Quadratic Convergence of Vortex Methods

By Vincenza Del Prete*

Dedicated to Professor Eugene Isaacson on the occasion of his 70th birthday

Abstract. We prove quadratic convergence for two-dimensional vortex methods with positive cutoffs. The result is established for flows with initial vorticity three times continuously differentiable and compact support. The proof is based on a refined version of a convergence result.

Introduction. The purpose of this paper is to prove that vortex methods with positive cutoffs can converge quadratically if the cutoff length is proportional to the mesh length and the flow is sufficiently smooth. This has been observed computationally by Hald and Del Prete [13], Beale and Majda [6] and Perlman [19].

The vortex method is a numerical technique for approximating the flow of an incompressible, inviscid fluid. The flow is described by Euler's equations. The method for the two-dimensional case was introduced by Chorin (see [8]). Various three-dimensional methods have been suggested and studied by Chorin [9], Beale and Majda [4], Greengard [12], Anderson and Greengard [2], Leonard [16], Raviart [20] and Beale [3]. Recently, Chiu and Nicolaides [7] investigated a vortex method with nonuniform mesh and a higher-order quadrature formula.

The convergence of the vortex method was first proved by Hald and Del Prete [13], but only for a short time interval. They assume that the initial vorticity is Hölder continuous and their class of cutoff includes some that are positive and singular. Positive cutoffs were not included in the theory of Hald [14], but were covered in the study of Beale and Majda [5]. They proved higher-order convergence for smooth flows and cutoffs that satisfy the so-called moment conditions and almost quadratic convergence for positive cutoffs. Our class of cutoffs cannot be compared with Beale and Majda's [5]. We assume more smoothness at the origin but allow a slow decay at infinity. In this paper we assume that the vorticity is three times continuously differentiable and prove quadratic convergence for our class of cutoffs. If the vorticity is two times continuously differentiable, we only obtain almost quadratic convergence. If the cutoff is positive, our result is better than the result of Beale and Majda [5]. On the other hand, Beale and Majda's theory gives higher rate of convergence for higher-order cutoffs. Our proof breaks down if the flow is not smooth. For such a flow Hald [15] has proved superlinear convergence for a large class of cutoffs.

It has been customary in previous papers [13], [14], [5], [1], [2], [15] to assume that the mesh length tends to zero faster than the cutoff length. It has been even

---

argued by Nakamura et al. [18] that this is necessary in order to get convergence. However, by using a new technique due to Beale [3] we obtain convergence in cases where the ratio of the mesh length and of the cutoff length is small but fixed. In his proof (for the three-dimensional case) Beale assumes implicity that the vorticity is at least five times continuously differentiable. Our proof follows Beale's closely, but we need only three continuous derivatives. Our main technical tool is a new estimate of the remainder of Taylor's formula. We also use a special case of a general stability result due to Hald [13]. To estimate the discretization error, we use a result of Cottet and Raviart [11] based on the Bramble-Hilbert lemma.

1. Notation and Statement of Results. The flow of an incompressible, inviscid two-dimensional fluid can be described by Euler's equation

$$\omega_t + (u \cdot \nabla)\omega = 0.$$

Here u is the velocity field with div $u = 0$ and $\omega = \text{curl } u$ is the vorticity and t is the time. To describe the evolution of the flow, we use the flow map $\Phi: \mathbf{R}^2 \times [0, T] \to \mathbf{R}^2$. Here $\Phi(\alpha, t)$ is the position at time t of a particle which at time $t = 0$ is at the point α. We denote the function $\alpha \to \Phi(\alpha, t)$ by Φ_t. It can be shown that the flow map satisfies the uncountably many ordinary differential equations

$$(1.1) \qquad \frac{d}{dt}\Phi(\alpha, t) = \int K(\Phi(\alpha, t) - \Phi(\alpha', t))\omega(\alpha')\, d\alpha',$$

$$\Phi(\alpha, 0) = \alpha.$$

Here, $\alpha = (\alpha_1, \alpha_2)^T$, $K(x) = (2\pi r^2)^{-1}x^\perp$ where $r^2 = x_1^2 + x_2^2$ and $x^\perp = (-x_2, x_1)^T$ where ^{T} means transpose. The function $\omega(\alpha) = \omega(\alpha, 0)$ is the initial vorticity. In addition, the velocity field and the vorticity distribution are given by

$$u(x, t) = \int K(x - \Phi(\alpha', t))\omega(\alpha')\, d\alpha',$$

$$\omega(x, t) = \int \delta(x - \Phi(\alpha', t))\omega(\alpha')\, d\alpha',$$

where δ denotes the delta function.

To solve Eqs. (1.1), we introduce the grid points $\alpha_j = jh$ where h is the mesh length, $j = (j_1, j_2)$ and j_1, j_2 are integers. The vortex method is an approximation to Eqs. (1.1), namely

$$(1.2) \qquad \frac{d}{dt}\tilde{\Phi}(\alpha_i, t) = \sum_{j \in J} K_\delta(\tilde{\Phi}(\alpha_i, t) - \tilde{\Phi}(\alpha_j, t))\kappa_j,$$

$$\tilde{\Phi}(\alpha_i, 0) = \alpha_i.$$

Here, $\kappa_j = \omega(\alpha_j)h^2$. The kernel $K_\delta = K * \varphi_\delta$ where $\varphi_\delta(x) = \delta^{-2}\varphi(x/\delta)$ is a radially symmetric approximation to the delta function. This approximation must satisfy further conditions which we shall specify later. We assume that the vorticity has compact support. The set J consists of the indices j such that the squares with center at α_j and side length h intersect the initial support of ω. The approximate velocity field and the approximate vorticity distribution are given by

$$\tilde{u}(x, t) = \sum_{j \in J} K_\delta(x - \tilde{\Phi}(\alpha_j, t))\kappa_j, \qquad \tilde{\omega}(x, t) = \sum_{j \in J} \varphi_\delta(x - \tilde{\Phi}(\alpha_j, t))\kappa_j.$$

Throughout this paper we shall assume that the solution of the differential equation is either two or three times continuously differentiable. Namely, letting $m = 2$ or $m = 3$, we shall assume that the flow satisfies

Assumption 1. The vorticity distribution, the velocity field and the flow map are m times continuously differentiable with respect to the space variable. The vorticity has compact support.

We believe that our result can be extended to the case of vorticity without compact support. This could be done, for example, by using Cottet and Raviart's technique [11] which assumes that the vorticity decays sufficiently fast at infinity. Our choice has been motivated by the fact that in numerical experiments one always handles a finite number of vortices.

Assumption 1 for $m = 3$ will be satisfied if the initial vorticity has compact support and the third derivatives are Hölder continuous. We assume that the support of $\omega(x, t)$ for $0 \leq t \leq T$ is contained in the set Ω and let D be the diameter of Ω. We now introduce the norms and seminorms:

$$\|\omega\|_{C^\lambda(D)} = \|\omega\|_\infty + D^\lambda \sup_{x \neq y} \frac{|\omega(x) - \omega(y)|}{|x - y|^\lambda},$$

$$\|u\|_{C^m} = \sum_{j=0}^{m} \max_{|\nu|=j} \|\partial^\nu u\|_\infty,$$

$$|\Phi|_{C^m} = \sum_{j=1}^{m} \max_{|\nu|=j} \|\partial^\nu \Phi\|_\infty.$$

Here, $0 < \lambda < 1$ and $\|\omega\|_{p,\Omega} = (\int_\Omega |\omega|^p \, dx)^{1/p}$. If $\Omega = \mathbf{R}^2$ then we drop the last subscript. We denote the set of functions of C^m which have compact support by C_c^m. By using the above notation we can reformulate

Assumption 1. There exists a constant C such that
(i) the initial vorticity $\omega \in C_c^m(\mathbf{R}^2)$ and $2(1 + D)\|\omega\|_{C^m} \leq C$.
(ii) $u \in C^m(\mathbf{R}^2)$ and $2(1 + D)\|u(t)\|_{C^m} \leq C$ for $0 \leq t \leq T$ and div $u = 0$.
(iii) $\Phi' \in C^{m-1}(\mathbf{R}^2)$, det $\Phi'(\alpha, t) = 1$ and $2(1 + D)|\Phi_t|_{C^m} \leq C$ for $0 \leq t \leq T$ and Φ is a differentiable function of t.

To estimate the error e in the vortex method, we introduce the discrete p norm

$$\|e\|_{p,h} = \left(\sum_{j \in J} |e_j|^p h^2 \right)^{1/p}.$$

In addition, $\|e\|_{\infty,h} = \max_{j \in J} |e_j|$. We assume that the cutoff function φ is a smooth radial function that vanishes at infinity, and that its integral is equal to 1. The conditions on the cutoff may be given in terms of the shape factor $f(r) = \int_{|x| < r} \varphi(x) \, dx$. Note that $K_\delta(x) = K(x) f(|x|/\delta)$. Throughout this paper we make

Assumption 2. Let $m = 2$ or 3.
(i) $f(r)/r^2$ is $m + 1$ times continuously differentiable as a function of r^2.
(ii) $f(r)$ tends to 1 as $r \to \infty$.
(iii) $|f^{(j)}(r)| \leq ar^{-4}$ for $r > 1$ and $j = 1, 2, \ldots, m + 1$.

Condition (iii) has been chosen for convenience. Our proof is valid if $|f^{(1)}| \leq$ const $r^{-(3+\varepsilon)}$ and $|f^{(4)}| \leq$ const r^{-4} for $r > 1$.

Below are three cutoffs which satisfy Assumption 2. The Gaussian cutoff was considered by Beale and Majda [5], the second is the two-dimensional version of a cutoff considered by Beale [3]. The last is new.

$$\varphi = \pi^{-1} e^{-r^2}, \qquad\qquad f = 1 - e^{-r^2},$$
$$\varphi = \pi^{-1}(1 + r^4)^{-3/2}, \qquad f = r^2(1 + r^4)^{-1/2},$$
$$\varphi = 2\pi^{-1}r^2(1 + r^4)^{-2}, \qquad f = r^4(1 + r^4)^{-1}.$$

We can now present

THEOREM 1. *Let $1 \leq p < \infty$. If Assumptions 1 and 2 are satisfied for $m = 2$ or $m = 3$, then there exist three constants $h_1 < h_2$ and C_1 such that if $\varepsilon = h/\delta < h_2$ then*

$$\|\tilde{\Phi}(t) - \Phi(t)\|_{p,h} \leq C_1 (h/\varepsilon)^2 (1 + (3 - m)|\log(h/\varepsilon)|)$$

for all $h < h_1$ and $0 \leq t \leq T$. The constants C_1 and h_2 depend only on C, p, T, D and the shape factor f, while h_1 depends on C, p, T, D, f and ε.

Remark. The theorem contains two results. If the initial vorticity is three times continuously differentiable, then the method converges quadratically. On the other hand, if the initial vorticity is twice continuously differentiable, we obtain only almost quadratic convergence. The proof of Theorem 1 is presented in Section 3. It uses that consistency plus stability implies convergence.

To formulate these results, we need a notation for the approximate velocity field. Let

$$v(x) = u(x, t) = \int K(x - \Phi_t(\alpha))\omega(\alpha)\, d\alpha,$$
$$V[\Psi; x] = \sum_{j \in J} K_\delta(x - \Psi(\alpha_j))\kappa_j.$$

To simplify our notation, we denote $V[\Phi_t; x]$ by $V(x)$ and $V[\Psi; \Psi(\alpha)]$ by $V[\Psi](\alpha)$. We will also denote $v \circ \Phi_t$ by $v[\Phi_t]$. Note that v is the exact velocity field u and that $V[\tilde{\Phi}_t, x]$ is the computed velocity field $\tilde{u}$. To estimate the difference $u - \tilde{u}$ we shall bound the consistency error $v[\Phi_t] - V[\Phi_t]$ and the stability error $V[\Phi_t] - V[\tilde{\Phi}_t]$. Our main result concerns the consistency.

LEMMA 1. *Let Assumptions 1 and 2 be satisfied with $m = 2, 3$ and let h and δ be two independent parameters; then there exists a constant C_0 such that*

$$\|v[\Phi_t] - V[\Phi_t]\|_{\infty,h} \leq C_0(\delta^2 + h^m \delta^{(2-m)}(1 + (3 - m)|\log \delta|))$$

for all $0 \leq t \leq T$. The constant C_0 depends on C, T, D, and the shape factor f.

Remark. The lemma shows that the consistency error is of order $\delta^2 + h^3\delta^{-1}$ if $m = 3$, of order $\delta^2 + h^2(1 + |\log \delta|)$ if $m = 2$. The proof of this result is an adaptation of Beale's [3] improved consistency lemma for a three-dimensional vortex method. As in Beale's proof, we use that the kernel K_δ in the vortex method is an odd function. However, the proof is further complicated because we do not assume that the vorticity is a smooth function. Our basic technical tool is a new version of

Taylor's formula. We also need a stability result. The following proposition is a special case of a result by Hald [15].

LEMMA 2. *Assume that $\|\omega\|_{C^\lambda(D)}$ and $\|\partial^\alpha\Phi\|_{C^\lambda(D)}$ are less than C, where $0 \le \lambda \le 1$ and $|\alpha| = 1$. Let Assumption 2 be satisfied and set $c = (4\sqrt{2}C)^{-1}$. There exist two constants C_2 and h_2 such that if h and h/δ are less than h_2 and*

$$\|\tilde{\Phi}_t - \Phi_t\|_{\infty,h} \le \tfrac{1}{2}ch,$$

then

$$\|V[\tilde{\Phi}_t] - V[\Phi_t]\|_{p,h} \le C_2\|\tilde{\Phi}_t - \Phi_t\|_{p,h},$$

where $1 < p < \infty$. The constants C_2 and h_2 depend on C, D, λ, p and the shape factor f.

Remark. Since $\|\omega\|_{C^\lambda(D)}$ is less than $2(1+D)\|\omega\|_{C^1}$ and a similar statement can be made for the first derivatives of the flow map, it follows that the assumptions in Lemma 2 for the vorticity and the flow map hold if Assumption 1 is satisfied. We remark that Hald's stability result includes a condition on the cutoff. But this condition is fulfilled when our Assumption 2 is satisfied.

2. Proof of the Consistency Result.

To prove the consistency lemma, we observe that the consistency error separates into two parts, namely the error $\{A\}$ from the discretization and the error $\{B\}$ due to smoothing:

$$
\begin{aligned}
(2.1) \quad V(x) - v(x) &= \sum_j K_\delta(x - \Phi_t(\alpha_j))\omega_j h^2 - \int K_\delta(x - \Phi_t(\alpha))\omega(\alpha)\,d\alpha \\
&\quad + \int [K_\delta(x - \Phi_t(\alpha)) - K(x - \Phi_t(\alpha))]\omega(\alpha)\,d\alpha \\
&= \{A\} + \{B\}.
\end{aligned}
$$

We observe that the smoothing error $\{B\}$ can be written as $u * \varphi_\delta - u$. To estimate the smoothing error we will use

LEMMA 3. *Let φ be a function in $L^1(\mathbf{R}^2)$ such that $|x|^2\varphi \in L^1(\mathbf{R}^2)$, $\int \varphi(x)\,dx = 1$ and $\int x^\alpha\varphi(x) = 0$ for $|\alpha| = 1$. If $g \in C^2(\mathbf{R}^2)$ then*

$$\sup_{x\in\mathbf{R}^2} |(g * \varphi_\delta)(x) - g(x)| \le \sup_{x\in\mathbf{R}^2, |\alpha|=2} |\partial^\alpha g(x)| \, \||x|^2\varphi\|_1 \, \delta^2.$$

Remark. We shall apply Lemma 3 to the velocity u. Note that Assumption 2 for the shape factor f implies that the cutoff function φ satisfies all the conditions in Lemma 3. In particular, (iii) implies that $|x|^2\varphi \in L^1$. The condition $\int x^\alpha\varphi\,dx = 0$ follows from the fact that φ is radial. To prove the lemma, we simply expand g in a Taylor series.

To estimate the discretization error, we use the following quadrature formula.

LEMMA 4. *Let l be an integer greater than or equal to 2. Assume that $\partial^\beta g \in L^1(\mathbf{R}^2)$ for $|\beta| \le l$. Then*

$$\left| \int g(x)\,dx - \sum_j g(jh)h^2 \right| \le \text{const } h^l \sum_{|\beta|=l} \|\partial^\beta g\|_1.$$

Remark. The result is due to Cottet and Raviart [11] (see also Cottet [10, Lemma 2.5 and the proof of Lemma 3.4]). The proof is based on the Bramble-Hilbert lemma and the fact that the space of functions which have derivatives in L^1 up to the order l, $l \leq 2$, is continuously imbedded in $C^0(\mathbf{R}^2)$. A simple proof for $l \geq 3$ based on the Poisson summation formula has been given by Anderson and Greengard [2].

We shall apply Lemma 4 to the function $g = K_\delta \circ \Phi_t \omega$, and hence we need bounds for K_δ and its derivatives. We observe first that $K_\delta(x)x^T$ is a two by two matrix.

LEMMA 5. *Let Assumption 1 be satisfied. For any R,*

$$\int_{|x|<R} |\partial^\beta (K_\delta(x)x^T)| \, dx \leq \begin{cases} c & \text{if } |\beta| = 0, 1, \\ c(1 + |\log\delta|) & \text{if } |\beta| = 2, \\ c\delta^{2-|\beta|} & \text{if } |\beta| = 3. \end{cases}$$

The constant depends on β and R but not on δ.

Proof. Write

$$\int_{|x|<R} |\partial^\beta (K_\delta(x)x^T)| \, dx = \int_{|x|<\delta} + \int_{\delta<|x|<R}.$$

Assumption 2 implies that

$$|\partial^\beta (K_\delta(x)x^T)| \leq \begin{cases} \text{const}\, \delta^{-|\beta|}, & |x| < \delta, \\ \text{const}|x|^{-|\beta|}, & |x| > \delta, \end{cases}$$

for $|\beta| \leq 4$. So the first integral is less than a constant times $\delta^{2-|\beta|}$. The second integral can be estimated with a constant if $|\beta| = 0$ or $|\beta| = 1$, with a constant times $(1 + |\log\delta|)$ if $|\beta| = 2$, and finally with a constant times $\delta^{2-|\beta|}$ if $|\beta| = 3$. This completes the proof.

We shall also need a special version of the Taylor formula for functions in C^m where the remainder is expressed as a tensor whose components are m times continuously differentiable away from the origin and satisfy suitable growth conditions.

LEMMA 6. *Let $f \in C^m(\mathbf{R}^n)$ and $1 \leq k \leq m$, m greater than or equal to 1. Let P_{k-1} be the Taylor polynomial of degree $k - 1$ of f centered at zero. Then there exist functions $\vartheta_\beta \in C^m(\mathbf{R}^n - 0)$, $|\beta| = k$, such that*

$$f(x) - P_{k-1}(x) = \sum_{|\beta|=k} \vartheta_\beta(x)x^\beta,$$

$$|\partial^\gamma \vartheta_\beta(x)| \leq \text{const} \|f\|_{C^m} |x|^{-|\gamma|}$$

for $x \neq 0$ and $0 \leq |\gamma| \leq m$, where the constant depends only on k and n.

Remark. Note that Lemma 6 does not follow from Taylor's formula with integral remainder because in that case the functions ϑ_β for $|\beta| = k$ are merely in $C^{m-k}(\mathbf{R}^n)$. The basic idea is this: if $\vartheta_1 = x_1 r^{-2}(f(x) - f(0))$ and $\vartheta_2 = x_2 r^{-2}(f(x) - f(0))$, then $f(x) - f(0) = x_1\vartheta_1 + x_2\vartheta_2$.

Proof. To get the expression for $f - P_{k-1}$, we simply choose

$$\vartheta_\beta = (f(x) - P_{k-1}(x)) \left(\frac{x^\beta}{\sum_{|\beta|=k} x^{2\beta}} \right).$$

To estimate the derivatives of ϑ_β, we use the Leibniz formula

$$\partial^\gamma \vartheta_\beta(x) = \sum_{\nu \le \gamma} \binom{\gamma}{\nu} \partial^\nu (f(x) - P_{k-1}(x)) \partial^{\gamma-\nu} \left(\frac{x^\beta}{\sum_{|\beta|=k} x^{2\beta}} \right).$$

Since $\partial^\nu P_{k-1}$ is the Taylor polynomial of degree $k - 1 - |\nu|$ of $\partial^\nu f$, it follows from Taylor's formula that

$$|\partial^\nu (f(x) - P_{k-1}(x))| \le \|f\|_{C^m} |x|^{k-|\nu|}.$$

Since $x^\beta / \sum_{|\beta|=k} x^{2\beta}$ is homogeneous of degree $-k$ and $\sum_{|\beta|=k} x^{2\beta} \ge n^{1-k} |x|^{2k}$, we see that

$$\left| \partial^{\gamma-\nu} \frac{x^\beta}{\sum_{|\beta|=k} x^{2\beta}} \right| \le c(k,n) |x|^{-k-(|\gamma|-|\nu|)}.$$

The proof is completed by inserting the bounds in the Leibniz formula.

Proof of Lemma 1. Let $m = 3$. Throughout this proof we assume that the time t is fixed. First we consider the smoothing error $\{B\}$ in Eq. (2.1). We recall that Assumption 2 implies that the cutoff φ satisfies the condition of Lemma 3. Thus it follows from Lemma 3 with $g = u$ that

$$(2.2) \qquad |u * \varphi_\delta - u| \le \text{const } \delta^2,$$

where the constant depends on the cutoff φ and the velocity field u.

We consider now the discretization error $\{A\}$ at a fixed point $x = \Phi_t(\alpha_i)$. Since $\alpha_j - \alpha_i = -\alpha_{j-i}$, it follows that the discretization error can be written as

$$\{A\} = \sum_{j \in J-i} K_\delta(\Phi_t(\alpha_i) - \Phi_t(\alpha_i + \alpha_j)) \omega(\alpha_i + \alpha_j) h^2$$
$$- \int K_\delta(\Phi_t(\alpha_i) - \Phi_t(\alpha_i + \alpha)) \omega(\alpha_i + \alpha) \, d\alpha.$$

We introduce the map $\Psi(\alpha) = \Phi_t(\alpha_i) - \Phi_t(\alpha_i + \alpha)$. Note that $\Psi(0) = 0$ and that Ψ has the same regularity properties as Φ_t. We will expand the functions $K_\delta \circ \Psi$ and ω around α_i, but we suppress the dependence on α_i by shifting the coordinate system such that the grid point α_i falls at the origin. The set Ω is shifted correspondingly. Since $K_\delta(x) = (2\pi r^2)^{-1} f(r/\delta) x^\perp$, we see that the discretization error is

$$\{A\} = \sum_j F(\alpha_j) h^2 - \int F(\alpha) \, d\alpha = E(F),$$

where

$$F(\alpha) = \frac{1}{2\pi} f\left(\frac{\Psi(\alpha)}{\delta} \right) \frac{\Psi(\alpha)^\perp}{|\Psi(\alpha)|^2} \omega(\alpha).$$

To study the discretization error, we linearize $\Psi(\alpha)$. By using the Taylor formula in Lemma 6 we decompose $\Psi(\alpha)$ into the sum of a linear part plus a remainder. Hence we shall write F as the sum of two terms: F_0 which contains the linear term and $F_1 = F - F_0$. The error is partitioned in $E(F_0)$ and $E(F_1)$ and will be estimated later. Since $\Psi(0) = 0$, Lemma 6 implies

$$\Psi(\alpha) = A\alpha + \vartheta(\alpha) \cdot (\alpha, \alpha),$$

where A is the Jacobian matrix of Ψ at zero, and $\vartheta(\alpha) \cdot (\beta, \beta)$ is a multilinear form in β with two components. The regularity of the flow (Assumption 1) implies that there exists a constant ρ_1 independent of $x = \Phi_t(\alpha_i)$ such that

$$(2.3) \qquad |\Psi(\alpha) - A\alpha| \leq |\Psi(\alpha)|/2.$$

Let $\tilde{\mu}$ be a smooth function such that $0 \leq \tilde{\mu} \leq 1$ and assume that $\tilde{\mu}(r) = 0$ for $r < 1$ and $\tilde{\mu}(r) = 1$ for $r > 2$. Set $\mu(r) = \tilde{\mu}(r/\rho_1)$ and let

$$\Psi_0(\alpha) = A\alpha + \mu(|\alpha|)\vartheta(\alpha) \cdot (\alpha, \alpha) = A\alpha + \mu(|\alpha|)(\Psi(\alpha) - A\alpha).$$

Note that $\Psi_0(0) = 0$ and that Ψ_0 is as regular as Ψ (i.e., $|\Psi_0|_{C^3} < \infty$, $\det \Psi_0' > $ const > 0). If ρ_1 is sufficiently small, then Ψ_0 is invertible by the implicit function theorem. We also have

$$\Psi(\alpha) = A\alpha + \mu(|\alpha|)\vartheta(\alpha) \cdot (\alpha, \alpha) + (1 - \mu(|\alpha|))\vartheta(\alpha) \cdot (\alpha, \alpha)$$
$$= \Psi_0(\alpha) + (1 - \mu(|\alpha|))(\Psi(\alpha) - A\alpha).$$

Notice that $\Psi - \Psi_0$ is equal to $\Psi(\alpha) - A\alpha$ if $|\alpha| \leq \rho_1$, and equal to zero for $|\alpha| > 2\rho_1$. So from (2.3) we get the following inequality, which will be used later,

$$(2.4) \qquad |\Psi(\alpha) - \Psi_0(\alpha)| \leq |\Psi(\alpha)|/2$$

for each $\alpha \in \mathbf{R}^2$. We can now define

$$F_0(\alpha) = f\left(\frac{|\Psi_0(\alpha)|}{\delta}\right) \frac{\Psi_0(\alpha)^\perp}{|\Psi_0(\alpha)|^2} \frac{\omega(\alpha)}{2\pi}$$

and

$$F_1(\alpha) = \left\{ f\left(\frac{|\Psi(\alpha)|}{\delta}\right) \frac{\Psi(\alpha)^\perp}{|\Psi(\alpha)|^2} - f\left(\frac{|\Psi_0(\alpha)|}{\delta}\right) \frac{\Psi_0(\alpha)^\perp}{|\Psi_0(\alpha)|^2} \right\} \frac{\omega(\alpha)}{2\pi}.$$

To estimate $E(F_0)$, we use that $K_\delta \circ \Psi_0$ is an odd function of α for $|\alpha| < \rho_1$ because here $\Psi_0(\alpha) = A\alpha$. Next we express $\omega(\alpha)$ as an even function of α with support in $|\alpha| < \rho_1$ plus another function that vanishes at zero. More precisely, we write $\omega = \hat{\sigma} + \sigma^T \Psi_0$. Here $\hat{\sigma}$ consists of the leading terms in the Taylor series of the even part of ω, multiplied by a smooth radial function ν which has support in $|\alpha| < 2\rho_1$ and is equal to 1 for $|\alpha| < \rho_1$. We may choose $\nu(\alpha) = 1 - \tilde{\nu}(|\alpha|/\rho_1)$, where $\tilde{\nu}(r)$ is a smooth function which vanishes for $r > 2$ and is equal to 1 for $r < 1$. We will prove that $\sigma \in C^3(\mathbf{R}^2 - 0)$. In addition, $\partial^\beta \sigma \in L^\infty(\mathbf{R}^2)$ for $|\beta| \leq 2$ and $\partial^\beta \sigma \in L^1(\mathbf{R}^2)$ for $|\beta| = 3$. Let P_2 be the second-order Taylor polynomial of ω at zero. Since $\alpha = A^{-1}\Psi_0(\alpha)$ and $\Psi_0^T \Psi_0 = |\Psi_0|^2$, we define $\hat{\sigma}$ and σ so that

$$\omega(\alpha) = \nu(\alpha)(\omega(0) + \tfrac{1}{2}(\omega''(0)\alpha, \alpha)) + \nu(\alpha)\omega'(0)\alpha + \omega(\alpha) - \nu(\alpha)P_2(\alpha)$$

$$(2.5) \qquad = \hat{\sigma}(\alpha) + \nu(\alpha)\omega'(0)A^{-1}\Psi_0(\alpha) + (\omega(\alpha) - \nu(\alpha)P_2(\alpha)) \frac{\Psi_0^T(\alpha)}{|\Psi_0(\alpha)|^2} \Psi_0(\alpha)$$

$$= \hat{\sigma}(\alpha) + \sigma^T(\alpha)\Psi_0(\alpha).$$

The first term $\nu\omega'(0)A^{-1}$ in the definition of σ is a smooth function of α with compact support. Thus, to estimate the derivatives of σ, we must study the derivatives of $\tilde{\sigma} = (\omega - \nu P_2)|\Psi_0|^{-2}\Psi_0$. The Leibniz formula yields

$$\partial^\beta \tilde{\sigma} = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma(\omega - \nu P_2)\partial^{\beta - \gamma}(|\Psi_0|^{-2}\Psi_0).$$

Our first claim is that

$$(2.6) \qquad |\partial^\gamma(\omega - \nu P_2)| \leq \begin{cases} \text{const}|\alpha|^{3-|\gamma|} & \text{if } |\alpha| < \rho_1, \\ \text{const} & \text{if } |\alpha| > \rho_1, \end{cases}$$

where the constant depends on ρ_1 and C, D from Assumption 1. To prove (2.6), we observe that $\nu = 1$ if $|\alpha| < \rho_1$ and that $\partial^\beta(\omega - P_2)$ is the remainder in the Taylor formula of order $2 - |\beta|$ for $\partial^\beta \omega$. If $\rho_1 \leq |\alpha| \leq 2\rho_1$ then both $\partial^\gamma P_2$ and $\partial^{\beta-\gamma}\nu$ are bounded by constants that depend on ρ_1. The estimate for $|\alpha| > 2\rho_1$ follows from Assumption 1. Our next claim is that

$$(2.7) \qquad |\partial^{\beta-\gamma}(|\Psi_0|^{-2}\Psi_0)| \leq \begin{cases} \text{const}|\alpha|^{|\gamma|-|\beta|-1} & \text{if } |\alpha| < \rho_1, \\ \text{const} & \text{if } |\alpha| > \rho_1. \end{cases}$$

To prove (2.7), we set $G(x) = |x|^{-2}x$ and observe that $\partial^\beta G$ is a homogeneous function of degree $-1 - |\beta|$. If $|\alpha| \leq \rho_1$ then $\Psi_0(\alpha) = A\alpha$, and it follows from Assumption 1 that $\det A = 1$ and that the elements in A are bounded by C. We obtain the first estimate in (2.7) by differentiating the composite function $G \circ \Psi_0$ and using the homogeneity of G. If $|\alpha| > \rho_1$ then (2.4) implies that $|\Psi_0| > |\Psi|/2$. Since $\alpha - 0 = \Psi^{-1}(\Psi(\alpha)) - \Psi^{-1}(0)$, it follows from the mean value theorem that $|\alpha| \leq \text{const}|\Psi(\alpha)|$, where the constant is less than C from Assumption 1. Consequently, $|\Psi_0(\alpha)| > \text{const}\,\rho_1$. So the proof of (2.7) is completed by differentiating $G \circ \Psi_0$ again and using that $|\Psi_0|_{C^3}$ is finite.

By using (2.6) and (2.7) in Leibniz' formula we see that $\partial^\beta \tilde\sigma$ are uniformly bounded for $|\beta| = 0, 1, 2$ and that $\partial^\beta \tilde\sigma$ is integrable for $|\beta| = 3$. Here we have used that ω and ν have compact support, namely $\Omega - \alpha_i$ and the sphere with center 0 and radius $2\rho_1$.

We are now ready to estimate the discretization error for F_0. By inserting (2.5) in the expression for F_0 we see that $F_0 = F_{00} + F_{01}$, where F_{00} is an odd function and

$$F_{01}(\alpha) = \frac{1}{2\pi} f\left(\frac{|\Psi_0(\alpha)|}{\delta}\right) \frac{\Psi_0(\alpha)^\perp \Psi_0(\alpha)^T}{|\Psi_0(\alpha)|^2} \sigma(\alpha) = k(\alpha)\sigma(\alpha).$$

Thus, $k(\alpha) = K_\delta(y)y^T$ where $y = \Psi_0(\alpha)$.

Since F_{00} is odd, $E(F_{00}) = 0$. We will show that $E(F_{01}) \leq \text{const}\, h^3 \delta^{-1}$ by using Lemma 4 together with the estimate

$$(2.8) \qquad \max_{|\beta|=3} \|\partial^\beta F_{01}\|_1 \leq \text{const}\, \delta^{-1}.$$

It follows from Leibniz' formula that

$$\int |\partial^\beta F_{01}(\alpha)|\, d\alpha \leq \int |k(\alpha)\partial^\beta \sigma(\alpha)|\, d\alpha + \sum_{0 < \gamma \leq \beta} \binom{\beta}{\gamma} \int |\partial^\gamma k(\alpha)|\, |\partial^{\beta-\gamma}\sigma(\alpha)|\, d\alpha.$$

We observe now that the support of σ is contained in a sphere B with center at the origin (of the shifted coordinate system) and with radius $D + 1 + 2\rho_1$. Since the derivatives $\partial^{\beta-\gamma}\sigma$ for $\gamma > 0$ are bounded and $|K_\delta(y)y^T|$ is less than $(2\pi)^{-1}\|f\|_\infty$, we see that

$$(2.9) \qquad \int |\partial^\beta F_{01}(\alpha)|\, d\alpha \leq \text{const} \left[\int_B |\partial^\beta \sigma|\, d\alpha + \sum_{0 < \gamma \leq \beta} \int_B |\partial^\gamma k(\alpha)|\, d\alpha \right],$$

where the constant depends on the shape factor f and the derivatives of σ. We have already shown that the integral $\int |\partial^\beta \sigma| \, d\alpha$ in (2.9) is bounded. Since $k(\alpha) = K_\delta(y)y^T$ with $y = \Psi_0(\alpha)$, it follows from the chain rule that $\partial^\gamma k(\alpha)$ can be expressed as a sum of derivatives of $K_\delta(y)y^T$, evaluated at $y = \Psi_0(\alpha)$, times derivatives of Ψ_0 of order less than or equal to $|\gamma|$. Choose R such that $|y| \leq R$ for all $y \in \Psi_0(B)$. By using the change of variables $\alpha = \Psi_0^{-1}(y)$ we conclude from Lemma 5 that

$$\int_B |\partial^\gamma k(\alpha)| \, d\alpha \leq \text{const} \sum_{0 < |\eta| \leq |\gamma|} \int_{\Psi_0(B)} |\partial^\eta (K_\delta(y)y^T)| \, dy$$
$$\leq \text{const} \, \delta^{-1},$$

where the constant depends on ρ_1 and R and the bounds for the derivatives of Ψ_0. By inserting this result in (2.9) we obtain (2.8). Lemma 4 and (2.8) imply that $E(F_{01}) \leq \text{const} \, h^3 \delta^{-1}$, and since $E(F_{00}) = 0$ it follows that $E(F_0) \leq \text{const} \, h^3 \delta^{-1}$.

To estimate $E(F_1)$ by Lemma 4, we must prove that

$$(2.10) \qquad \max_{|\beta|=3} \|\partial^\beta F_1\|_1 \leq \text{const} \, \delta^{-1}.$$

Let $g(\eta) = |\eta|^{-2} f(\eta) \eta^\perp$. We can then express F_1 as

$$F_1 = (g(\Psi/\delta) - g(\Psi_0/\delta))\omega/(2\pi\delta).$$

To estimate the derivatives of F_1, we need bounds for the derivatives of g. It follows from the mean value theorem that $g(\eta) - g(\varsigma) = \hat{g}(\eta, \varsigma)(\eta - \varsigma)$, where

$$\hat{g}(\eta, \varsigma) = \int_0^1 g'(s\eta + (1-s)\varsigma) \, ds$$

and $g' = (\partial_1 g, \partial_2 g)$. Note that $\hat{g}$ is a two by two matrix. Assumption 2 implies that $\hat{g}$ is in C^3, and by differentiating with respect to η and ς we see that

$$(2.11) \qquad \partial_\eta^\beta \partial_\varsigma^\gamma \hat{g}(\eta, \varsigma) = \int_0^1 \partial^{\beta+\gamma} g'(s\eta + (1-s)\varsigma) s^{|\beta|}(1-s)^{|\gamma|} \, ds,$$

for $|\beta| + |\gamma| \leq 3$. Thus, $|\partial_\eta^\beta \partial_\varsigma^\gamma \hat{g}(\eta, \varsigma)| \leq \text{const}$ if $|\eta|$ and $|\varsigma|$ are less than a constant L. We now replace η and ς in $\hat{g}$ by $\delta^{-1}\Psi(\alpha)$ and $\delta^{-1}\Psi_0(\alpha)$. Let $|\gamma| \leq 3$. By using the chain rule we see that $\partial_\alpha^\gamma \hat{g}(\delta^{-1}\Psi, \delta^{-1}\Psi_0)$ consists of a sum of derivatives of $\hat{g}(\eta, \varsigma)$ with respect to η and ς of order $|\beta|$ greater or equal to 1 and less than or equal to $|\gamma|$ evaluated at $\eta = \delta^{-1}\Psi$ and $\varsigma = \delta^{-1}\Psi_0$, multiplied by $\delta^{-|\beta|}$ times a function which is a product of derivatives of Ψ and Ψ_0 of order greater or equal to 1 and less than or equal to $|\gamma|$. We can now show that

$$(2.12) \qquad \left| \partial_\alpha^\gamma \hat{g}\left(\frac{\Psi(\alpha)}{\delta}, \frac{\Psi_0(\alpha)}{\delta}\right) \right| \leq \begin{cases} \text{const} \, \delta^{-|\gamma|} & \text{if } |\alpha| \leq \delta, \\ \text{const} \, \delta^2 |\alpha|^{-|\gamma|-2} & \text{if } R_1 > |\alpha| > \delta, \end{cases}$$

where $R_1 = 1 + D$. Let $|\alpha| \leq \delta$. Assumption 1 and the regularity of Ψ_0 imply that $|\Psi(\alpha) - \Psi(0)| \leq L|\alpha| \leq L\delta$ and $|\Psi_0(\alpha)| \leq L\delta$. This implies that the derivatives of $\hat{g}(\eta, \varsigma)$ at $\eta = \delta^{-1}\Psi$ and $\varsigma = \delta^{-1}\Psi_0$ are bounded, and the first statement in (2.12) therefore follows from the chain rule. To prove the second inequality in (2.12),

we let $|\alpha| > \delta$. It follows from the regularity of Ψ^{-1} and Ψ_0^{-1} that there exists a positive constant L_1 such that $|\Psi(\alpha)| \geq L_1|\alpha| \geq L_1\delta$ and $|\Psi_0(\alpha)| \geq L_1|\alpha| \geq L_1\delta$. Since $\eta^\perp|\eta|^{-2}$ is homogeneous of degree -1 we find from Assumption 2 that $|\partial^\beta g(\eta)| \leq \text{const}|\eta|^{-1-|\beta|}$ if $|\eta| > L_1$ and $0 \leq |\beta| \leq 4$. Thus, if $|\eta| \geq L_1$ and $|\eta - \varsigma| < |\eta|/2$, we can estimate the integral in (2.11) and obtain

$$(2.13) \qquad |\partial^{\beta+\gamma}\hat{g}(\eta,\varsigma)| \leq \text{const}|\eta|^{-2-|\beta|-|\gamma|}$$

for $0 \leq |\beta| + |\gamma| \leq 3$. Now let $\eta = \delta^{-1}\Psi(\alpha)$ and $\varsigma = \delta^{-1}\Psi_0(\alpha)$ and observe that inequality $|\eta - \varsigma| < |\eta|/2$ is satisfied because of (2.4). If $|\alpha| > \delta$ then $|\eta| > L_1$, and we conclude from the chain rule and (2.13) that

$$\left|\partial_\alpha^\gamma \hat{g}\left(\frac{\Psi(\alpha)}{\delta}, \frac{\Psi_0(\alpha)}{\delta}\right)\right| \leq \text{const} \sum_{1 \leq |\beta| \leq |\gamma|} \left(\frac{|\Psi(\alpha)|}{\delta}\right)^{-2-|\beta|} \delta^{-|\beta|}$$

$$\leq \text{const}\,\delta^2 \sum_{1 \leq |\beta| \leq |\gamma|} |\alpha|^{-2-|\beta|} \leq \text{const}\,\delta^2|\alpha|^{-2-|\gamma|}.$$

In the last inequality we have used the fact that $|\alpha| \leq R_1$. This concludes the proof of the second inequality in (2.12).

We will next study the derivatives of F_1. Define G such that

$$F_1(\alpha) = \frac{1}{2\pi\delta^2}\hat{g}\left(\frac{\Psi(\alpha)}{\delta}, \frac{\Psi_0(\alpha)}{\delta}\right)(\Psi(\alpha) - \Psi_0(\alpha))\omega(\alpha)$$

$$= \delta^{-2}\hat{g}\left(\frac{\Psi(\alpha)}{\delta}, \frac{\Psi(\alpha)}{\delta}\right)\chi(\alpha)|\alpha|^2 = |\alpha|^2 G(\alpha),$$

where $\chi(\alpha) = (2\pi|\alpha|^2)^{-1}(\Psi(\alpha) - \Psi_0(\alpha))\omega(\alpha)$. Let $|\beta| = 3$. Since $\partial^\beta|\alpha|^2 = 0$, it follows from Leibniz' formula that

$$(2.14) \qquad \partial^\beta F_1(\alpha) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^{\beta-\gamma}|\alpha|^2 \partial^\gamma G(\alpha).$$

To estimate the derivative of G, we will use that

$$\partial^\gamma G(\alpha) = \delta^{-2} \sum_{\lambda \leq \gamma} \binom{\gamma}{\lambda} \partial^{\gamma-\lambda}\hat{g}\left(\frac{\Psi(\alpha)}{\delta}, \frac{\Psi_0(\alpha)}{\delta}\right) \partial^\lambda \chi(\alpha).$$

We already have bounds for the derivatives of $\hat{g}$, namely (2.12), and we will now prove that

$$(2.15) \qquad |\partial^\lambda \chi(\alpha)| \leq \text{const}|\alpha|^{-|\lambda|}$$

for $|\alpha| \leq R_1$ and $|\lambda| \leq 3$. We observe first that $\chi(\alpha) = (2\pi)^{-1}(1 - \mu(|\alpha|))\omega(\alpha) \cdot |\alpha|^{-2}(\Psi(\alpha) - A\alpha)$. The first factor is in C_c^3. From Lemma 6 we have

$$|\alpha|^{-2}(\Psi(\alpha) - A\alpha) = \sum_{|\beta|=2} \vartheta_\beta(\alpha)\frac{\alpha^\beta}{|\alpha|^2},$$

where $|\partial^\mu \vartheta_\beta(\alpha)| \leq \text{const}|\alpha|^{-|\mu|}$. Since $|\alpha|^{-2}\alpha^\beta$ is homogeneous of degree 0, we conclude that $|\partial^{\beta-\gamma}(|\alpha|^{-2}(\Psi(\alpha) - A\alpha))| \leq \text{const}|\alpha|^{-|\beta-\gamma|}$, and Eq. (2.15) follows.

By combining (2.12) and (2.15) we see that the derivatives of G can be estimated by

$$|\partial^\gamma G(\alpha)| \leq \begin{cases} \text{const}\,\delta^{-2}|\alpha|^{-|\gamma|} & \text{if } |\alpha| \leq \delta, \\ \text{const}\,|\alpha|^{-2-|\gamma|} & \text{if } R_1 > |\alpha| > \delta. \end{cases}$$

By inserting the bounds for $\partial^\gamma G$ in (2.14) we obtain the pointwise estimate

$$|\partial^\beta F_1(\alpha)| \leq \begin{cases} \text{const}\,\delta^{-2}|\alpha|^{-1} & \text{if } |\alpha| \leq \delta, \\ \text{const}\,|\alpha|^{-3} & \text{if } R_1 > |\alpha| > \delta. \end{cases}$$

We can now integrate $\partial^\beta F_1$ over $\mathbf{R}^2$, and since F_1 vanishes for $|\alpha| \geq R_1$, we have proved (2.10). By combining (2.10) with Lemma 4 we obtain $|E(F_1)| \leq \text{const}\,h^3\delta^{-1}$. Since $F = F_0 + F_1$, we have estimated the discretization error and hence completed the proof of Lemma 1.

To prove the lemma for $m = 2$, we use Lemma 4 for $l = 2$ and follow closely the previous proof. So we end up with the estimates

$$\max_{|\beta|=2} \|\partial^\beta F_{01}\|_1 \leq \text{const}(1 + |\log\delta|), \qquad \max_{|\beta|=2} \|\partial^\beta F_1\|_1 \leq \text{const}(1 + |\log\delta|),$$

which replace inequalities (2.8) and (2.10). The above estimates lead to $E(F) \leq \text{const}\,h^2(1 + |\log\delta|)$, and from this the assertion follows.

3. Proof of the Convergence Theorem.

The proof of the convergence theorem is based on Lemma 1 and Lemma 2, and we need only to prove the theorem for p sufficiently large, see Beale and Majda [5, p. 46]. Let $p > 2$ and let h_2 and c be the constants that appear in the statement of the stability lemma. Let $e(t) = \Phi_t - \tilde{\Phi}_t$. Let $h < h_2$ and $h/\delta = \varepsilon$ where ε is a fixed number less than h_2. We make the following claim. If

$$(3.1) \qquad\qquad \|e(t)\|_{p,h} \leq \tfrac{1}{2}ch^{1+2/p}$$

holds for $0 \leq t \leq t^*$ and $t^* < T$, then

$$(3.2) \qquad\qquad \|e(t)\|_{p,h} \leq C_1(h/\varepsilon)^2,$$

where the constant C_1 does not depend on t^* or ε. Since $\|e(t)\|_{\infty,h} \leq h^{-2/p}\|e(t)\|_{p,h}$ it follows from (3.1) that $\|e(t)\|_{\infty,h} \leq \tfrac{1}{2}ch$ for $0 \leq t < t^*$. We can therefore apply Lemma 2 and get

$$(3.3) \qquad\qquad \|V[\Phi_t] - V[\tilde{\Phi}_t]\|_{p,h} \leq C_2\|\Phi_t - \tilde{\Phi}_t\|_{p,h}.$$

Since $\|e\|_{p,h}$ is less than $\|e\|_{\infty,h} \cdot \|1\|_{p,h}$ it follows from Lemma 1 that there exists a constant C_0 such that

$$(3.4) \qquad \|v[\Phi_t] - V[\Phi_t]\|_{p,h} \leq C_0\left(\frac{h^2}{\varepsilon^2} + h^2\varepsilon\right)\|1\|_{p,h} \leq C'(h/\varepsilon)^2,$$

where C' depends on p and the support of ω. By combining (1.1), (1.2) and (3.3), (3.4) we get

$$(3.5) \qquad \|\dot{e}(t)\|_{p,h} = \|v[\Phi_t] - V[\tilde{\Phi}_t]\|_{p,h} \leq C'(h/\varepsilon)^2 + C_2\|e(t)\|_{p,h}.$$

Let now $F(t) = \|e(t)\|_{p,h}$. Since $\dot{F}(t) \leq \|\dot{e}(t)\|_{p,h}$, inequality (3.5) implies that $F(t) \leq C_1(h/\varepsilon)^2$ for $0 \leq t < t^*$, where $C_1 = C'(\exp\{C_2T\} - 1)C_2^{-1}$. Notice that

C_1 does not depend on ε. We have therefore established (3.2). Let now

$$E = \{t \in [0,T]: \|e(s)\|_{p,h} \le \tfrac{1}{2}ch^{1+2/p},\ 0 \le s \le t\},$$

and observe that $E \ne \varnothing$ since $e(t)$ is a continuous function that vanishes at $t = 0$. Let $T^* = \sup E$. We will show later that $T^* = T$. This implies that $\|e(t)\|_{p,h} \le \tfrac{1}{2}ch^{1+2/p}$ for $0 \le t \le T$. By the claim this implies that $\|e(t)\|_{p,h} \le C_1(h/\varepsilon)^2$ for $0 \le t \le T$. So the proof is completed provided $T^* = T$. If $T^* < T$ then (3.1) and therefore (3.2) holds for $0 \le t < T^*$. Since $p > 2$ there exists a constant $h_1 < h_2$ such that $C_1(h/\varepsilon)^2 < \tfrac{1}{4}ch^{1+2/p}$ for $h < h_1$. Notice that h_1 depends on ε. Let $h < h_1$. Then

$$(3.6) \qquad\qquad \|e(t)\|_{p,h} < \tfrac{1}{4}ch^{1+2/p}$$

for $0 \le t < T^*$. Since $\|e(t)\|_{p,h}$ is a continuous function, $T^* \in E$ and (3.1) will therefore hold in a larger interval. This contradicts the definition of T^*. Hence $T = T^*$ and this completes the proof.

Department of Mathematics
University of California
Berkeley, California 94720
E-mail: mauceri@igecuniv(bitnet)

1. C. R. ANDERSON, *Vortex Methods for Flows with Variable Density*, Ph.D. Thesis, Univ. of California, Berkeley, 1983.

2. C. ANDERSON & C. GREENGARD, "On vortex methods", *SIAM J. Numer. Anal.*, v. 22, 1985, pp. 413–440.

3. J. T. BEALE, "A convergent 3-D vortex method with grid-free stretching," *Math. Comp.*, v. 46, 1986, pp. 401–424.

4. J. T. BEALE & A. MAJDA, "Vortex methods. I: Convergence in three dimensions," *Math. Comp.*, v. 39, 1982, pp. 1–27.

5. J. T. BEALE & A. MAJDA, "Vortex methods. II: Higher order accuracy in two and three dimensions," *Math. Comp.*, v. 39, 1982, pp. 29–52.

6. J. T. BEALE & A. MAJDA, "Higher order accurate vortex methods with explicit velocity kernels", *J. Comput. Phys.*, v. 58, 1985, pp. 188–208.

7. C. CHIU & R. A. NICOLAIDES, "Convergence of a higher-order vortex method for two-dimensional Euler equations," *Math. Comp.*, v. 51, 1988, pp. 507–534.

8. A. J. CHORIN, "Numerical study of slightly viscous flow," *J. Fluid Mech.*, v. 57, 1973, pp. 785–796.

9. A. J. CHORIN, "Vortex models and boundary layer instability," *SIAM J. Sci. Statist. Comput.*, v. 1, 1980, pp. 1–21.

10. G.-H. COTTET, Thèse de 3ème cycle, l'Université Pierre et Marie Curie, Paris 6, 1982.

11. G.-H. COTTET & P.-A. RAVIART, "Particle methods for one-dimensional Vlasov-Poisson equations," *SIAM J. Numer. Anal.*, v. 21, 1984, pp. 52–76.

12. C. GREENGARD, *Three Dimensional Vortex Methods*, Ph.D. Thesis, Univ. California, Berkeley, 1984.

13. O. HALD & V. M. DEL PRETE, "Convergence of vortex methods for Euler's equations," *Math. Comp.*, v. 32, 1978, pp. 791–809.

14. O. H. HALD, "Convergence of vortex methods for Euler's equations, II," *SIAM J. Numer. Anal.*, v. 16, 1979, pp. 726–755.

15. O. H. HALD, "Convergence of vortex methods for Euler's equations, III," *SIAM J. Numer. Anal.*, v. 24, 1987, pp. 538–582.

16. A. LEONARD, "Vortex methods for flow simulation," *J. Comput. Phys.*, v. 37, 1980, pp. 289–335.

17. F. MILINAZZO & P. G. SAFFMAN, "The calculation of large Reynolds number two-dimensional flow using discrete vortices with random walk," *J. Comput. Phys.*, v. 23, 1977, pp. 380–392.

18. Y. NAKAMURA, A. LEONARD & P. SPALART, *Vortex Simulation of an Inviscid Shear Layer*, AIAA/ASME 3rd joint Thermophysics, Fluids, Plasma and Heat Transfer Conference, AIAA-82-0948, 1982.

19. M. B. PERLMAN, "On the accuracy of vortex methods," *J. Comput. Phys.*, v. 59, 1985, pp. 200–223.

20. P. A. RAVIART, *An Analysis of Particle Methods*, Lecture Notes in Math., vol. 1127, Springer-Verlag, Berlin and New York, 1985.