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Noninterpolatory Integration Rules for Cauchy Principal Value Integrals

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Dedicated to the memory of Peter Henrici

Abstract. Let $w(x)$ be an admissible weight on $[-1,1]$ and let $\{p_n(x)\}_0^\infty$ be its associated sequence of orthonormal polynomials. We study the convergence of noninterpolatory integration rules for approximating Cauchy principal value integrals

$$I(f;\lambda) := \fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}\,dx, \qquad \lambda \in (-1,1).$$

This requires investigation of the convergence of the expansion

$$I(f;\lambda) \sim \sum_{k=0}^{\infty}(f,p_k)q_k(\lambda), \qquad \lambda \in (-1,1),$$

in terms of the functions of the second kind $\{q_k(\lambda)\}_0^\infty$ associated with w, where

$$(f,p_k) := \int_{-1}^{1} w(x)f(x)p_k(x)\,dx \quad \text{and} \quad q_k(\lambda) := \fint_{-1}^{1} w(x)\frac{p_k(x)}{x-\lambda}\,dx,$$

$k = 0,1,2,\ldots, \lambda \in (-1,1)$.

1. Introduction. In the third volume of his monumental work, *Applied and Computational Complex Analysis*, Henrici [8, pp. 139–142] gave an algorithm for the numerical evaluation of *Cauchy principal value* (CPV) integrals. This algorithm was presented in a more explicit form in a recent paper, by one of the authors [15]. In neither case were convergence questions considered. In this paper, we shall analyze the convergence questions arising from the use of this algorithm.

Consider the CPV integral of the form

$$I(f;\lambda) := \fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}\,dx, \qquad -1 < \lambda < 1, \tag{1}$$

where w is an admissible weight function, $w \in \mathscr{A}$, that is, $w(x)$ is nonnegative and integrable in $[-1,1]$ and

$$m_0 := \int_{-1}^{1} w(x)\,dx > 0. \tag{2}$$

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For such w, there exist sequences of *orthonormal polynomials*

$$\{p_n(x) := p_n(w,x) := k_n x^n + \cdots, \ k_n > 0\}, \tag{3}$$

with respect to the inner product

$$(f,g) := \int_{-1}^{1} w(x)f(x)g(x)\,dx, \tag{4}$$

satisfying a three-term recurrence relation

$$xp_n(x) = \alpha_{n+1}p_{n+1}(x) + \beta_{n+1}p_n(x) + \alpha_n p_{n-1}(x), \qquad n = 0,1,2,\ldots, \tag{5}$$

where

$$\alpha_n := k_{n-1}/k_n, \quad n \geq 1; \qquad \beta_{n+1} := (xp_n, p_n), \quad n \geq 0,$$
$$p_{-1}(x) \equiv 0 \quad \text{and} \quad p_0(x) \equiv k_0 = m_0^{-1/2}.$$

If we define $q_n(\lambda)$, the *function of the second kind*, by

$$q_n(\lambda) := q_n(w,\lambda) := I(p_n;\lambda) := ⨍_{-1}^{1} w(x)\frac{p_n(x)}{x-\lambda}\,dx, \qquad -1 < \lambda < 1, \tag{6}$$

then the $q_n(\lambda)$ satisfy the same recurrence relation as the $\{p_n(x)\}$, namely

$$\lambda q_n(\lambda) = \alpha_{n+1}q_{n+1}(\lambda) + \beta_{n+1}q_n(\lambda) + \alpha_n q_{n-1}(\lambda), \qquad n = 0,1,2,\ldots, \tag{7}$$

with starting values $q_{-1}(\lambda) \equiv -1$, $q_0(\lambda) \equiv I(p_0;\lambda)$ and $\alpha_0 := m_0^{1/2}$. If we denote by

$$a_k := (f, p_k) \tag{8}$$

the Fourier coefficient of $p_k(x)$ in the formal expansion of $f(x)$,

$$f(x) \sim \sum_{k=0}^{\infty} a_k p_k(x), \tag{9}$$

then we can write a formal expansion for $I(f;\lambda)$ in terms of the $q_n(\lambda)$,

$$I(f;\lambda) \sim \sum_{k=0}^{\infty} a_k q_k(\lambda). \tag{10}$$

Hence, an approximation to $I(f;\lambda)$ will be given by the truncated sum

$$S_N(f;\lambda) := \sum_{k=0}^{N} a_k q_k(\lambda). \tag{11}$$

If we now have a sequence of integration rules

$$Q_m(g) := \sum_{i=1}^{m} w_{im} g(x_{im}), \tag{12}$$

which converges to

$$I(g) := \int_{-1}^{1} w(x)g(x)\,dx \tag{13}$$

for all $g \in C[-1,1]$ or all $g \in R[-1,1]$, the space of bounded Riemann integrable functions on $[-1,1]$, and if we approximate the Fourier coefficients a_k by

$$a_{km} := Q_m(fp_k), \tag{14}$$

then, in general, we obtain a noninterpolatory integration rule for $I(f;\lambda)$, namely

$$Q_m^N(f;\lambda) := \sum_{k=0}^{N} a_{km} q_k(\lambda). \tag{15}$$

The approximations $Q_m^N(f;\lambda)$ can be evaluated in a stable manner using backward recursion by the algorithm given in [15], provided that we have the value of $q_0(\lambda)$. We can also express $Q_m^N(f;\lambda)$ in a Lagrangian form that is more useful in the numerical solution of integral equations:

$$Q_m^N(f;\lambda) = \sum_{i=1}^{m} w_{im}^N(\lambda) f(x_{im}), \tag{16}$$

where the weights

$$w_{im}^N(\lambda) := w_{im} \sum_{k=0}^{N} p_k(x_{im}) q_k(\lambda), \qquad i = 1, 2, \ldots, m, \tag{17}$$

can also be evaluated in a stable manner by the backward recursion algorithm [15].

As indicated above, this general approach to the numerical evaluation of CPV integrals appears in Henrici [8, pp. 139–142]. However, there is no discussion there of convergence or of the integration rules $Q_m(g)$. In fact, it is precisely the freedom in the choice of these rules, subject only to the condition that they converge to $I(g)$ for all $g \in C[-1,1]$ or all $g \in R[-1,1]$, that affords this method for evaluating CPV integrals considerable interest. Thus, if f is well behaved in most of the interval $[-1,1]$, but is irregular over a small subinterval $[a,b] \subset [-1,1]$, then we can concentrate most of our integration points x_{im} in $[a,b]$.

This was also done by Gerasoulis [7] using a different approach, and the results he achieved were a considerable improvement over those achieved using a conventional spacing of integration points. There have been many approaches to noninterpolatory integration of CPV integrals [4], [14], [17], but these two are the only ones that cater to the situation indicated above.

In Section 2, we state and prove Theorems 1 to 5, which deal with convergence of $S_N(f;\lambda)$ to $I(f;\lambda)$. In Section 3, we state and prove Theorems 6 to 8, which deal with the convergence of $Q_m^N(f;\lambda)$ to $I(f;\lambda)$ as m and $N \to \infty$. It turns out that in the general case we shall be able to prove convergence only for the iterated limit

$$\lim_{N\to\infty} \lim_{m\to\infty} Q_m^N(f;\lambda). \tag{18}$$

In fact, we shall show that we cannot in general expect convergence of the double limit. However, in certain cases where we can convert the double limit to a single limit in which m depends on N in some specific manner, we shall again be able to prove convergence. A similar approach was used by Dagnino [3] in studying the convergence of noninterpolatory product integration rules.

2. Convergence Results for $S_N(f;\lambda)$. Before we can study the convergence of $Q_m^N(f;\lambda)$ to $I(f;\lambda)$, we must establish the convergence of $S_N(f;\lambda)$ to $I(f;\lambda)$. To this end, we shall use the methods presented in Natanson [11] and Freud [5] for proving convergence of orthonormal expansions. Since the proofs in [11] depend on the Christoffel-Darboux formula

$$\sum_{k=0}^{N} p_k(x)p_k(y) = \alpha_{N+1}\frac{p_{N+1}(x)p_N(y) - p_N(x)p_{N+1}(y)}{x-y}, \tag{19}$$

we shall first establish an analogous formula for the sum

$$K_N(x,\lambda) := \sum_{k=0}^{N} p_k(x)q_k(\lambda). \tag{20}$$

Throughout, C, C_1, $C_2, \dots$, and B, B_1, $B_2, \dots$ denote positive constants independent of N, m, x and λ.

LEMMA 1. *Let $\{p_n\}_0^\infty$ be a sequence of orthonormal polynomials on $[-1,1]$, with respect to $w \in \mathscr{A}$, and let $q_n(\lambda) := I(p_n;\lambda)$, $n = 1,2,3,\dots$, exist for a given $\lambda \in (-1,1)$. Then, for $N = 1,2,3,\dots$,*

$$K_N(x,\lambda) = \frac{\alpha_{N+1}\{p_{N+1}(x)q_N(\lambda) - p_N(x)q_{N+1}(\lambda)\} + 1}{x-\lambda}. \tag{21}$$

Proof. We have from (5) and (7) that for $k = 0,1,2,\dots$,

$$xp_k(x) = \alpha_{k+1}p_{k+1}(x) + \beta_{k+1}p_k(x) + \alpha_k p_{k-1}(x), \tag{22}$$

and

$$\lambda q_k(\lambda) = \alpha_{k+1}q_{k+1}(\lambda) + \beta_{k+1}q_k(\lambda) + \alpha_k q_{k-1}(\lambda). \tag{23}$$

Multiply (22) by $q_k(\lambda)$ and multiply (23) by $p_k(x)$; then subtract the two and sum from $k = 0$ to N. This yields

$$\begin{aligned}(x-\lambda)K_N(x,\lambda) &= \alpha_{N+1}\{p_{N+1}(x)q_N(\lambda) - p_N(x)q_{N+1}(\lambda)\}\\ &\quad - \alpha_0\{p_0(x)q_{-1}(\lambda) - p_{-1}(x)q_0(\lambda)\}.\end{aligned} \tag{24}$$

Since $p_{-1}(x) \equiv 0$, $q_{-1}(\lambda) \equiv -1$ and $\alpha_0 = m_0^{1/2} = 1/p_0$, (21) follows. □

COROLLARY 1. *The sum $K_N(x,\lambda)$ can also be written as*

$$\begin{aligned}K_N&(x,\lambda)\\ &= \alpha_{N+1}\left\{\frac{p_{N+1}(x)(q_N(\lambda) - q_N(x)) - p_N(x)(q_{N+1}(\lambda) - q_{N+1}(x))}{x-\lambda}\right\}\\ &= \alpha_{N+1}\left\{\frac{q_{N+1}(\lambda)(p_N(\lambda) - p_N(x)) - q_N(\lambda)(p_{N+1}(\lambda) - p_{N+1}(x))}{x-\lambda}\right\}.\end{aligned} \tag{25}$$

Proof. If we set $x = \lambda$ in (24), we find that

$$\begin{aligned}\alpha_{N+1}&\{p_{N+1}(x)q_N(x) - p_N(x)q_{N+1}(x)\}\\ &= -1 = \alpha_{N+1}\{p_{N+1}(\lambda)q_N(\lambda) - p_N(\lambda)q_{N+1}(\lambda)\}.\end{aligned}$$

Substituting into (21) yields (25). □

Before proving some convergence theorems for $S_N(f;\lambda)$, we recall some definitions and results connected with the existence of $I(f;\lambda)$ [1]. We say that a function f is *of Dini type on an interval I* of length $l(I)$, and write $f \in DT(I)$, if

$$\int_0^{l(I)} \omega_I(f;t)t^{-1}\,dt < \infty, \tag{26}$$

where $\omega_I(f;t)$ is the ordinary modulus of continuity of f on I, defined by

$$\omega_I(f;t) := \sup_{\substack{|x-y|\le t\\ x,y\in I}} |f(x)-f(y)|, \qquad t>0. \tag{27}$$

Obviously, if $f \in DT(I)$, then $f \in C(I)$. Furthermore, it can easily be shown that if $f \in DT(I)$, then f satisfies the *Dini-Lipschitz condition on* I, that is

$$\lim_{t\to 0+} \omega_I(f;t)\log t = 0. \tag{28}$$

Finally, it is well known that if $\lambda \in (-1,1)$ and if for some small enough $\varepsilon > 0$, $f \in DT(\lambda-\varepsilon, \lambda+\varepsilon) \cap R[-1,1]$ and $w \in DT(\lambda-\varepsilon, \lambda+\varepsilon) \cap \mathscr{A}$, then $I(f;\lambda)$ exists. Hence, to ensure the existence of $I(f;\lambda)$ for all $\lambda \in (-1,1)$, it is sufficient to require that $f \in R[-1,1]$ and $w \in \mathscr{A}$ belong to $DT(-1,1)$.

We are now ready to prove some convergence results about $S_N(f;\lambda)$ corresponding to the convergence theorems for orthonormal expansions in [11]. As usual, for $w \in \mathscr{A}$ and $0 < p < \infty$, we let

$$L_{p,w} := \left\{ g\colon [-1,1] \to \mathbf{R} \,|\, g \text{ is measurable and } \int_{-1}^{1} w(x)|g(x)|^p\,dx < \infty \right\}. \tag{29}$$

THEOREM 1. *Assume that for some* $\lambda \in (-1,1)$, $I(f;\lambda)$ *exists, that*

$$\sup_k |q_k(\lambda)| \le B < \infty, \tag{30}$$

and that

$$\varphi_\lambda(x) := (f(x)-f(\lambda))/(x-\lambda), \qquad x \in [-1,1], \tag{31}$$

belongs to $L_{2,w}$. *Then*

$$\lim_{N\to\infty} S_N(f;\lambda) = I(f;\lambda). \tag{32}$$

Proof. Multiply (21) by $w(x)(f(x)-f(\lambda))$ and integrate between -1 and 1. We obtain

$$\begin{aligned} &S_N(f;\lambda) - f(\lambda)q_0(\lambda)/p_0 \\ &\quad = \alpha_{N+1}\{c_{N+1}q_N(\lambda) - c_N q_{N+1}(\lambda)\} + I(f;\lambda) - f(\lambda)q_0(\lambda)/p_0, \end{aligned} \tag{33}$$

where $c_k := (\varphi_\lambda, p_k)$ is the kth Fourier coefficient of φ_λ with respect to p_k. Since $\varphi_\lambda \in L_{2,w}$, $c_k \to 0$ as $k \to \infty$. Hence, since $\alpha_{N+1} \le 1$ [5, p. 41], while (30) holds, we obtain (32). □

An important special case of this theorem is that of the *generalized smooth Jacobi weight* (we write $w \in$ GSJ), studied by Nevai [13, p. 673], among others. It is defined by

$$w(x) := \psi(x) \prod_{j=0}^{m+1} |x-t_j|^{\gamma_j}, \qquad x \in [-1,1], \tag{34}$$

where $m \ge 0$, $-1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\gamma_j > -1$, $j = 0,1,2,\ldots,m+1$, $\psi \in DT(-1,1)$ and $\psi(x) > 0$ in $[-1,1]$. Clearly, if

$$\mathscr{D} := [-1,1]\backslash\{t_0, t_1, \ldots, t_{m+1}\}, \tag{35}$$

then $w \in \mathscr{A} \cap DT(\mathscr{D})$, so that if $\lambda \in \mathscr{D}$ and $f \in DT(\lambda-\varepsilon,\ \lambda+\varepsilon) \cap R[-1,1]$ for some $\varepsilon > 0$, then $I(f;\lambda)$ exists. Furthermore, Criscuolo and Mastroianni [2] have shown that if $w \in$ GSJ, then (30) holds for $\lambda \in \mathscr{D}$, and uniformly in any closed subset of $\mathscr{D}$. Hence, we have the following corollary:

COROLLARY 2. *If* $w \in \text{GSJ}$ *and* $f \in DT(\lambda - \varepsilon,\ \lambda + \varepsilon) \cap R[-1,1]$ *for some* $\lambda \in \mathscr{D}$ *and some small enough* $\varepsilon > 0$, *while* $\varphi_\lambda \in L_{2,w}$, *then* (32) *holds.*

In the sequel, we use the norm $\|f\| := \max_{[-1,1]} |f(x)|$ for any $f \in C[-1,1]$.

THEOREM 2. *If* (30) *holds for some* $\lambda \in (-1,1)$, *if*

$$\sup_k \|p_k(x)\| < \infty, \tag{36}$$

and if $\varphi_\lambda \in L_{1,w}$, *then* (32) *holds.*

Proof. By Theorem 3 in [11, p. 69], the Fourier coefficients c_k of φ_λ (defined by (31)) converge to 0 as $k \to \infty$ under the hypotheses of the theorem. Furthermore, since $\varphi_\lambda \in L_{1,w}$, $I(f;\lambda)$ exists, as shown by the identity

$$I(f;\lambda) = \int_{-1}^{1} w(x)\varphi_\lambda(x)\,dx + f(\lambda)q_0(\lambda)/p_0. \tag{37}$$

Hence (32) follows from (33). □

COROLLARY 3. *Assume that* $w \in \text{GSJ}$, *where* $\gamma_0, \gamma_{m+1} \leq -1/2$ *and* $\gamma_j \leq 0$, $j = 1, 2, \ldots, m$. *Further assume that* $\lambda \in \mathscr{D}$, *and that* $\varphi_\lambda \in L_{1,w}$. *Then* (32) *holds.*

Proof. By Nevai [13, p. 674, (16)], there exists $C > 0$ such that for $x \in [-1,1]$ and $k = 1, 2, 3, \ldots$,

$$|p_k(x)| \leq C\{[w(x)(1-x^2)^{1/2}]^{-1/2} + 1\}. \tag{38}$$

Hence, under the hypotheses of the corollary, (36) is true. Furthermore, as above, (30) is true for all $\lambda \in \mathscr{D}$. Hence, by Theorem 2, (32) holds. □

Theorems 1 and 2 are of a local nature, since they depend on the behavior of the Fourier coefficients c_k of $\varphi_\lambda(x)$. The following is a global theorem, and its proof requires much more delicate analysis. The proof is modelled on the proof of Theorem 2 in [11, p. 95].

THEOREM 3. *If* $f \in DT[-1,1]$ *and* $w \in \text{GSJ}$, *then* (32) *holds uniformly for* λ *in each compact subset of* $\mathscr{D}$.

Proof. We first remark that $I(f;\lambda)$ exists for all $\lambda \in \mathscr{D}$ and that f satisfies the Dini-Lipschitz condition (28) on $J := [-1,1]$. We shall start by proving that

$$L_N(\lambda) := \int_{-1}^{1} w(x)|K_N(x,\lambda)|\,dx, \tag{39}$$

is $O(\log N)$, uniformly in a given compact subset $\mathscr{K}$ of $\mathscr{D}$. We first establish this bound for the case $m = 0$ in (34), that is when $w(x)$ has no zeros or infinities in $(-1,1)$. To this end, we write $L_N(\lambda)$ as the sum of five integrals

$$\begin{aligned} L_N(\lambda) &= \int_{-1}^{-1+h/2} + \int_{-1+h/2}^{\lambda-1/N} + \int_{\lambda-1/N}^{\lambda+1/N} + \int_{\lambda+1/N}^{1-h/2} + \int_{1-h/2}^{1} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

and choose N sufficiently large so that $[\lambda - 1/N, \lambda + 1/N] \subset \mathscr{D}$ for all $\lambda \in \mathscr{K}$ and choose $h > 0$ so small that $\mathscr{K} \subset [-1+h, 1-h]$. We consider first I_1 and use (21) for $K_N(x,\lambda)$. Now, for $x \in [-1, -1+h/2]$ and $\lambda \in \mathscr{K}$, $|x-\lambda| \geq h/2$. Further, since (30) holds uniformly for $\lambda \in \mathscr{K}$, since $\alpha_{N+1} \leq 1$, and since

$$\begin{aligned}\int_{-1}^{1} w(x)|p_k(x)|\,dx &\leq \left\{\int_{-1}^{1} w(x)\,dx\right\}^{1/2} \left\{\int_{-1}^{1} w(x)p_k^2(x)\,dx\right\}^{1/2} \\ &= \left\{\int_{-1}^{1} w(x)\,dx\right\}^{1/2},\end{aligned}$$

it follows that $I_1 = O(1)$. Similarly, $I_5 = O(1)$. For $x \in [\lambda + 1/N, 1 - h/2]$, it follows from (38) and the fact that (30) holds uniformly for $\lambda \in \mathscr{K}$, that

$$|K_N(x,\lambda)| \leq C/|x-\lambda|,$$

where C is independent of N, x and λ. Hence,

$$\begin{aligned} I_4 &\leq C^{-1} \int_{\lambda+1/N}^{1} \frac{w(x)}{x-\lambda}\,dx \\ &\leq \int_{-1}^{1} \left|\frac{w(x)-w(\lambda)}{x-\lambda}\right| dx + w(\lambda) \int_{\lambda+1/N}^{1} \frac{dx}{x-\lambda} = O(\log N).\end{aligned}$$

Similarly, $I_2 = O(\log N)$. Finally, since

$$|K_N(x,\lambda)| \leq (N+1) \sup_k |p_k(x)| \sup_k |q_k(\lambda)|,$$

we obtain $I_3 = O(1)$. Combining these estimates, we obtain

$$\sup_{\lambda \in \mathscr{K}} |L_N(\lambda)| \leq C_1 \log N, \tag{40}$$

for some C_1 independent of N. For the general case, we let h be the distance of $\mathscr{K}$ from the set $T := \{t_0, t_1, \ldots, t_{m+1}\}$ and denote by U the subset of $[-1,1]$ such that the distance of T to U is at most $h/2$. As before, we can show that

$$\int_U w(x)|K_N(x,\lambda)|\,dx = O(1),$$

and that

$$\int_{V_N} w(x)|K_N(x,\lambda)|\,dx = O(\log N),$$

where $V_N := [-1,1] \backslash ([\lambda - 1/N, \lambda + 1/N] \cup U)$. If we choose N large enough so that $1/N < h$, we obtain (40).

Next, let P_N^* be the polynomial of best approximation to f in the uniform norm, let $r_N := f - P_N^*$, and let $E_N(f) := \|r_N\|$. Since f satisfies (28) on J, it follows from Jackson's Theorems that

$$\lim_{N \to \infty} E_N(f) \log N = 0. \tag{41}$$

Now, for any $g \in C[-1,1]$, we have

$$|S_N(g;\lambda)| = |(g, K_N(x,\lambda))| \leq \|g\| L_N(\lambda).$$

Hence, uniformly for $\lambda \in \mathscr{K}$, we have from (40) and (41),

$$\lim_{N\to\infty} |S_N(r_N;\lambda)| = 0.$$

Since

$$I(P_N^*;\lambda) = S_N(P_N^*;\lambda),$$

we have

$$I(f;\lambda) - S_N(f;\lambda) = I(r_N;\lambda) - S_N(r_N;\lambda),$$

and it thus remains to show that

$$\lim_{N\to\infty} |I(r_N;\lambda)| = 0.$$

Now

$$\begin{aligned} I(r_N;\lambda) &= \int_{-1}^{1} w(x)\frac{r_N(x)-r_N(\lambda)}{x-\lambda}\,dx + r_N(\lambda)q_0(\lambda)/p_0 \\ &= \int_{-1}^{1} w(x)\frac{r_N(x)-r_N(\lambda)}{x-\lambda}\,dx + o(1). \end{aligned}$$

Furthermore, as in [1],

$$\begin{aligned} \int_{-1}^{1} w(x)\frac{r_N(x)-r_N(\lambda)}{x-\lambda}\,dx &= \int_{-1}^{\lambda-1/N} + \int_{\lambda-1/N}^{\lambda+1/N} + \int_{\lambda+1/N}^{1} \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Here

$$|J_1| \le 2E_N(f)\int_{-1}^{\lambda-1/N} \frac{w(x)}{|x-\lambda|}\,dx = E_N(f)O(\log N) = o(1) \quad \text{as } N\to\infty.$$

Similarly, $J_3 = o(1)$ as $N \to \infty$. Finally,

$$\begin{aligned} &\int_{\lambda-1/N}^{\lambda+1/N} w(x)\frac{r_N(x)-r_N(\lambda)}{x-\lambda}\,dx \\ &\quad = \int_{\lambda-1/N}^{\lambda+1/N} w(x)\frac{f(x)-f(\lambda)}{x-\lambda}\,dx - \int_{\lambda-1/N}^{\lambda+1/N} w(x)\frac{P_N^*(x)-P_N^*(\lambda)}{x-\lambda}\,dx. \end{aligned}$$

Since $f \in DT[-1,1]$, the first integral on the right-hand side is $o(1)$. As for the second integral, we have from [9] that

$$\begin{aligned} \left|\frac{P_N^*(x)-P_N^*(\lambda)}{x-\lambda}\right| &\le \max\{|P_N^{*\prime}(t)|: t \in [\lambda-1/N, \lambda+1/N]\} \\ &\le CN\omega(f;1/N). \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\lambda-1/N}^{\lambda+1/N} w(x)\left|\frac{P_N^*(x)-P_N^*(\lambda)}{x-\lambda}\right|\,dx \\ &\qquad \le 2C\omega(f;1/N)\max\{w(x): x \in [\lambda-1/N, \lambda+1/N]\} \\ &\qquad \to 0 \quad \text{as } N\to\infty, \end{aligned}$$

since $w(x)$ is uniformly bounded above for $\lambda \in \mathscr{K}$ and N large enough. This completes our proof. □

Remark. Theorem 3 is similar to Theorem 2.2 in [1]. By following the proof of Theorem 3, we can prove a result similar to Theorem 2.1 in [1], namely, that if f satisfies (28) on J, if $w \in$ GSJ, and if for some $\lambda \in \mathscr{D}$, $I(f;\lambda)$ exists, then (32) holds. The proof of Theorem 3 holds in this case too, except that we must show that

$$J_N := \int_{\lambda-1/N}^{\lambda+1/N} w(x)\frac{f(x)-f(\lambda)}{x-\lambda}\,dx = o(1), \qquad N\to\infty.$$

Since

$$J_0 := \int_{-1}^{1} w(x)\frac{f(x)-f(\lambda)}{x-\lambda}\,dx = I(f;\lambda) - f(\lambda)I(1;\lambda),$$

and both $I(f;\lambda)$ and $I(1;\lambda)$ exist, it follows that J_0 exists. Hence $J_N = o(1)$, and the proof is complete. □

We now give some additional conditions for (32) to hold, which impose less restrictions on the weight function $w \in \mathscr{A}$, but require more smoothness of f. To this end, we first prove a lemma:

LEMMA 2. *Let $w \in \mathscr{A}$, and assume that for some $\lambda \in (-1,1)$,*

$$\Gamma(\lambda) := \int_{-1}^{1}\left|\frac{w(x)-w(\lambda)}{x-\lambda}\right| dx < \infty, \tag{42}$$

while for some positive ε, B_1 and B_2

$$B_1 \le w(x) \le B_2 \qquad \text{for } |x-\lambda| \le 2\varepsilon. \tag{43}$$

Then there exists a constant $B_3 > 0$ such that

$$T_{n-1}(\lambda) := \sum_{k=0}^{n-1} q_k^2(\lambda) \le B_3 n, \qquad n = 1,2,3,\ldots. \tag{44}$$

If $\Gamma(\lambda)$ is uniformly bounded and (43) holds uniformly for $\lambda \in [a-\varepsilon, b+\varepsilon] \subset [-1,1]$, then (44) holds uniformly for $\lambda \in [a,b]$.

Proof. We first establish the following analogue of the Christoffel function extremum problem, noting that in essence, it is contained in [6]: Defining

$$\rho_n(w;\lambda) := \inf\left\{\frac{I(P^2)}{(I(P;\lambda))^2} : P \in \mathscr{P}_{n-1}, I(P;\lambda) \ne 0\right\}, \tag{45}$$

where $\mathscr{P}_m$ denotes the set of all polynomials of degree $\le m$, we have

$$\rho_n(w;\lambda) = 1/T_{n-1}(\lambda). \tag{46}$$

To see this, we note that for any $P \in \mathscr{P}_{n-1}$, we can write

$$P(x) = \sum_{k=0}^{n-1} a_k p_k(x), \qquad \text{where } a_k := (P, p_k),\ k = 0,1,2,\ldots n-1.$$

Hence

$$|I(P;\lambda)| = \left|\sum_{k=0}^{n-1} a_k q_k(\lambda)\right| \le \left\{\sum_{k=0}^{n-1} a_k^2\right\}^{1/2} \{T_{n-1}(\lambda)\}^{1/2}$$
$$= \{I(P^2)\}^{1/2}\{T_{n-1}(\lambda)\}^{1/2},$$

so that

$$\rho_n(w,\lambda) \ge 1/T_{n-1}(\lambda).$$

On the other hand,

$$\hat{P}(x) := \sum_{k=0}^{n-1} p_k(x) q_k(\lambda) \in \mathscr{P}_{n-1},$$

and satisfies

$$I(\hat{P}; \lambda) = T_{n-1}(\lambda) = I(\hat{P}^2).$$

Then (46) follows.

We now use (46) to prove (44). Choose ε such that $[\lambda - 2\varepsilon,\ \lambda + 2\varepsilon] \subset [-1, 1]$. Now for any $P \in \mathscr{P}_{n-1}$,

$$
\begin{aligned}
|I(P;\lambda)| &= \left| \int_{|x-\lambda|\le\varepsilon} P(x)\frac{w(x)-w(\lambda)}{x-\lambda}\,dx + w(\lambda) \,⨍_{|x-\lambda|\le\varepsilon} \frac{P(x)}{x-\lambda}\,dx \right. \\
&\qquad \left. + \int_{|x-\lambda|\ge\varepsilon} \frac{w(x)P(x)}{x-\lambda}\,dx \right| \\
&\le \Gamma(\lambda) \max_{|x-\lambda|\le\varepsilon} |P(x)| + w(\lambda) \left| ⨍_{|x-\lambda|\le\varepsilon} \frac{P(x)}{x-\lambda}\,dx \right| + \varepsilon^{-1} I(|P|).
\end{aligned} \tag{47}
$$

Next, let χ be the characteristic function of $[\lambda - \varepsilon,\ \lambda + \varepsilon]$, that is, $\chi(x) := 1$ in $[\lambda - \varepsilon,\ \lambda + \varepsilon]$ and $\chi(x) := 0$ elsewhere. We have from (45),

$$
\begin{aligned}
\left| ⨍_{|x-\lambda|\le\varepsilon} \frac{P(x)}{x-\lambda}\,dx \right|^2 &\le \rho_n(\chi;\lambda)^{-1} \int_{|x-\lambda|\le\varepsilon} P^2(x)\,dx \\
&\le B_1^{-1}\rho_n(\chi;\lambda)^{-1} \int_{|x-\lambda|\le\varepsilon} P^2(x)w(x)\,dx \le B_1^{-1}\rho_n(\chi;\lambda)^{-1} I(P^2).
\end{aligned} \tag{48}
$$

Furthermore, by standard estimates for Christoffel functions for the Legendre weight (cf. [12]),

$$\max_{|x-\lambda|\le\varepsilon} (P(x))^2 \le Cn \int_{\lambda-2\varepsilon}^{\lambda+2\varepsilon} P^2(t)\,dt \le CnB_1^{-1} I(P^2). \tag{49}$$

Combining (47), (48) and (49), and using the Cauchy-Schwarz inequality, we obtain

$$|I(P;\lambda)| \le B_4\{n^{1/2}\Gamma(\lambda) + \rho_n^{-1/2}(\chi;\lambda) + 1\} I(P^2)^{1/2}.$$

But

$$\rho_n(\chi;\lambda)^{-1} = \sum_{k=0}^{n-1} q_k^2(\chi;\lambda) \le B_5 n,$$

since $q_k(\chi;\lambda)$ is the function of the second kind associated with the Legendre weight shifted to $[\lambda - \varepsilon,\ \lambda + \varepsilon]$, so that $q_k(\chi;\lambda) = O(1)$. Hence

$$|I(P;\lambda)| \le B_6 n^{1/2} I(P^2)^{1/2},$$

so that

$$1/T_{n-1}(\lambda) = \rho_n(w;\lambda) \ge B_7/n.$$

If the assumptions on λ hold uniformly in $[a - \varepsilon,\ b + \varepsilon]$, it is not difficult to modify the proof to hold uniformly in $[a, b]$. □

We now prove the analogue of Theorem IV.1.2 in Freud [5, p. 139].

THEOREM 4. *Let $w \in \mathscr{A}$ and assume that for some $\lambda \in (-1,1)$, (42) holds, while (43) holds for some positive ε, B_1 and B_2. Define for $n = 1,2,3,\ldots$,*

$$E_n^{(2)}(f;w) := \inf_{P\in\mathscr{P}_n} (f-P,\ f-P)^{1/2}. \tag{50}$$

Then, if

$$\sum_{n=1}^{\infty} E_n^{(2)}(f;w)n^{-1/2} < \infty, \tag{51}$$

(32) holds. If $\Gamma(\lambda)$ is uniformly bounded for $\lambda \in [a-\varepsilon,\ b+\varepsilon] \subset [-1,1]$, while (43) holds uniformly for $\lambda \in [a-\varepsilon,\ b+\varepsilon]$, then (32) holds uniformly in $[a,b]$.

Proof. First recall the notation (8). For any positive integer m,

$$\begin{aligned}\sum_{k=2^m+1}^{2^{m+1}} |a_k q_k(\lambda)| &\le \left\{\sum_{k=2^m+1}^{2^{m+1}} a_k^2\right\}^{1/2} \left\{\sum_{k=2^m+1}^{2^{m+1}} q_k^2(\lambda)\right\}^{1/2} \\ &\le \left\{\sum_{k=2^m+1}^{\infty} a_k^2\right\}^{1/2} \left\{\sum_{k=0}^{2^{m+1}} q_k^2(\lambda)\right\}^{1/2} \\ &= E_{2^m}^{(2)}(f;w) \left\{\sum_{k=0}^{2^{m+1}} q_k^2(\lambda)\right\}^{1/2} \\ &\le CE_{2^m}^{(2)}(f;w)2^{m/2},\end{aligned}$$

where the last inequality follows from (44). Since $E_k^{(2)}(f;w)$ is nonincreasing with k,

$$\begin{aligned}2^{m/2}E_{2^m}^{(2)}(f;w) &\le 2^{m/2}\left\{2^{-m+1}\sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f;w)\right\} \\ &= 2^{1-m/2}\sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f;w) \le 2\sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f;w)k^{-1/2}.\end{aligned}$$

Hence,

$$\sum_{k=2}^{\infty} |a_k q_k(\lambda)| = \sum_{m=1}^{\infty}\sum_{k=2^{m-1}+1}^{2^m} |a_k q_k(\lambda)| \le B\sum_{k=1}^{\infty} E_k^{(2)}(f;w)k^{-1/2}. \quad \square$$

The next theorem is the analogue of Theorem IV.1.3 in Freud [5, p. 140].

THEOREM 5. *Let w and λ be as in Theorem 4. Let $f \in C[-1,1]$ and for $J := [-1,1]$, suppose that $w_J(f;\delta)$ satisfies for some $\eta > 0$,*

$$\lim_{\delta\to 0+} w_J(f;\delta)\delta^{-1/2}|\log\delta|^{1+\eta} = 0. \tag{52}$$

Then (32) holds. If $\Gamma(\lambda)$ is uniformly bounded for $\lambda \in [a-\varepsilon,\ b+\varepsilon] \subset [-1,1]$, while (43) holds uniformly for $\lambda \in [a-\varepsilon, b+\varepsilon]$, then (32) holds uniformly in $[a,b]$.

Proof. By Jackson's Theorem,

$$E_k^{(2)}(f;w) \le B_1 w_J(f;k^{-1}) \le B_1 k^{-1/2}|\log k|^{-1-\eta}. \quad \square$$

3. Convergence Results for $Q_m^N(f;\lambda)$. We are now ready to prove our convergence theorems for $Q_m^N(f;\lambda)$. First a result on the iterated limit.

THEOREM 6. *Assume that $f \in R[-1,1]$, that $I(f;\lambda)$ exists and that $w \in \mathscr{A}$ and $\lambda \in [-1,1]$ are such that* (32) *holds. Let $\{Q_m(\cdot)\}_{m=1}^{\infty}$ be a sequence of integration rules such that for all $g \in R[-1,1]$,*

$$\lim_{m\to\infty} Q_m(g) = I(g).$$

Then

$$\lim_{N\to\infty}\lim_{m\to\infty} Q_m^N(f;\lambda) = I(f;\lambda). \tag{53}$$

Proof. It suffices to show that for each fixed N,

$$\lim_{m\to\infty} Q_m^N(f;\lambda) = S_N(f;\lambda),$$

since

$$I(f;\lambda) = S_N(f;\lambda) + \sum_{k=N+1}^{\infty} a_k q_k(\lambda) = S_N(f;\lambda) + o(1).$$

For fixed N, we choose m sufficiently large so that

$$|a_{km} - a_k| \le \varepsilon \max_{0\le k\le N} |q_k(\lambda)|/(N+1), \qquad k = 0,1,2,\ldots,N,$$

yielding the theorem. □

Even though we have convergence of the iterated limit (53), we cannot in general have convergence of the double limit (that is the limit with m and $N \to \infty$ simultaneously), as illustrated by the following simple example:

Example 1. Let

$$w(x) := (1-x^2)^{-1/2} \quad \text{and} \quad f(x) \equiv 1, \qquad x \in (-1,1),$$

and let $Q_m(\cdot)$ be the Gauss-Chebyshev rule

$$Q_m(g) := \frac{\pi}{m}\sum_{i=1}^{m} g\left(\cos\frac{2i-1}{2m}\pi\right).$$

Then, with $N = 2m$, we have that

$$Q_m^{2m}(f;\lambda) = \sum_{k=0}^{2m} Q_m(fp_k)q_k(\lambda) = \sum_{k=0}^{2m} Q_m(p_k)q_k(\lambda).$$

Since $Q_m(g)$ is exact for all $g \in \mathscr{P}_{2m-1}$,

$$Q_m(p_k) = I(p_k) = \int_{-1}^{1} w(x)p_k(x)\,dx, \qquad 0 \le k \le 2m-1,$$

so that

$$Q_m(p_0) = p_0\pi \quad \text{and} \quad Q_m(p_k) = 0, \qquad k = 1,2,\ldots,2m-1.$$

Furthermore,

$$Q_m(p_{2m}) = \frac{\pi}{m}\sum_{i=1}^{m}\left(\frac{2}{\pi}\right)^{1/2} T_{2m}\left(\cos\frac{2i-1}{2m}\pi\right)$$
$$= (2\pi)^{1/2}m^{-1}\sum_{i=1}^{m}\cos(2i-1)\pi = -(2\pi)^{1/2}.$$

Hence

$$Q_m^{2m}(f;\lambda) = \pi^{1/2}q_0(\lambda) - (2\pi)^{1/2}q_{2m}(\lambda).$$

But (see, for example, [8, p. 148])

$$q_0(\lambda) = I(f;\lambda) = 0,$$

so

$$Q_m^{2m}(f;\lambda) - I(f;\lambda) = -(2\pi)^{1/2}q_{2m}(\lambda),$$

which does not go to zero for any nonzero $\lambda \in (-1,1)$ as $m \to \infty$, inasmuch as $q_{2m}(\lambda) = (2/\pi)^{1/2}U_{2m+1}(\lambda)$, where $U_{2m+1}(\lambda)$ is the Chebyshev polynomial of the second kind of degree $2m+1$.

Example 1 shows that at least in general, converting the iterated limit to a single limit does not lead to convergence. However, there are cases where this procedure will work. One simple example occurs when $m = N+1$ and $Q_m(\cdot)$ is the Gauss integration rule with respect to w. In this case, it turns out that

$$Q_{N+1}^N(f;\lambda) = I(L_{N+1};\lambda), \tag{54}$$

where L_{N+1} is the Lagrange interpolation polynomial of degree $\le N$ interpolating f at the zeros of p_{N+1}. This follows since

$$\begin{aligned} I(L_{N+1};\lambda) &= \sum_{k=0}^{N}(L_{N+1},p_k)q_k(\lambda) = \sum_{k=0}^{N} Q_{N+1}(L_{N+1}p_k)q_k(\lambda) \\ &= \sum_{k=0}^{N} Q_{N+1}(fp_k)q_k(\lambda) = Q_{N+1}^N(f;\lambda) \end{aligned} \tag{55}$$

(see, for example, [16, pp. 1250–1251]). Since it has been shown in [1] that for $w \in \text{GSJ}$,

$$\lim_{N\to\infty} I(L_{N+1},\lambda) = I(f;\lambda),$$

we have that for the sequence of Gauss rules $\{Q_m(\cdot)\}_{m=1}^{\infty}$ associated with $w \in \text{GSJ}$,

$$\lim_{N\to\infty} Q_{N+1}^N(f;\lambda) = I(f;\lambda).$$

We can generalize this result to any sequence of integration rules $\{Q_m(\cdot)\}_{m=1}^{\infty}$ that is ultimately exact for all polynomials, that is $Q_m(g) = I(g)$ for all $g \in \mathscr{P}_n$ and all $m \ge m(n)$. A particular instance of this, that allows points to be concentrated in regions where the behaviour of f is problematic, is rules exact for piecewise polynomials of increasing degree.

In the general situation, if the weights w_{im} and the points x_{im} in a sequence of rules $\{Q_m(\cdot)\}_{m=1}^{\infty}$ are such that

$$\sum_{i=1}^{m}|w_{im}^N(\lambda)| = O(\log N), \tag{56}$$

if $f \in DT[-1,1]$ and if $w \in \mathrm{GSJ}$, then we have that

$$\lim_{N\to\infty} Q^N_{m(2N)}(f;\lambda) = I(f;\lambda). \tag{57}$$

Here $m(2N)$ denotes the least integer m such that $Q_m(g) = I(g)$ for all $g \in \mathscr{P}_{2N}$. The proof follows standard lines, namely

$$I(f;\lambda) = I(P^*_N;\lambda) + I(r_N;\lambda), \tag{58}$$

where, as above, $P^*_N \in \mathscr{P}_N$ is the polynomial of best approximation to f in the uniform norm and $r_N := f - P^*_N$. Since, by hypothesis, $Q_m(gp_k) = I(gp_k)$ for all $k \le N$, all $m \ge m(2N)$ and all $g \in \mathscr{P}_N$, it follows that

$$Q^N_m(f;\lambda) = I(P^*_N;\lambda) + Q^N_m(r_N;\lambda). \tag{59}$$

Hence

$$\begin{aligned}|I(f;\lambda) - Q^N_m(f;\lambda) &\le |I(r_N;\lambda)| + |Q^N_m(r_N;\lambda)| \\ &\le |I(r_N;\lambda)| + \sum_{i=1}^m |w^N_{im}(\lambda)|\,\|r_N\|.\end{aligned}$$

As in the proof of Theorem 3 above, $I(r_N;\lambda) = o(1)$, and since f satisfies (28), the second term is also $o(1)$ from (56), proving (57).

What about conditions on w_{im} and x_{im} that ensure (56)? We shall prove

LEMMA 3. *With the above notation, if*

$$h_{im} := x_{i+1,m} - x_{im} \le C_1/N \tag{60}$$

for some $C_1 > 0$, uniformly for all i and $m \ge m(2N)$, while

$$|w_{im}|/w(x_{im}) \le C_2(h_{i-1,m} + h_{im}), \tag{61}$$

then (56) *holds whenever $w \in \mathrm{GSJ}$ and $\lambda \in \mathscr{D}$.*

Proof. As in the proof of Theorem 3, we consider first the case $m = 0$ in (34), and we decompose the sum on the left-hand side of (56) into five sums:

$$\begin{aligned}\sum_{i=1}^m |w^N_{im}(\lambda)| &= \sum_{x_{im}\le -1+\delta} + \sum_{\substack{\lambda - x_{im} > 2C_1/N \\ x_{im} > -1+\delta}} + \sum_{|x_{im}-\lambda|\le 2C_1/N} \\ &\quad + \sum_{\substack{x_{im}-\lambda > 2C_1/N \\ x_{im} < 1-\delta}} + \sum_{x_{im}\ge 1-\delta} \\ &=: \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 + \sum\nolimits_5,\end{aligned}$$

where δ is some sufficiently small positive number. Now by (17) and (21), and the uniform boundedness of $\{q_k(\lambda)\}_0^\infty$,

$$\begin{aligned}\sum\nolimits_1 &\le (1-\delta+\lambda)^{-1} O(1) \sum w_{im}\{|p_N(x_{im})| + |p_{N+1}(x_{im})|\} \\ &\le O(1) \int_{-1}^{-1+\delta+C_1/N} w(x)\{|p_N(x)| + |p_{N+1}(x)|\}\,dx \\ &\qquad\qquad \text{(by Theorem 5 in [10, p. 534])} \\ &\le O(1)\left\{\int_{-1}^1 w(x)\,dx\right\}^{1/2} = O(1).\end{aligned}$$

Similarly, $\sum_5 = O(1)$. Next,

$$\begin{aligned}\sum\nolimits_2 &= O(1)\sum |w_{im}|/(\lambda - x_{im})\\ &\le O(1)\int_{-1}^{\lambda - C_1/N} dx/(\lambda - x) = O(\log N),\end{aligned}$$

by uniform boundedness of p_N, q_N and w. Similarly, $\sum_4 = O(\log N)$. Finally,

$$\begin{aligned}\sum\nolimits_3 &= O(1)\sum_{|x_{im}-\lambda|\le 2C_1/N} |w_{im}|\sum_{k=0}^{N} |p_k(x_{im})q_N(\lambda)|\\ &= O(1)(N+1)\sum_{|x_{im}-\lambda|\le 2C_1/N} |w_{im}|\\ &= O(1),\end{aligned}$$

which proves the lemma for the case $m = 0$. For the general case, we enclose each of the interior singularities of w in a small interval avoiding λ and treat the w_{im} associated with these intervals in the same manner as $\sum_1$. □

The assertion (57) is a special case of the following theorem:

THEOREM 7. *Suppose that for* $m = 1, 2, 3, \ldots$, *the rule* $Q_m(\cdot)$ *has precision* $\pi_m > N_m$, *that* $t_m := \min\{N_m, \pi_m - N_m\}$ *satisfies*

$$\lim_{m\to\infty} t_m = \infty, \tag{62}$$

and that

$$\sum_{i=1}^{m} |w_{im}^{N_m}(\lambda)| \le C\log t_m, \qquad m = 1, 2, 3, \ldots. \tag{63}$$

Assume that $f \in C[-1,1]$ *satisfies* (28) *with* $I = [-1,1]$, *that* $I(f;\lambda)$ *exists, that* $q_0(\lambda)$ *is finite and that* $w(x)$ *is bounded above in a neighborhood of* λ. *Then*

$$\lim_{m\to\infty} Q_m^{N_m}(f;\lambda) = I(f;\lambda). \tag{64}$$

Proof. If $P \in \mathscr{P}_{t_m}$, then

$$\begin{aligned}Q_m^{N_m}(P;\lambda) &= \sum_{k=0}^{N_m} Q_m(Pp_k)q_k(\lambda) = \sum_{k=0}^{N_m}(P, p_k)q_k(\lambda)\\ &= \sum_{k=0}^{t_m}(P, p_k)q_k(\lambda) = I(P;\lambda),\end{aligned}$$

since $t_m \le N_m$. Then, if $P_m^* \in \mathscr{P}_{t_m}$ is the polynomial of best approximation to f in the uniform norm, and if $r_m := f - P_m^*$, then as above, for m sufficiently large so that $[\lambda - 1/t_m, \lambda + 1/t_m] \subset [-1, 1]$,

$$\begin{aligned}
|Q_m^{N_m}(f;\lambda) - I(f;\lambda)| &= |Q_m^{N_m}(r_m;\lambda) - I(r_m;\lambda)| \\
&\le \sum_{i=1}^{m} |w_{im}^{N_m}(\lambda)|\,\|r_m\| + \int_{|\lambda - x| \ge 1/t_m} w(x)\frac{|r_m(x)|}{|x-\lambda|}\,dx \\
&\quad + \left| \fint_{|\lambda - x| \le 1/t_m} \frac{w(x) r_m(x)}{x - \lambda}\,dx \right| \\
&\le C \log t_m \omega(f; t_m^{-1}) + C_1 \|r_m\| \log t_m \\
&\quad + \left| \int_{|\lambda - x| \le 1/t_m} w(x) \frac{f(x) - f(\lambda)}{x - \lambda}\,dx \right| \\
&\quad + \left| \int_{|\lambda - x| \le 1/t_m} w(x) \frac{P_m^*(x) - P_m^*(\lambda)}{x - \lambda}\,dx \right| \\
&\quad + |r_m(\lambda)| \left| \fint_{|\lambda - x| \le 1/t_m} \frac{w(x)}{x - \lambda}\,dx \right| \\
&\le o(1) + o(1) + o(1) + o(1) + o(1),
\end{aligned}$$

by the arguments used in the proof of Theorem 3 and the fact that w is bounded above near λ. □

We conclude with an almost trivial theorem that gives necessary and sufficient conditions for the convergence of a sequence of approximations $\{Q_m^{N_m}(f;\lambda)\}_{m=1}^{\infty}$. It shows that we must choose N_m in such a way that $Q_m(fp_k)$ is small for all k large enough with $k \le N_m$:

THEOREM 8. *Assume that for all* $g \in R[-1,1]$,

$$\lim_{m\to\infty} Q_m(g) = I(g),$$

that $I(f;\lambda)$ *exists and that* (32) *holds. Then, given a sequence* $\{(m, N_m)\}_{m=1}^{\infty}$ *of pairs of positive integers with*

$$\lim_{m\to\infty} N_m = \infty,$$

we have that

$$\lim_{m\to\infty} Q_m^{N_m}(f;\lambda) = I(f;\lambda)$$

if and only if for every $\varepsilon > 0$ *we can find a positive integer* K *such that for all large enough* m,

$$\left| \sum_{k=K}^{N_m} Q_m(fp_k) q_k(\lambda) \right| < \varepsilon. \tag{65}$$

Proof. For any fixed J and all m large enough,

$$\begin{aligned}
Q_m^{N_m}(f;\lambda) - I(f;\lambda) &= \sum_{k=0}^{N_m} Q_m(fp_k)q_k(\lambda) - \sum_{k=0}^{\infty}(f,p_k)q_k(\lambda) \\
&= \sum_{k=0}^{K-1} \{Q_m(fp_k) - (f,p_k)\} q_k(\lambda) \\
&\quad - \sum_{k=K}^{\infty} (f,p_k)q_k(\lambda) + \sum_{k=K}^{N_m} Q_m(fp_k)q_k(\lambda).
\end{aligned}$$

Here, as in the proof of Theorem 6, the first term in this last right-hand side is $o(1)$ as $m \to \infty$. Further, given $\varepsilon > 0$, we can find a K such that the absolute value of the second term in this last right-hand side is bounded above by ε. Hence for m large enough,

$$\left| Q_m^{N_m}(f;\lambda) - I(f;\lambda) - \sum_{k=K}^{N_m} Q_m(fp_k)q_k(\lambda) \right| < 2\varepsilon,$$

which proves the theorem. □

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