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On the Convergence of a Time Discretization Scheme for the Navier-Stokes Equations*

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Abstract. A linearized version of the implicit Euler scheme is considered for the approximation of the solutions to the Navier-Stokes equations in a two-dimensional domain. The rate of convergence in the H^1-norm is established.

1. Introduction. We are concerned with the discretization in time of the Navier-Stokes equations in a bounded two-dimensional domain:

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t}(t,x) - \Delta u(t,x) + \nabla p(t,x) + (u\cdot\nabla)u(t,x) = 0, \qquad x\in\Omega,\ t>0,\\
&\operatorname{div} u(t,x) = 0, \qquad x\in\Omega,\ t>0,\\
&u(t,x) = 0, \qquad x\in\partial\Omega,\ t>0,\\
&u(0,x) = u_0(x), \qquad x\in\Omega.
\end{aligned}
\tag{1.1}
$$

Here, $u(t,x) = (u_1(t,x), u_2(t,x))$ is the velocity, $p(t,x)$ is the pressure, Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, and u_0 is the initial velocity field.

As in Fujita and Kato [5], [15] and Temam [24], we cast (1.1) as an evolution equation in the appropriate Hilbert space:

$\mathscr{V} = \{v = (v_1, v_2)\colon v_1, v_2 \in C_0^\infty(\Omega),\ \operatorname{div} v = 0\}$;

H = closure of $\mathscr{V}$ in $L^{2,2}(\Omega)$, the space of $\mathbf{R}^2$-valued functions, each component of which is in $L^2(\Omega)$, equipped with the inner product

$$(u,v) = \int_\Omega \sum_{i=1}^2 u_i(x)v_i(x)\,dx$$

and the induced norm $\|u\| = (u,u)^{1/2}$;

V = closure of $\mathscr{V}$ in $H_0^{1/2}(\Omega)$, the Sobolev space of $\mathbf{R}^2$-valued functions, each component of which is in $H_0^1(\Omega)$, equipped with the inner product

$$(u,v)_1 = \int_\Omega \sum_{i,j=1}^2 \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx$$

and the induced norm $\|u\|_1 = (u,u)_1^{1/2}$.

Similarly, the spaces $H^{s,2}(\Omega)$ and the norms $\|\cdot\|_s$ are defined in terms of the standard Sobolev spaces.

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We let $P\colon L^{2,2}(\Omega) \to H$ denote the orthogonal projection and define the Stokes operator $A\colon D(A) \subset H \to H$, $D(A) = V \cap H^{2,2}(\Omega)$ by $Au = -P\Delta u$, $u \in D(A)$. We note that $\|u\|_s$ and $\|A^{s/2}u\|$ are equivalent, $u \in D(A^{s/2})$, $0 \le s \le 2$ [7].

Within this framework, (1.1) is expressed as the evolution equation in H: $u(t) \in D(A)$, $t \ge 0$, and

$$\frac{du}{dt}(t) + Au(t) + B(u(t), u(t)) = 0, \qquad t > 0, \quad u(0) = u_0, \tag{1.2}$$

where $B(u, v) = P(u \cdot \nabla)v$.

The application of a linearized version of the implicit Euler scheme to (1.2) determines the sequence $u_{k,n} \in D(A)$, $n = 0, 1, 2, \ldots$, such that

$$\overline{\partial}_t u_{k,n} + Au_{k,n} + B(u_{k,n-1}, u_{k,n}) = 0, \qquad n = 1, 2, \ldots, \qquad u_{k,0} = u_0, \tag{1.3}$$

where $k > 0$ is the time step and

$$\overline{\partial}_t u_{k,n} = \frac{u_{k,n} - u_{k,n-1}}{k}.$$

We will establish the following result:

THEOREM. *If $u_0 \in D(A)$ and $t = nk$, then*

$$\|u_{k,n} - u(t)\|_1 \le \frac{Ce^{-\delta t}}{t^{1/2}} k \tag{1.4}$$

for $k < k_0$, where C, δ and k_0 are positive constants depending on the data u_0 and Ω only.

Here and in the sequel, C, δ and k_0 will denote possibly different constants which depend only on the data. This convention renders the proofs of results such as the above theorems more readable. In any case, the interested reader should have no difficulty in tracing the dependence of the various constants on the data.

The above result parallels results pertaining to the approximation by the implicit Euler scheme of the analytic semigroup generated by the positive definite selfadjoint operator A, as discussed, for example, by Fujita and Mizutani [6] and Thomée [25].

The convergence in the L^2-norm of the scheme described by (1.3) has been discussed by Girault and Raviart [8]. They have established the L^2-convergence of the scheme in terms of the smoothness properties of the solution. Under the same conditions as in the theorem, and using the same techniques, we are able to show that

$$\|u_{k,n} - u(t)\| \le Ce^{-\delta t} k, \qquad t = nk, \tag{1.5}$$

for $k < k_0$, where C, δ and k_0 are positive constants depending on the data u_0 and Ω only. We do not include the proof since it is straightforward, once (1.4) is established, and (1.5) is not sufficiently novel.

Earlier, Temam [22] derived a priori bounds for the scheme (1.3) and concluded qualitative convergence without a convergence rate in weak norms. Rannacher [19] gave $O(\Delta t)$-error estimates for the explicit Euler scheme. Recently, Heywood and Rannacher [12] obtained local and global $O(\Delta t^2)$-error estimates for the Crank-Nicolson scheme under realistic assumptions concerning the smoothness of the solution. It is relatively easier to establish the rate of convergence of higher-order schemes by assuming the solution to be sufficiently regular. However, as has been emphasized by Heywood and Rannacher in a series of papers [9], [10], [11], [12], and discussed also by Rautmann [21] and Temam [23], [24], such regularity assumptions may entail global compatibility conditions which are not met or which are not verifiable, in general. Higher-order results as in [2], [8] and [16] involve such conditions, and future work should attempt to clarify whether the anticipated orders of such schemes are realized under realistic assumptions on the data.

Our approach is based on the Fujita-Kato approach to the Navier-Stokes equations [5], [15], and has been inspired by Okamoto's papers [17], [18] on the spatial discretization of (1.2). We have not considered fully discrete schemes since the technicalities, which are considerable, vary depending on the spatial discretization schemes that are utilized, and may obscure the essential goal of the paper, i.e., the demonstration of the convergence of the linearized implicit Euler scheme (1.3) at the predicted rate for $u_0 \in D(A)$. By the same token, we have not included nonhomogeneous boundary data or a forcing term in (1.1). Under appropriate technical assumptions, the basic result (1.4) may be extended to the nonhomogeneous cases. If $\Omega \subset \mathbf{R}^3$, the counterpart of our theorem may be established over a time interval $(0,T]$ in which a bound on $\|u(t)\|_1$ and $\|u_{n,k}\|_1$ $(t = nk)$ may be assumed. In the 2-dimensional case, the required a priori estimates are available for $\|u(t)\|_1$ and will be established in the next section for $\|u_{n,k}\|_1$.

2. Some a priori Estimates.

LEMMA 1. *If $\{u_{k,n}\}_{n=0}^{\infty}$ is the solution of the linearized implicit Euler scheme (1.3), the following a priori estimates are valid:*

$$\|u_{k,n}\|^2 + 2\sum_{j=1}^{n} \|A^{1/2}u_{k,j}\|^2 k \le \|u_0\|^2, \qquad n = 1,2,\dots, \tag{2.1}$$

$$\|A^{1/2}u_{k,n}\| \le C(\|A^{1/2}u_0\|, \Omega)e^{-\delta t}, \qquad 0 < k < k_0, \tag{2.2}$$

where $t = nk, C, \delta$ and k_0 are positive constants which depend on the data u_0 and Ω.

Proof. We form the inner product of (1.3) with $u_{k,n}$ and obtain

$$(\overline{\partial}_t u_{k,n}, u_{k,n}) + (Au_{k,n}, u_{k,n}) + (B(u_{k,n-1}, u_{k,n}), u_{k,n}) = 0. \tag{2.3}$$

Since

$$(B(u_{k,n-1}, u_{k,n}), u_{k,n}) = b(u_{k,n-1}, u_{k,n}, u_{k,n}) = 0, \tag{2.4}$$

as in Temam [22, p. 163], and

$$(\overline{\partial}_t u_{k,n}, u_{k,n}) = \tfrac{1}{2}\overline{\partial}_t\|u_{k,n}\|^2 + \tfrac{k}{2}\|\overline{\partial}_t u_{k,n}\|^2, \tag{2.5}$$

as in Thomée [25, p. 157], (2.3) yields

$$\tfrac{1}{2}\overline{\partial}_t\|u_{k,n}\|^2 + \|A^{1/2}u_{k,n}\|^2 \le 0, \tag{2.6}$$

so that

$$\|u_{k,n}\|^2 + 2\sum_{j=1}^{n} \|A^{1/2}u_{k,j}\|^2 k \le \|u_0\|^2,$$

i.e., (2.1) is established.

In order to establish the a priori bound (2.2) on $\|A^{1/2}u_{k,n}\|$, we form the inner product of (1.3) with $Au_{k,n}$ and obtain

$$(\overline{\partial}_t u_{k,n}, Au_{k,n}) + \|Au_{k,n}\|^2 + (B(u_{k,n-1}, u_{k,n}), Au_{k,n}) = 0,$$

so that

$$\begin{aligned}\tfrac{1}{2}\overline{\partial}_t\|A^{1/2}u_{k,n}\|^2 + \|Au_{k,n}\|^2 &\le \|B(u_{k,n-1}, u_{k,n})\|\,\|Au_{k,n}\| \\ &\le C\|u_{k,n-1}\|^{1/2}\|A^{1/2}u_{k,n-1}\|^{1/2}\|A^{1/2}u_{k,n}\|^{1/2}\|Au_{k,n}\|^{3/2},\end{aligned} \tag{2.7}$$

where we have used the inequality

$$\|v\|_{L^4} \le C\|v\|^{1/2}\|v\|_1^{1/2}, \qquad v \in H^1(\Omega),\ \Omega \subset \mathbf{R}^2$$

[22, p. 291].

Making use of Young's inequality, (2.7) leads to

$$\begin{aligned}\|A^{1/2}u_{k,n}\|^2 + k\|Au_{k,n}\|^2 \le{}& \|A^{1/2}u_{k,n-1}\|^2 \\ &+ Ck\|u_{k,n-1}\|^2\|A^{1/2}u_{k,n-1}\|^2\|A^{1/2}u_{k,n}\|^2,\end{aligned}$$

and summing over n, using the inequality $\|A^{1/2}u_{k,n}\| \le C\|Au_{k,n}\|$, we obtain

$$\begin{aligned}\|A^{1/2}u_{k,n}\|^2 \le{}& \|A^{1/2}u_0\|^2 \\ &+ Ck\sum_{m=1}^{n} [\|u_{k,m-1}\|^2\|A^{1/2}u_{k,m-1}\|^2 - 1]\|A^{1/2}u_{k,m}\|^2.\end{aligned} \tag{2.8}$$

The inequalities (2.1), (2.8) and the discrete Gronwall lemma, as for example in [13], lead to the a priori estimate (2.2). □

In addition to the estimates on the solution of (1.2) that we will be able to refer to, we will need the following estimate:

LEMMA 2. *If $u_0 \in D(A)$, then*

$$\|A^{1/2}D_t u(t)\| \le \frac{C(\|Au_0\|, \Omega)}{t^{1/2}} e^{-\delta t}, \qquad t > 0.$$

We omit the proof since it is readily obtainable using the techniques of [5] and [18].

3. The Error Estimate. In this section we will prove the theorem in Section 1. We will freely use the results of Fujita and Kato [5], [15], Fujita and Morimoto [7], Temam [22], [24], and Foias and Temam [4] with regard to the fractional powers of the Stokes operator A and the properties of the trilinear form $b(u, v, w) = (B(u, v), w)$. The a priori estimates that have been established by Okamoto [18] for the solution of (1.2) are essential as well. The reader will notice the parallels between our treatment of time discretization and Okamoto's treatment of spatial discretization.

We restate the theorem:

THEOREM. *If $u_0 \in D(A)$ and $t = nk$, $n = 1, 2, \ldots$, then*

$$\|u_{k,n} - u(t)\|_1 \leq \frac{C(\|Au_0\|, \Omega)e^{-\delta t}}{t^{1/2}} k. \tag{3.1}$$

Proof. We have

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} B(u(s), u(s))\, ds, \qquad t = nk, \tag{3.2}$$

$$u_{k,n} = E_k^n u_0 - \sum_{j=1}^{n} E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k, \tag{3.3}$$

where $E_k = (I + kA)^{-1}$, I denoting the identity.

As in Thomée [25],

$$\|A^{1/2}(E_k^n - e^{-tA})u_0\| \leq \frac{C(\|Au_0\|, \Omega)e^{-\delta t}}{t^{1/2}} k, \tag{3.4}$$

where, as always, $\delta > 0$ also depends on the data. Thus,

$$\begin{aligned} \|A^{1/2}(u(t) - u_{k,n})\| \leq \frac{Ce^{-\delta t}}{t^{1/2}} k + \bigg\| A^{1/2} \int_0^t e^{-(t-s)A} B(u(s), u(s))\, ds \\ - A^{1/2} \sum_{j=1}^{n} E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k \bigg\|. \end{aligned} \tag{3.5}$$

We write

$$\begin{aligned} &\int_0^t e^{-(t-s)A} B(u(s), u(s))\, ds - \sum_{j=1}^{n} E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k \\ &\quad = \left[\int_0^t e^{-(t-s)A} B(u(s), u(s))\, ds - \sum_{j=1}^{n} E_k^{n-j+1} B(u_{j-1}, u_j)k \right] \\ &\quad\quad + \left[\sum_{j=1}^{n} E_k^{n-j+1} (B(u_{j-1}, u_j) - B(u_{k,j-1}, u_{k,j}))k \right], \end{aligned} \tag{3.6}$$

where u_j denotes $u(jk)$.

We write the last expression of (3.6) as

$$\begin{aligned} &\sum_{j=1}^{n} E_k^{n-j+1} (B(u_{j-1}, u_j) - B(u_{k,j-1}, u_{k,j}))k \\ &\quad = \sum_{j=1}^{n} E_k^{n-j+1} B(u_{j-1} - u_{k,j-1}, u_j)k + \sum_{j=1}^{n} E_k^{n-j+1} B(u_{k,j-1}, u_j - u_{k,j})k \\ &\quad = \sum_{j=1}^{n-1} E_k^{n-j} B(u_j - u_{k,j}, u_{j+1})k + \sum_{j=1}^{n-1} E_k^{n-j+1} B(u_{k,j-1}, u_j - u_{k,j})k \\ &\quad\quad + E_k B(u_{k,n-1}, u_n - u_{k,n})k, \end{aligned} \tag{3.7}$$

since $u_{k,0} = u_0$. We have

$$\begin{aligned}
&\|A^{1/2}E_kB(u_{k,n-1}, u_n - u_{k,n})\|k \\
&\quad = \|A^{3/4}E_kA^{-1/4}B(u_{k,n-1}, u_n - u_{k,n})\|k \\
&\quad \le \frac{C}{k^{3/4}}\|A^{1/4}u_{k,n-1}\| \cdot \|A^{1/2}(u_n - u_{k,n})\|k \le Ck^{1/4}\|A^{1/2}(u_n - u_{k,n})\|
\end{aligned} \tag{3.8}$$

thanks to the estimate (2.2) on $\|A^{1/2}u_{k,n}\|$.

We consider next

$$\begin{aligned}
&\sum_{j=1}^{n-1} \|A^{1/2}E_k^{n-j}B(u_j - u_{k,j}, u_{j+1})\|k \\
&\quad = \sum_{j=1}^{n-1} \|A^{3/4}E_k^{n-j}A^{-1/4}B(u_j - u_{k,j}, u_{j+1})\|k \\
&\quad \le Ce^{-\delta t}\sum_{j=1}^{n-1} \frac{e^{jk\delta}}{(nk - jk)^{3/4}}\|A^{1/4}(u_j - u_{k,j})\|\,\|A^{1/2}u_{j+1}\|k \\
&\quad \le Ce^{-\delta t}\sum_{j=1}^{n-1} \frac{e^{jk\delta}}{(nk - jk)^{3/4}}\|A^{1/2}(u_j - u_{k,j})\|k,
\end{aligned} \tag{3.9}$$

by virtue of the estimate (2.2) on $\|A^{1/2}u_n\|$.

Similarly,

$$\begin{aligned}
&\sum_{j=1}^{n-1} \|A^{1/2}E_k^{n-j+1}B(u_{k,j-1}, u_j - u_{k,j})\|k \\
&\quad \le Ce^{-\delta t}\sum_{j=1}^{n-1} \frac{e^{jk\delta}}{(nk - jk)^{3/4}}\|A^{1/2}(u_j - u_{k,j})\|k.
\end{aligned} \tag{3.10}$$

From (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) we obtain, for sufficiently small k,

$$\begin{aligned}
e^{\delta t}\|A^{1/2}(u_n - u_{k,n})\| \le \frac{Ck}{t^{1/2}} &+ C\sum_{j=1}^{n-1} \frac{e^{\delta jk}}{(t - jk)^{3/4}}\|A^{1/2}(u_j - u_{k,j})\|k \\
&+ e^{\delta t}\left\| A^{1/2}\int_0^t e^{-(t-s)A}B(u(s), u(s))\,ds \right. \\
&\left. \qquad - A^{1/2}\sum_{j=1}^{n} E_k^{n-j+1}B(u_{j-1}, u_j)k \right\|.
\end{aligned} \tag{3.11}$$

Thanks to a generalization of Gronwall's lemma as in Okamoto [18] and Amann [1], the theorem will be established once we show that

$$\begin{aligned}
&\left\| A^{1/2}\int_0^t e^{-(t-s)A}B(u(s), u(s))\,ds - A^{1/2}\sum_{j=1}^{n} E_k^{n-j+1}B(u_{j-1}, u_j)k \right\| \\
&\qquad \le \frac{Ce^{-\delta t}}{t^{1/2}}k.
\end{aligned} \tag{3.12}$$

We write

$$
\begin{aligned}
&\int_0^t e^{-(t-s)A} B(u(s), u(s))\, ds - \sum_{j=1}^{n} E_k^{n-j+1} B(u_{j-1}, u_j) k \\
&\quad = \int_0^t e^{-(t-s)A} (B(u(s), u(s)) - B(u(t), u(t)))\, ds \\
&\quad - \sum_{j=1}^{n} E_k^{n-j+1} (B(u_{j-1}, u_j) - B(u(t), u(t))) k \\
&\quad + \left[\int_0^t e^{-(t-s)A} - \sum_{j=1}^{n} E_k^{n-j+1} k \right] B(u(t), u(t)).
\end{aligned} \tag{3.13}
$$

We will deal with the last line of (3.13) first. As in Kato [14, p. 489],

$$
\int_0^t e^{-(t-s)A}\, ds = (I - e^{-tA}) A^{-1}. \tag{3.14}
$$

It is also easily verified that

$$
\sum_{j=1}^{n} E_k^{n-j+1} k = (I - E_k^n) A^{-1}. \tag{3.15}
$$

By (3.14) and (3.15),

$$
\begin{aligned}
&\left\| A^{1/2} \left[\int_0^t e^{-(t-s)A} - \sum_{j=1}^{n} E_k^{n-j+1} k \right] B(u(t), u(t)) \right\| \\
&\quad = \| A^{1/2} (E_k^n - e^{-tA}) A^{-1} B(u(t), u(t)) \| \le \frac{Ce^{-\delta t}}{t^{1/2}} \| B(u(t), u(t)) \| k \\
&\quad \le \frac{Ce^{-\delta t}}{t^{1/2}} \| Au(t) \| \, \| A^{1/2} u(t) \| k \le \frac{C(\|Au_0\|, \Omega)}{t^{1/2}} e^{-\delta t},
\end{aligned} \tag{3.16}
$$

thanks to the error estimates on the approximation of $\exp(-tA)$ as in Thomée [25] and the a priori estimates established by Okamoto [18].

By (3.13) and (3.16), the inequality (3.12) will be established once we estimate

$$
\begin{aligned}
&\left\| A^{1/2} \int_0^t e^{-(t-s)A} (B(u(s), u(s)) - B(u(t), u(t)))\, ds \right. \\
&\qquad \left. - \sum_{j=1}^{n} E_k^{n-j+1} (B(u_{j-1}, u_j) - B(u(t), u(t))) k \right\|.
\end{aligned}
$$

To this end, we write

$$
\begin{aligned}
&\int_0^t e^{-(t-s)A}(B(u(s),u(s)) - B(u(t),u(t)))\,ds \\
&\qquad - \sum_{j=1}^{n} E_k^{n-j+1}(B(u_{j-1},u_j) - B(u(t),u(t)))k \\
&\quad = \sum_{j=1}^{n} \int_{(j-1)k}^{jk} [e^{-(t-s)A} - e^{-(t-(j-1)k)A}](B(u(s),u(s)) - B(u(t),u(t)))\,ds \\
&\qquad + \sum_{j=1}^{n} \int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}(B(u(s),u(s)) - B(u_{j-1},u_j))\,ds \\
&\qquad + \sum_{j=1}^{n} [e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}](B(u_{j-1},u_j) - B(u(t),u(t)))k \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}
\tag{3.17}
$$

We will establish that

$$
\|A^{1/2}I_l\| \le \frac{Ce^{-\delta t}}{t^{1/2}}k, \qquad l = 1,2,3, \tag{3.18}
$$

and this will conclude the proof.

In order to estimate $\|A^{1/2}I_1\|$ we first note that

$$
\begin{aligned}
e^{-(t-s)A} - e^{-(t-(j-1)k)A} &= e^{-(t-s)A} - e^{-(t-s)A}e^{-(s-(j-1)k)A} \\
&= e^{-(t-s)A}[I - e^{-(s-(j-1)k)A}]
\end{aligned}
$$

for $s \in [(j-1)k, jk)$. Therefore,

$$
\begin{aligned}
&\|A^{1/2}[e^{-(t-s)A} - e^{-(t-(j-1)k)A}]G\| \\
&\quad = \|A^{3/2}e^{-(t-s)A}A^{-1}(I - e^{-(s-(j-1)k)A})G\| \\
&\quad \le \frac{Ce^{-\delta(t-s)}}{(t-s)^{3/2}}\|G\|k, \qquad s \in [(j-1)k, jk),
\end{aligned}
\tag{3.19}
$$

as in Crouzeix and Thomée [3, proof of Theorem 1].

Making use of Lemma 2 and [18],

$$
\begin{aligned}
\|B(u(s),u(s)) - B(u(t),u(t))\| &\le C\|Au(t)\|\,\|A^{1/2}(u(s) - u(t))\| \\
&\le C(\|Au_0\|,\Omega)e^{-\delta s}\frac{(t-s)}{s^{1/2}}.
\end{aligned}
\tag{3.20}
$$

Equations (3.17), (3.19) and (3.20) yield

$$
\begin{aligned}
\|A^{1/2}I_1\| &\le Ce^{-\delta t}k\sum_{j=1}^{n}\int_{(j-1)k}^{jk}\frac{1}{(t-s)^{1/2}s^{1/2}}\,ds \\
&= Ce^{-\delta t}\int_0^t \frac{1}{(t-s)^{1/2}s^{1/2}}\,ds \le Ce^{-\delta t}k.
\end{aligned}
\tag{3.21}
$$

In order to estimate $\|A^{1/2}I_2\|$, we write

$$
\begin{aligned}
I_2 = &\int_0^k e^{-tA}B(u(s),u(s))\,ds - e^{-tA}B(u_0,u_1)k \\
&+\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u(s)-u_{j-1},u(s))\,ds \\
&+\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u_{j-1},u(s)-u_j)\,ds.
\end{aligned}
\tag{3.22}
$$

We first observe

$$
\|A^{1/2}e^{-tA}B(u(s),u(s))\| \le \frac{Ce^{-\delta t}}{t^{1/2}}\|A^{1/2}u(s)\|\,\|Au(s)\| \le \frac{C(\|Au_0\|,\Omega)e^{-\delta t}}{t^{1/2}},
$$

so that

$$
\left\|A^{1/2}\int_0^k e^{-tA}B(u(s),u(s))\,ds\right\| \le \frac{Ce^{-\delta t}}{t^{1/2}}k. \tag{3.23}
$$

Similarly,

$$
\|A^{1/2}B(u_0,u_1)\|k \le \frac{Ce^{-\delta t}}{t^{1/2}}k. \tag{3.24}
$$

By (3.22), (3.23) and (3.24) the result (3.18) for $l=1$ will have been established if such an estimate is proven for the remaining terms of (3.22). It will suffice treating

$$
\left\|A^{1/2}\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u(s)-u_{j-1},u(s))\,ds\right\|,
$$

since the last term is treated in a similar manner.

For $j=2,3,\dots,n$, $s\in[(j-1)k,jk)$,

$$
\begin{aligned}
&\|A^{1/2}e^{-(t-(j-1)k)}B(u(s)-u_{j-1},u(s))\| \\
&\le \frac{Ce^{-\delta t}e^{\delta(j-1)k}}{(t-(j-1)k)^{1/2}}\|Au(s)\|\,\|A^{1/2}(u(s)-u_{j-1})\| \\
&\le \frac{Ce^{-\delta t}k}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}},
\end{aligned}
\tag{3.25}
$$

again by [18] and Lemma 2.

By (3.25) we have

$$
\begin{aligned}
&\left\|A^{1/2}\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u(s)-u_{j-1},u(s))\,ds\right\| \\
&\le Ce^{-\delta t}k\left(\sum_{j=2}^{n}\frac{k}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}}\right) \\
&\le Ce^{-\delta t}k\int_0^t \frac{1}{(t-s)^{1/2}s^{1/2}}\,ds \le Ce^{-\delta t}k.
\end{aligned}
\tag{3.26}
$$

The last line of (3.22) is treated similarly, and we are left with the task of estimating $\|A^{1/2}I_3\|$. We write

$$\begin{aligned} I_3 &= [e^{-tA} - E_k^n](B(u_0,u_1) - B(u(t),u(t)))k \\ &+ \sum_{j=2}^{n}[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}]B(u_{j-1}-u(t),u_j)k \\ &+ \sum_{j=2}^{n}[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}]B(u(t),u_j-u(t))k. \end{aligned} \tag{3.27}$$

Each term of the first line of (3.27) is treated in a similar manner. For example,

$$\|A^{1/2}E_kB(u_0,u_1)\|k \le \frac{Ce^{-\delta t}}{t^{1/2}}\|B(u_0,u_1)\|k \le \frac{C(\|Au_0\|,\Omega)e^{-\delta t}}{t^{1/2}}k.$$

Now,

$$\begin{aligned} &\sum_{j=2}^{n}\|A^{1/2}[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}]B(u_{j-1}-u(t),u_j)\|k \\ &\le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\|B(u_{j-1}-u(t),u_j)\|k \\ &\le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\|A^{1/2}(u_{j-1}-u(t))\|\,\|Au_j\|k \\ &\le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\cdot\frac{(t-(j-1)k)}{((j-1)k)^{1/2}}e^{-\delta jk}k \\ &= Ce^{-\delta t}k\left(\sum_{j=2}^{n}\frac{1}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}}k\right) \le Ce^{-\delta t}k, \end{aligned} \tag{3.28}$$

again by [18], [25] and Lemma 2.

The last line of (3.27) is handled similarly and (3.18) is established for I_3 as well. As anticipated earlier, we thus conclude the proof of the theorem since $\|A^{1/2}(u(t)-u_{n,k})\|$ is equivalent to $\|u(t)-u_{n,k}\|_1$, as in [7]. □

4. Concluding Remarks. Our results are incomplete, just as those of [18], in that the rate of convergence has not been established for u_0, which is merely assumed to be in H, even though the solution exists for any $u_0 \in H$ [13] since $\Omega \subset \mathbf{R}^2$. It is of interest to consider this more general situation. From the practical standpoint it is perhaps of greater interest to establish higher-order convergence, at least for $u_0 \in D(A)$, as we mentioned at the beginning.

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