)\overrightarrow{A_3 A_1} + \beta \overrightarrow{A_3 A_2}].$$

Similarly, we obtain

$$(3.29) \quad (\mathbf{u}_1 + \mathbf{u}_2)^{(2)} = -2(\delta \overrightarrow{A_2 A_1} - \gamma \tau_2 \overrightarrow{A_2 B_3}),$$

$$(3.30) \quad (9\mathbf{u}_1 + \mathbf{u}_{12})^{(2)} = 12[\gamma(\tau_1 - 1)\overrightarrow{B_3 A_1} + \delta \overrightarrow{B_3 A_2}],$$

where

$$(3.31) \quad \gamma = (\mathbf{u}_1 \cdot \nabla \lambda_2)^{(3)}$$

and

$$(3.32) \quad \delta = (\mathbf{u}_3 \cdot \nabla \lambda_1)^{(2)}.$$

We remark that

$$\begin{aligned} \overrightarrow{A_2 B_3} &= \mu_1 \overrightarrow{A_2 A_1} + \mu_3 \overrightarrow{A_2 A_3}, \\ \overrightarrow{B_3 A_1} &= (1 - \mu_1) \overrightarrow{A_3 A_1} - \mu_2 \overrightarrow{A_3 A_2}, \\ \overrightarrow{B_3 A_2} &= -\mu_1 \overrightarrow{A_3 A_1} + (1 - \mu_2) \overrightarrow{A_3 A_2}. \end{aligned}$$

Thus Eq. (3.8) becomes (from (3.26) and (3.29))

$$\beta = \delta - \tau_2 \mu_1 \gamma,$$

$$\nu_2 \alpha = \tau_2 \mu_3 \gamma,$$

and Eq. (3.9) becomes

$$(\nu_1 - 1)\alpha = (1 - \mu_1)(\tau_1 - 1)\gamma - \mu_1 \delta,$$

$$\beta = -(\tau_1 - 1)\mu_2 \gamma + (1 - \mu_2)\delta.$$

We thus obtain the linear system

$$\begin{pmatrix} 0 & 1 & \mu_1\tau_2 & -1 \\ \nu_2 & 0 & -\mu_3\tau_2 & 0 \\ \nu_1 - 1 & 0 & -(\tau_1 - 1)(1 - \mu_1) & \mu_1 \\ 0 & 1 & \mu_2(\tau_1 - 1) & -(1 - \mu_2) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0.$$

The determinant of this linear system is

$$D = -(\nu_1 - 1)\tau_2\mu_3\mu_2 + \nu_2[(\tau_1 - 1)\mu_2 - \mu_1^2\tau_2].$$

Therefore, if  $D \neq 0$ , we have  $\alpha = \beta = \gamma = \delta = 0$ . We then deduce from (3.20) and (3.21) that  $\mathbf{u}_1^{(0)} = 0$ . Consequently,

$$\mathbf{u}_2^{(0)} = \mathbf{u}_3^{(0)} = \mathbf{u}_{12}^{(0)} = \mathbf{u}_{21}^{(0)} = \mathbf{u}_{31}^{(0)} = \mathbf{u}_{13}^{(0)} = \mathbf{u}_{23}^{(0)} = 0.$$

It then follows from (3.1) that  $\mathbf{u}_{32}^{(0)} = 0$  and from (3.16) and (3.17) that  $\mathbf{u}_{123}^{(0)} = 0$ . Hence  $\mathbf{u}_h^{(0)} = 0$ . In a similar way, we obtain  $\mathbf{u}_h^{(1)} = \mathbf{u}_h^{(2)} = \mathbf{u}_h^{(3)} = 0$ , so that  $\mathbf{u}_h = 0$ . Since this shows that  $\text{Ker}(\text{div}_h) = \{0\}$ , we get that  $\text{div}_h$  is one-to-one.

Finally, we consider the inequality stated in Proposition 3.1. To prove this result, we consider the previous equations with  $q \neq 0$ . First observe that Eqs. (3.1)–(3.15) give the values of

$$(\mathbf{u}_i \cdot \nabla \lambda_l)^{(k)}, (\mathbf{u}_{ij} \cdot \nabla \lambda_l)^{(k)}, \quad k = 0, 1, 2, 3,$$

as linear functions of  $(q_1, q_2, q_3)^{(k')}$ ,  $k' = 0, 1, 2, 3$ , and thus we obtain

$$\begin{aligned} |(\mathbf{u}_i \cdot \nabla \lambda_l)^{(k)}| &\leq \frac{C_1}{|D|} \max_{i,k'} |q_i^{(k')}|, \\ |(\mathbf{u}_{ij} \cdot \nabla \lambda_l)^{(k)}| &\leq \frac{C_2}{|D|} \max_{i,k'} |q_i^{(k')}|. \end{aligned}$$

We next use (3.16) and (3.17) to obtain

$$\begin{aligned} |(\mathbf{u}_{123} \cdot \nabla \lambda_1)^{(k)}| &\leq |q_{23}^{(k)}| + \frac{4C_2}{|D|} \max_{i,k'} |q_i^{(k')}|, \\ |(\mathbf{u}_{123} \cdot \nabla \lambda_2)^{(k)}| &\leq |q_{13}^{(k)}| + \frac{4C_2}{|D|} \max_{i,k'} |q_i^{(k')}|. \end{aligned}$$

By standard scaling arguments, we have that

$$\begin{aligned} |q_i^{(k)}| &\leq C \|q\|_{L^2(T_k)} / A_k^{1/2}, \\ |q_{ij}^{(k)}| &\leq C \|q\|_{L^2(T_k)} / A_k^{1/2}, \\ \|\text{Grad } \mathbf{u}\|_{L^2(T_k)} &\leq C(\theta) \left( \sum |\mathbf{u}_i| + \sum |\mathbf{u}_{ij}| + |\mathbf{u}_{123}| \right), \end{aligned}$$

where  $A_k$  denotes the area of  $T_k$ . The inequality stated in Proposition 3.1 follows immediately.

In order to understand the condition on the determinant  $D$  under which Proposition 3.1 is valid, we now examine this condition in more detail. For the configuration of triangles depicted in Figure 3.1, we have that  $\nu_1 < 0$ ,  $\mu_3 < 0$ ,  $\tau_1 < 0$ . If we assume that

$$\nu_2 \geq 0, \tau_2 \geq 0, \mu_2 \geq 0, \nu_2 + \mu_2 \neq 0, \nu_2 + \tau_2 \neq 0, \tau_2 + \mu_2 \neq 0,$$

then it is easy to see that  $D \neq 0$ . Some meshes not covered by Proposition 1 are:

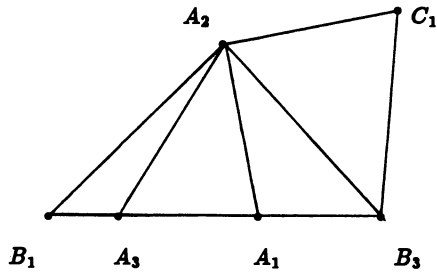


FIGURE 3.2.  $\nu_2 = \mu_2 = 0$

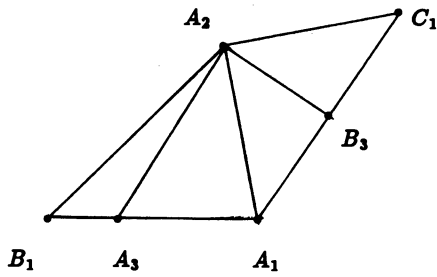


FIGURE 3.3.  $\nu_2 = \tau_2 = 0$

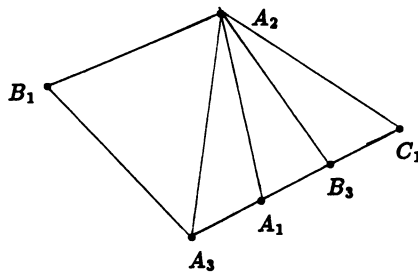


FIGURE 3.4.  $\tau_2 = \mu_2 = 0$

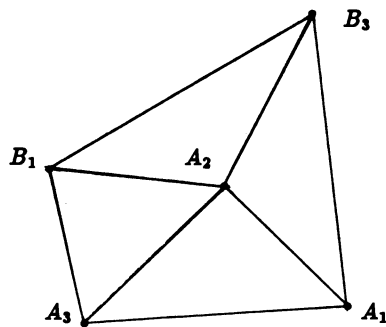


FIGURE 3.5

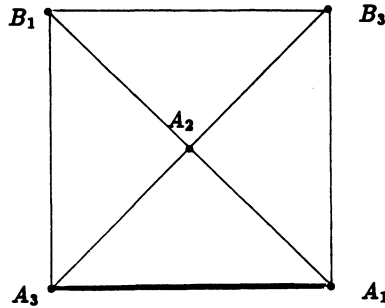


FIGURE 3.6

*Remark.* The limiting case  $C_1 = B_1$  is allowed. In this case,  $\mathbf{u}_h = 0$  at the Gauss points on  $\overrightarrow{B_1 A_2}$ . Therefore, the mesh depicted in Figure 3.5 and, in particular, the special case depicted in Figure 3.6 are covered by Proposition 3.1.

**4. Verification of Hypothesis H1—Mesh II.** We next consider the mesh in Figure 4.1.

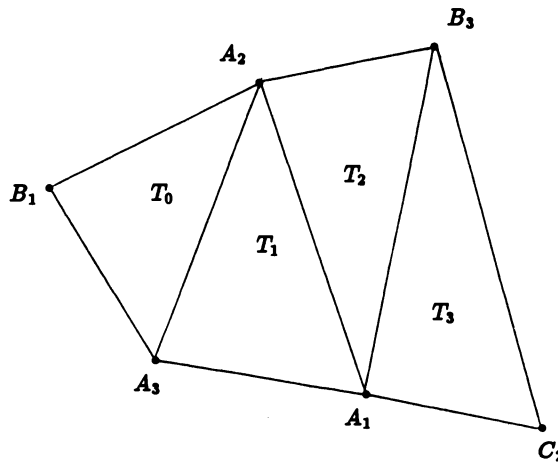


FIGURE 4.1

We use the same notation with the addition that  $(\xi_1, \xi_2, \xi_3)$  are the barycentric coordinates of  $C_2$  with respect to  $A_1, A_2, B_3$ . The verification of condition (2.5) is contained in the following proposition.

**PROPOSITION 4.1.** *If*

$$\Delta = (\nu_1 - 1)\xi_1\mu_2\mu_3 - \nu_2(\xi_2 - 1)\mu_1 + \nu_2\xi_1[(1 - \mu_1)^2 - \mu_3] \neq 0,$$

*then the operator  $\text{div}_h$  from  $\hat{V}_h$  into  $H_h$  is one-to-one and for all  $\mathbf{u}_h \in \hat{V}_h$ ,*

$$\|\mathbf{u}_h\|_{1,h,\Omega_r} \leq \frac{C(\theta)}{|\Delta|} \|\text{div}_h \mathbf{u}_h\|_{0,\Omega_r}.$$

*Proof.* We follow the proof of Proposition 3.1 up to Eq. (3.27). In triangle  $T_1$ , it follows from (3.2) that

$$9\mathbf{u}_2 + \mathbf{u}_{21} = 6(\mathbf{u}_2 + \mathbf{u}_1) - 9\mathbf{u}_1 - \mathbf{u}_{12}$$

and from (3.26) and (3.27) that

$$(4.1) \quad (9\mathbf{u}_2 + \mathbf{u}_{21})^{(1)} = 12\{\alpha\nu_2\vec{A_2A_3} + [\alpha(\nu_1 - 1) + \beta]\vec{A_1A_3}\}.$$

By symmetry, we obtain

$$(4.2) \quad (\mathbf{u}_1 + \mathbf{u}_2)^{(2)} = -2[\delta\vec{A_1A_2} - \gamma\xi_1\vec{A_1B_3}],$$

$$(4.3) \quad (9\mathbf{u}_1 + \mathbf{u}_{12})^{(2)} = 12\{\gamma\xi_1\vec{A_1B_3} + [\gamma(\xi_2 - 1) + \delta]\vec{A_2B_3}\}.$$

Then (3.8) and (3.9) become

$$\begin{aligned} \beta &= -\delta - \gamma\xi_1(\mu_1 - 1), \\ \nu_2\alpha &= \xi_1\mu_3\gamma, \\ (\nu_1 - 1)\alpha &= (\mu_1 - 1)\xi_1\gamma + \mu_1[(\xi_2 - 1)\gamma + \delta], \\ \beta &= \mu_2\xi_1\gamma + (\mu_2 - 1)[(\xi_2 - 1)\gamma + \delta]. \end{aligned}$$

We thus obtain the linear system

$$\begin{pmatrix} 0 & 1 & (\mu_1 - 1)\xi_1 & 1 \\ \nu_2 & 0 & -\mu_3\xi_1 & 0 \\ \nu_1 - 1 & 0 & \xi_1 - \mu_1(\xi_1 + \xi_2 - 1) & -\mu_1 \\ 0 & 1 & -\xi_1\mu_2 - (\mu_2 - 1)(\xi_2 - 1) & (1 - \mu_2) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0.$$

The remainder of the proof of Proposition 4.1 follows in an analogous fashion to that of Proposition 3.1.

We now examine the determinant  $\Delta$  in more detail in order to understand the condition under which Proposition 4.1 is valid. For the configuration of triangles depicted in Figure 4.1, we have that  $\nu_1 < 0$ ,  $\mu_3 < 0$ ,  $\xi_2 < 0$ . If we assume that

$$\xi_1 \geq 0, \mu_2 \geq 0, \mu_1 \geq 0, \nu_2 \geq 0, \nu_2 + \xi_1 \neq 0, \nu_2 + \mu_2 \neq 0, \xi_1 + \mu_1 \neq 0,$$

then it is easy to see that  $\Delta \neq 0$ . Some meshes not covered by Proposition 4.1 are

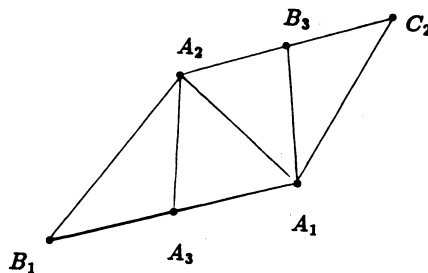


FIGURE 4.2.  $\nu_2 = \xi_1 = 0$

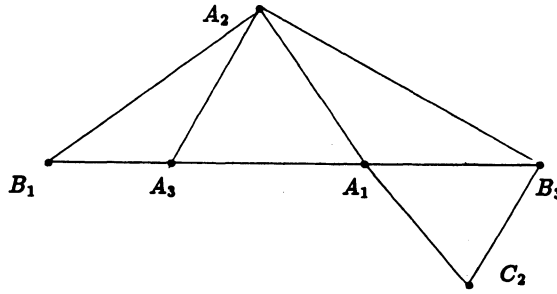


FIGURE 4.3.  $\nu_2 = \mu_2 = 0$

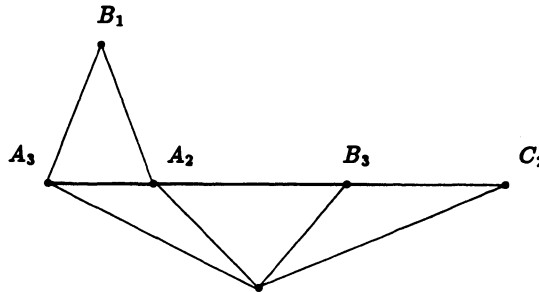


FIGURE 4.4.  $\xi_1 = \mu_1 = 0$

However, the standard regular mesh, depicted in Figure 4.5, is covered by Proposition 4.1.

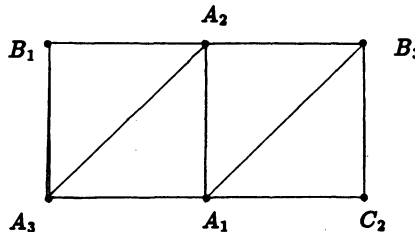


FIGURE 4.5

**5. Verification of Hypothesis H1.—Mesh III.** Next, we turn to the case of three triangles aligned as in Figure 5.1.

We set  $\Omega_r = T = T_1 \cup T_2 \cup T_3$ . Let  $(\nu_1, \nu_2, \nu_3)$  denote the barycentric coordinates of  $B$  with respect to the points  $A_1, A_2, A_3$  and let  $(\mu_1, \mu_2, \mu_3)$  denote the barycentric coordinates of a point  $x \in T$  with respect to  $A_1, A_2, A_3$ . To define the barycentric coordinates in each  $T_i$ , we also denote the vertex  $B$  in the triangle  $T_i$  by  $B_i$ . We then denote by  $\lambda_j^{(i)}$  the barycentric coordinate of a point  $x \in T_i$  with respect to the vertex of  $T_i$  with subscript  $j$ . Define the subspace  $\mathbf{X}_h$  of  $\mathbf{V}_h$  by

$$\mathbf{X}_h = \{ \mathbf{u}_h \in (C^0(\mathbb{R}^2))^2 : \mathbf{u}_h|_{T_i} \in (P_3)^2, \mathbf{u}_h(B) = 0, \mathbf{u}_h = 0 \text{ in } \Omega/T \}.$$

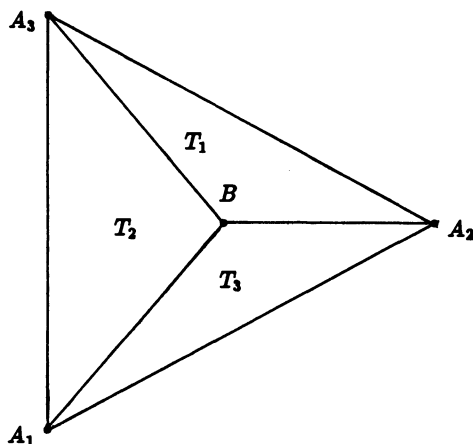


FIGURE 5.1

Note that  $\mathbf{u}_h \in \mathbf{X}_h$  may be represented in the triangle  $T_i$  by

$$\mathbf{u}_h = \sum_{\substack{j=1 \\ j \neq i}}^3 \mathbf{u}_{ij}^{(i)} \lambda_i^2 \lambda_j + \sum_{\substack{j=1 \\ j \neq i}}^3 \mathbf{u}_{ji}^{(i)} \lambda_j^2 \lambda_i + \mathbf{u}_{123}^{(i)} \lambda_1 \lambda_2 \lambda_3,$$

where we have omitted the superscript  $(i)$  on the barycentric coordinates. We will continue to omit superscripts (on  $\mathbf{u}_{ij}$  as well) when there is no ambiguity about the triangle. Using the continuity of  $\mathbf{u}_h$ , it follows by comparing the expressions for  $\mathbf{u}_h$  along common interior sides, that

$$(5.1) \quad \begin{aligned} \mathbf{u}_{13}^{(1)} &= \mathbf{u}_{23}^{(2)}, & \mathbf{u}_{31}^{(1)} &= \mathbf{u}_{32}^{(2)}, \\ \mathbf{u}_{12}^{(2)} &= \mathbf{u}_{13}^{(3)}, & \mathbf{u}_{21}^{(2)} &= \mathbf{u}_{31}^{(3)}, \\ \mathbf{u}_{23}^{(3)} &= \mathbf{u}_{21}^{(1)}, & \mathbf{u}_{32}^{(3)} &= \mathbf{u}_{12}^{(1)}. \end{aligned}$$

Using the above, it is easy to check that  $\dim \mathbf{X}_h = 18$ , while  $H_h$ , defined in Section 2, has dimension 15. To get a result analogous to Propositions 3.1 and 4.1, we define

$$\hat{\mathbf{V}}_h = \left\{ \mathbf{u}_h \in \mathbf{X}_h : \int_s \mathbf{u}_h \cdot \boldsymbol{\nu} \, d\sigma = 0, \text{ for all sides } s \text{ of } \Omega_r \right\},$$

where  $\boldsymbol{\nu}$  is the normal to  $s$ . Note that  $\int_s \mathbf{u}_h \cdot \boldsymbol{\nu} \, d\sigma = 0$  is equivalent to the conditions

$$(5.2) \quad (\mathbf{u}_{31}^{(1)} + \mathbf{u}_{13}^{(1)}) \cdot \nabla \lambda_2^{(1)} = (\mathbf{u}_{21}^{(1)} + \mathbf{u}_{12}^{(1)}) \cdot \nabla \lambda_3^{(1)} = 0,$$

$$(5.3) \quad (\mathbf{u}_{12}^{(2)} + \mathbf{u}_{21}^{(2)}) \cdot \nabla \lambda_3^{(2)} = (\mathbf{u}_{32}^{(2)} + \mathbf{u}_{23}^{(2)}) \cdot \nabla \lambda_1^{(2)} = 0,$$

$$(5.4) \quad (\mathbf{u}_{32}^{(3)} + \mathbf{u}_{23}^{(3)}) \cdot \nabla \lambda_1^{(3)} = (\mathbf{u}_{13}^{(3)} + \mathbf{u}_{31}^{(3)}) \cdot \nabla \lambda_2^{(3)} = 0.$$

We will then prove:

**PROPOSITION 5.1.** *The operator  $\text{div}$  from  $\hat{\mathbf{V}}_h$  into  $H_h$  is one-to-one and for all  $\mathbf{u}_h \in \hat{\mathbf{V}}_h$ ,*

$$(5.5) \quad \|\mathbf{u}_h\|_{1,h,\Omega_r} \leq C(\theta) \|\text{div}_h \mathbf{u}_h\|_{0,\Omega_r}.$$



Note that for this mesh, we are actually proving the inf-sup condition for *conforming* piecewise cubics.

Before proving Proposition 5.1, we gather some facts about the relationships among the various barycentric coordinates defined above. By writing  $B$  as a convex combination of the  $A_i$ , we easily obtain

$$\begin{aligned} \mu_1 &= \nu_1 \lambda_1^{(1)} = \nu_1 \lambda_2^{(2)} + \lambda_1^{(2)} = \nu_1 \lambda_3^{(3)} + \lambda_1^{(3)}, \\ \mu_2 &= \nu_2 \lambda_1^{(1)} + \lambda_2^{(1)} = \nu_2 \lambda_2^{(2)} = \nu_2 \lambda_3^{(3)} + \lambda_2^{(3)}, \\ \mu_3 &= \nu_3 \lambda_1^{(1)} + \lambda_3^{(1)} = \nu_3 \lambda_2^{(2)} + \lambda_3^{(2)} = \nu_3 \lambda_3^{(3)}. \end{aligned}$$

From these relations it follows that

$$\nu_3 \nabla \lambda_3^{(1)} = -\nu_1 \nabla \lambda_1^{(3)}, \quad \nu_1 \nabla \lambda_2^{(1)} = -\nu_2 \nabla \lambda_1^{(2)}, \quad \nu_2 \nabla \lambda_3^{(2)} = -\nu_3 \nabla \lambda_2^{(3)}.$$

By first expressing  $\lambda_i^{(j)}$  as a linear function of  $\mu_1, \mu_2, \mu_3$  and then using the fact that  $\sum_{i=1}^3 \nabla \lambda_i^{(j)} = 0$ , we obtain the further relations

$$(5.6) \quad \nabla \lambda_2^{(1)} = (\nabla \mu_2 + \nu_2 \nabla \lambda_3^{(1)}) / (1 - \nu_2),$$

$$(5.7) \quad \nabla \lambda_3^{(1)} = (\nabla \mu_3 + \nu_3 \nabla \lambda_2^{(1)}) / (1 - \nu_3),$$

$$(5.8) \quad \nabla \lambda_3^{(2)} = (\nabla \mu_3 + \nu_3 \nabla \lambda_1^{(2)}) / (1 - \nu_3),$$

$$(5.9) \quad \nabla \lambda_1^{(2)} = (\nabla \mu_1 + \nu_1 \nabla \lambda_3^{(2)}) / (1 - \nu_1),$$

$$(5.10) \quad \nabla \lambda_1^{(3)} = (\nabla \mu_1 + \nu_1 \nabla \lambda_2^{(3)}) / (1 - \nu_1),$$

$$(5.11) \quad \nabla \lambda_2^{(3)} = (\nabla \mu_2 + \nu_2 \nabla \lambda_1^{(3)}) / (1 - \nu_2).$$

Using these facts, we are now ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* From the equation  $\operatorname{div} \mathbf{u}_h = q$ , we get in triangle  $T_1$  the equations

$$(5.12) \quad \mathbf{u}_{12} \cdot \nabla \lambda_2 + \mathbf{u}_{13} \cdot \nabla \lambda_3 = q_1,$$

$$(5.13) \quad \mathbf{u}_{21} \cdot \nabla \lambda_1 = q_2,$$

$$(5.14) \quad \mathbf{u}_{31} \cdot \nabla \lambda_1 = q_3,$$

$$(5.15) \quad 2\mathbf{u}_{12} \cdot \nabla \lambda_1 + 2\mathbf{u}_{21} \cdot \nabla \lambda_2 + \mathbf{u}_{123} \cdot \nabla \lambda_3 = q_{12},$$

$$(5.16) \quad \mathbf{u}_{123} \cdot \nabla \lambda_1 = q_{23},$$

in triangle  $T_2$  the equations

$$(5.17) \quad \mathbf{u}_{12} \cdot \nabla \lambda_2 = q_1,$$

$$(5.18) \quad \mathbf{u}_{21} \cdot \nabla \lambda_1 + \mathbf{u}_{23} \cdot \nabla \lambda_3 = q_2,$$

$$(5.19) \quad \mathbf{u}_{32} \cdot \nabla \lambda_2 = q_3,$$

$$(5.20) \quad \mathbf{u}_{123} \cdot \nabla \lambda_2 = q_{13},$$

$$(5.21) \quad 2\mathbf{u}_{23} \cdot \nabla \lambda_2 + 2\mathbf{u}_{32} \cdot \nabla \lambda_3 + \mathbf{u}_{123} \cdot \nabla \lambda_1 = q_{23},$$

and in triangle  $T_3$  the equations

$$(5.22) \quad \mathbf{u}_{13} \cdot \nabla \lambda_3 = q_1,$$

$$(5.23) \quad \mathbf{u}_{23} \cdot \nabla \lambda_3 = q_2,$$

$$(5.24) \quad \mathbf{u}_{31} \cdot \nabla \lambda_1 + \mathbf{u}_{32} \cdot \nabla \lambda_2 = q_3,$$

$$(5.25) \quad \mathbf{u}_{123} \cdot \nabla \lambda_3 = q_{12},$$

$$(5.26) \quad 2\mathbf{u}_{31} \cdot \nabla \lambda_3 + 2\mathbf{u}_{13} \cdot \nabla \lambda_1 + \mathbf{u}_{123} \cdot \nabla \lambda_1 = q_{13}.$$

Now observe that  $\mathbf{u}_{21}^{(1)} = \mathbf{u}_{23}^{(3)}$  is determined from Eqs. (5.13) and (5.23),  $\mathbf{u}_{31}^{(1)} = \mathbf{u}_{32}^{(2)}$  is determined from Eqs. (5.14) and (5.19), and  $\mathbf{u}_{12}^{(2)} = \mathbf{u}_{13}^{(3)}$  is determined from Eqs. (5.17) and (5.22). More precisely, using the relationships among the barycentric coordinates, the functions  $\mathbf{u}_{ji}^{(i)} \cdot \nabla \lambda_k^{(i)}$ ,  $i = 1, 3, j = 1, 3, k = 1, 3, j \neq i$ , may be written as linear combinations of  $q_j^{(i)}$ ,  $i = 1, 3, j = 1, 3$ , and these  $\mathbf{u}_{ji}^{(i)} = 0$  if the above  $q_j^{(i)} = 0$ . Then  $\mathbf{u}_{ij}^{(i)} \cdot \nabla \lambda_k^{(i)}$ ,  $i \neq j \neq k$ , is determined by Eqs. (5.2), (5.3), and (5.4). We deduce from (5.6), (5.7), and (5.12) that in triangle  $T_1$

$$(5.27) \quad \begin{aligned} & \frac{1}{1-\nu_2} \mathbf{u}_{12} \cdot \nabla \mu_2 + \frac{1}{1-\nu_3} \mathbf{u}_{13} \cdot \nabla \mu_3 \\ &= q_1 - \frac{\nu_2}{1-\nu_2} \mathbf{u}_{12} \cdot \nabla \lambda_3^{(1)} - \frac{\nu_3}{1-\nu_3} \mathbf{u}_{13} \cdot \nabla \lambda_2^{(1)}. \end{aligned}$$

Similarly, in  $T_2$

$$(5.28) \quad \begin{aligned} & \frac{1}{1-\nu_3} \mathbf{u}_{23} \cdot \nabla \mu_3 + \frac{1}{1-\nu_1} \mathbf{u}_{21} \cdot \nabla \mu_1 \\ &= q_2 - \frac{\nu_3}{1-\nu_3} \mathbf{u}_{23} \cdot \nabla \lambda_1^{(2)} - \frac{\nu_1}{1-\nu_1} \mathbf{u}_{21} \cdot \nabla \lambda_3^{(2)} \end{aligned}$$

and in  $T_3$

$$(5.29) \quad \begin{aligned} & \frac{1}{1-\nu_1} \mathbf{u}_{31} \cdot \nabla \mu_1 + \frac{1}{1-\nu_2} \mathbf{u}_{32} \cdot \nabla \mu_2 \\ &= q_3 - \frac{\nu_1}{1-\nu_1} \mathbf{u}_{31} \cdot \nabla \lambda_2^{(3)} - \frac{\nu_2}{1-\nu_2} \mathbf{u}_{32} \cdot \nabla \lambda_1^{(3)}. \end{aligned}$$

Using the equivalences in (5.1), the above system of equations may be reduced to a linear system in the three unknowns  $\mathbf{u}_{12}^{(1)} \cdot \nabla \mu_2$ ,  $\mathbf{u}_{23}^{(2)} \cdot \nabla \mu_3$ ,  $\mathbf{u}_{31}^{(3)} \cdot \nabla \mu_1$ , with matrix

$$\begin{pmatrix} (1-\nu_2)^{-1} & (1-\nu_3)^{-1} & 0 \\ 0 & (1-\nu_3)^{-1} & (1-\nu_1)^{-1} \\ (1-\nu_2)^{-1} & 0 & (1-\nu_1)^{-1} \end{pmatrix}.$$

The determinant of this matrix is  $2[(1-\nu_1)(1-\nu_2)(1-\nu_3)]^{-1} \neq 0$ , which allows us to determine  $\mathbf{u}_{ij}^{(i)}$ . Finally, we use Eqs. (5.15), (5.16), (5.20), (5.21), (5.25), and (5.26) to determine  $\mathbf{u}_{123}^{(i)}$  as linear functions of the  $q_{ij}$  and  $q_i$ . Thus we see that the operator  $\text{div}$  from  $\hat{\mathbf{V}}_h$  into  $H_h$  is one-to-one. Inequality (5.5) follows by the same scaling argument used in the proof of Proposition 3.1.

**6. Additional Results.** In checking the validity of Hypothesis H1 for a given mesh, it may be the case that there are a few triangles which do not easily fit into one of the mesh types covered in the previous sections. In that case, the following lemmas may be useful.

LEMMA 6.1. Let  $T_0$  and  $T_1$  be two triangles with a common side  $s$  and  $\Omega_0 = T_0 \cup T_1$ . Then, for all  $q_h \in H_h(T_0)$ , there exists  $\mathbf{u}_h \in \mathbf{V}_h(\Omega_0)$  such that

$$\operatorname{div}_h \mathbf{u}_h = q_h \quad \text{in } T_0, \quad \int_s \mathbf{u}_h \, d\sigma = 0, \quad \text{and} \quad \|\mathbf{u}_h\|_{1,h,\Omega_0} \leq C(\theta) \|q_h\|_{0,T_0},$$

where  $C$  depends only on the smallest angle  $\theta$  of the triangles  $T_0$  and  $T_1$ .

*Proof.*

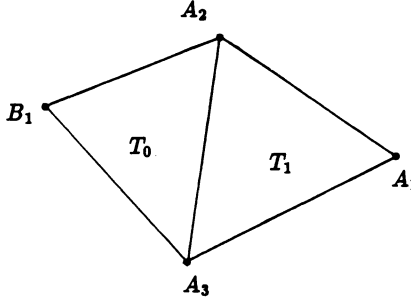


FIGURE 6.1

Using the notation of Proposition 3.1 and Figure 6.1, we see that the requirement that  $\mathbf{u}^{(0)} = 0$  at the Gauss points of  $\overrightarrow{B_1A_2}$  and  $\overrightarrow{B_1A_3}$  leads to the conditions

$$\mathbf{u}_2 = \mathbf{u}_3 = -\mathbf{u}_1, \quad \mathbf{u}_{13} = -\mathbf{u}_{31} = -9\mathbf{u}_1, \quad \mathbf{u}_{12} = -\mathbf{u}_{21} = -9\mathbf{u}_1,$$

and the requirement that  $\int_s \mathbf{u}_h \, d\sigma = 0$  on  $\overrightarrow{A_3A_2}$  can be implemented by setting

$$\mathbf{u}_{23} = 3\mathbf{u}_1 + \kappa, \quad \mathbf{u}_{32} = 3\mathbf{u}_1 - \kappa,$$

with  $\kappa$  to be determined. Setting  $\alpha = \mathbf{u}_1 \cdot \nabla \lambda_2$ , the equation  $\operatorname{div}_h \mathbf{u}_h = q_h$  in  $T_0$  leads to the equations

$$\begin{aligned} \mathbf{u}_1 \cdot \nabla \lambda_1 &= q_1/12, \\ \kappa \cdot \nabla \lambda_3 &= q_2 - q_1/2 + 6\alpha, \\ \kappa \cdot \nabla \lambda_2 &= -q_3 + q_1 + 6\alpha, \\ \mathbf{u}_{123} \cdot \nabla \lambda_2 &= q_{13} + 3q_1 + 18\alpha, \\ \mathbf{u}_{123} \cdot \nabla \lambda_3 &= q_{12} + 3q_1/2 - 18\alpha. \end{aligned}$$

It is easy to check that choosing  $\alpha = -q_1/24$  and defining  $\mathbf{u}$  in triangle  $T_1$  by

$$\mathbf{u}_i^{(1)} = \mathbf{u}_i^{(0)}, \quad \mathbf{u}_{ij}^{(1)} = \mathbf{u}_{ij}^{(0)}, \quad \mathbf{u}_{123}^{(1)} = \mathbf{u}_{123}^{(0)}$$

gives the desired result.

LEMMA 6.2. If  $\overline{\Omega}_s = \overline{\Omega}_r \cup T$ , where  $T$  has a common side with  $\Omega_r$ , and if Hypothesis H1 holds for  $\Omega_r$ , then Hypothesis H1 holds for  $\Omega_s$ .

*Proof.* Let  $q_h \in H_h(\Omega_s)$ . Lemma 6.1 then implies that there exists  $\mathbf{u}_0 \in \mathbf{V}_h(\Omega_s)$  such that

$$\operatorname{div}_h \mathbf{u}_0 = q_h \quad \text{in } T \quad \text{and} \quad \|\mathbf{u}_0\|_{1,h,\Omega_s} \leq C_1 \|q_h\|_{0,\Omega_s}.$$

Setting  $q_1 = q_h - \operatorname{div}_h \mathbf{u}_0$ , we observe that  $\int_s \mathbf{u}_h \, d\sigma = 0$  implies that  $q_1 \in H_h(\Omega_r)$ .

Hence, there exists  $\mathbf{u}_1 \in \mathbf{V}_h(\Omega_r)$  with  $\operatorname{div}_h \mathbf{u}_1 = q_1$  and

$$\|\mathbf{u}_1\|_{1,h,\Omega_s} \leq \|q_1\|_{0,\Omega_r}/\gamma_r \leq (1 + C_1\sqrt{2})\|q_h\|_{0,\Omega_s}/\gamma_r.$$

Taking  $\mathbf{u}_h = \mathbf{u}_0 + \mathbf{u}_1$ , we get that  $\operatorname{div}_h \mathbf{u}_h = q_h$  and

$$\|\mathbf{u}_h\|_{1,h,\Omega_s} \leq (C_1 + [1 + C_1\sqrt{2}]/\gamma_r)\|q_h\|_{0,\Omega_s} = \|q_h\|_{0,\Omega_s}/\gamma_s,$$

with  $\gamma_s = \gamma_r/(1 + C_1\sqrt{2} + C_1\gamma_r)$ .

*Remark.* Finally, since we have not found a counterexample, we conjecture that the spaces  $\mathbf{W}_h$  and  $Q_h$  form a stable Stokes pair for any triangulation of a convex polygon satisfying the minimal angle condition and containing an interior vertex.

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