Incomplete Hyperelliptic Integrals
and Hypergeometric Series

By J. F. Loiseau, J. P. Codaccioni, and R. Caboz

Abstract. We consider the incomplete hyperelliptic integral

$$H(a, X) = \int_0^X \frac{dx}{\sqrt{a - \lambda_2 x^2 - \lambda_n x^n}}$$

with $a > 0$, $\lambda_2 > 0$, $n > 2$, where X belongs to the connected component of $\{x | \lambda_2 x^2 + \lambda_n x^n < a\}$ containing the origin.

Continuing previous work on the complete hyperelliptic integral, we express in this paper $H(a, X)$ as a convergent series of hypergeometric type. A brief survey of some applications to algebraic equations and mechanics is then given.

1. Introduction. In a previous paper [8], we considered the complete hyperelliptic integral

$$(1) \qquad J(a) = \int_{\alpha(a)}^{\beta(a)} \frac{dx}{\sqrt{a - P_n(x)}}$$

where

  (i) $a > 0$,
  (ii) $P_n(x) = \sum_{k=2}^{n} \lambda_k x^k$, $n > 2$, $\lambda_2 > 0$,
  (iii) $[\alpha(a), \beta(a)]$ is the connected component of $\{x | P_n(x) \leq a\}$ containing the origin (see Figure 1).

In the present paper, we deal with the incomplete integral

$$(2) \qquad H(a, X) = \int_0^X \frac{dx}{\sqrt{a - P_n(x)}}$$

with conditions (i), (ii) and (iii) and $X \in (\alpha(a), \beta(a))$.

Since, for $x \in [0, X]$, P_n is monotone in x, we consider the inverse function x_+ (resp. x_-) to the function P_n for $X > 0$ (resp. $X < 0$).

Changing variables in Eq. (2), we find

$$(3) \qquad H_\pm(a, X) = \pm \int_0^{P_n(X)} \frac{x'_\pm(u)}{\sqrt{a - u}} \, du.$$

Computation of $x_\pm$, the inverse function of P_n, reduces to the problem of solving the algebraic equation

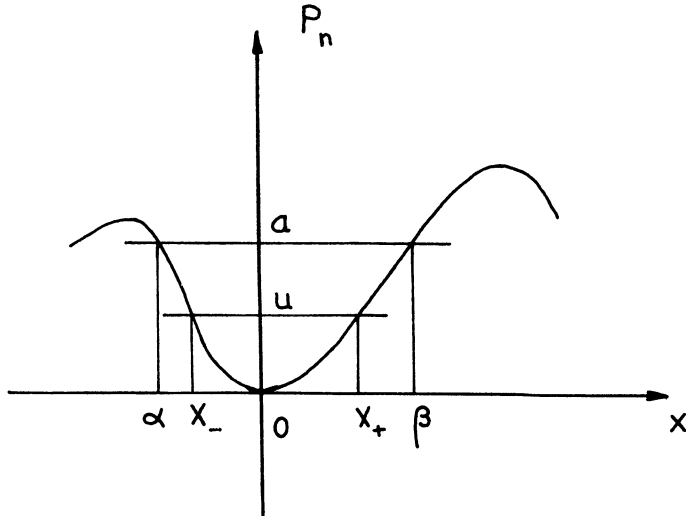$$(4) \qquad \lambda_n x^n + \lambda_{n-1} x^{n-1} + \cdots + \lambda_2 x^2 - u = 0$$

FIGURE 1

Typical graph of P_n: $x \to P_n(x)$.

or, more precisely, to determine as functions of u the two solutions of (4) belonging to the interval $[\alpha(a), \beta(a)]$.

For $n = 2$ we have the trivial case $x_{\pm}(u) = \pm\sqrt{u/\lambda_2}$, and the $H_{\pm}(a, X)$ reduce to inverse trigonometric functions. For $n > 2$ the only algebraically solvable cases are $n = 3$ and $n = 4$, but the expressions of the solutions of (4) are complicated enough to be of no help for computing $H_{\pm}(a, X)$. If we try to find x by inversion of series, the lack of the linear term $\lambda_1 x$ allows us only to develop in powers of $\sqrt{u}$. In that case we have first to express $\sqrt{P_n(x)}$ as an infinite power series and then to invert it to get finally $x(\sqrt{u})$. Unfortunately, the algorithm does not give the general term of the series and none of the ways we tried to express x as a convergent series succeeded, except for one particular case: when $P_n(x)$ contains only one superquadratic term:

$$(5) \qquad P_n(x) = \lambda_2 x^2 + \lambda_n x^n, \qquad n > 2.$$

In this case, the computation is developed in Section 2 and used in Section 3 to compute the corresponding integral $H_{\pm}(a, X)$. A few consequences of these results are then studied in the subsequent two sections.

2. Computation of $x_{\pm}(u)$. We now limit ourselves to the case of Eq. (5). Setting $x_{\pm} = \pm\sqrt{z}$, we derive

$$(6) \qquad z = \frac{u}{\lambda_2} - (\pm 1)^n \frac{\lambda_n}{\lambda_2} z^{n/2}.$$

The Lagrange-Bürmann theorem ([12, p. 133]) states:

Let θ be a function of z analytic on and inside a contour (Γ) surrounding a point A, and let t be such that the inequality

$$(a) \qquad |t\theta(z)| < |z - A|$$

is satisfied at all points z on the perimeter of (Γ). Then the equation

$$(b) \qquad z = A + t\theta(z)$$

has one solution in the interior of (Γ), *and further, any function f of z, analytic on and inside* (Γ) *can be expanded as a power series in t by the Lagrange formula*

(c)
$$f(z) = f(A) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{dA^{n-1}} [f'(A)\theta(A)^n].$$

This theorem will be applied in the case

(7)
$$\begin{cases} \theta(z) = z^{n/2}, \\ f(z) = \pm\sqrt{z}, \\ A = u/\lambda_2, \\ t = -(\pm 1)^n \lambda_n/\lambda_2, \end{cases}$$

for which Eq. (b) of the Lagrange-Bürmann theorem is exactly Eq. (6).

First, we emphasize that here f and θ are real-valued functions defined on $[0, +\infty)$ and for which a domain is an interval of $\mathbf{R}$ and the corresponding contour (Γ) a set containing only two elements: the endpoints of this interval. Condition (a) is then satisfied when it is proved to be true for the two endpoints of an interval containing A.
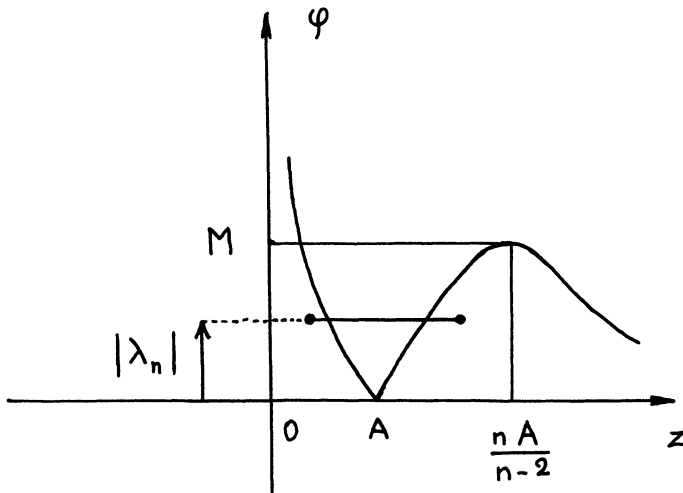

FIGURE 2

Graph of $\varphi: z \to \varphi(z)$ *of Eq.* (9).

We have $u \in [0, a]$, i.e., $A \in [0, a/\lambda_2]$, and condition (a) of the Lagrange-Bürmann theorem reduces to

(8)
$$|\lambda_n| \frac{z^{n/2}}{\lambda_2} < |z - A|.$$

Setting

(9)
$$\varphi(z) = \lambda_2 \frac{|z - A|}{z^{n/2}},$$

condition (8) reads

(10)
$$\varphi(z) > |\lambda_n|.$$

The graph of φ, represented in Figure 2, admits a local maximum

$$(11) \qquad M(A) = \frac{2\lambda_2(n-2)^{(n-2)/2}}{A^{(n-2)/2}n^{n/2}}$$

for the value $z = nA/(n-2)$. The condition (10) must be verified for two points surrounding A and for every $A \in [0, a/\lambda_2]$, so that we require

$$(12) \qquad M(A) > |\lambda_n|$$

for every $A \in [0, a/\lambda_2]$. As M is a decreasing positive function of A, condition (12) will be verified for every $A \in [0, a/\lambda_2]$ as soon as $M(a/\lambda_2) > |\lambda_n|$, i.e.,

$$(13) \qquad \eta(a) \equiv \frac{|\lambda_n|}{2} \left(\frac{n}{\lambda_2}\right)^{n/2} \left(\frac{a}{n-2}\right)^{(n-2)/2} < 1.$$

Condition (13) has been encountered previously in the study of the complete hyperelliptic integral $J(a)$, in the particular case of a single superquadratic term [8]: it is the condition required for the hypergeometric series representing $J(a)$ to converge.

Now Eq. (c) of the Lagrange-Bürmann theorem allows us to write $x_{\pm}(u)$ as the Lagrange series

$$(14) \qquad \begin{aligned} x_{\pm}(u) = &\pm \sum_{k=0}^{\infty} (-1)^{k(n_{\pm}+1)} \frac{2^{(nk+1)/2}}{k(n-2)+1} \frac{\Gamma\left(\frac{kn+1}{2}\right)}{\Gamma\left(\frac{k(n-2)+1}{2}\right)} \\ &\times \frac{(\lambda_n/\lambda_2)^k}{k!} \left(\frac{u}{\lambda_2}\right)^{(k(n-2)+1)/2} \end{aligned}$$

with $n_{\pm}$ standing for $2n$ in x_+ and n in x_-.

Some tedious algebraic manipulations [4] finally yield

$$(15) \qquad \begin{aligned} x_+(u) &= -x_-(u) \\ &= \left(\frac{u}{\lambda_2}\right)^{1/2} {}_{n/2}F_{n/2-1}\left[\begin{array}{c} \dfrac{1}{n}, \dfrac{3}{n}, \ldots, \dfrac{n-1}{n} \\ \dfrac{3}{n-2}, \dfrac{5}{n-2}, \ldots, \dfrac{n-1}{n-2} \end{array}; \eta(u)\right] \end{aligned}$$

if n is even, and

$$(16a) \qquad x_{\pm}(u) = \pm x_1(u) + \lambda_n x_2(u),$$

with

$$(16b_1) \quad x_1(u) = \left(\frac{u}{\lambda_2}\right)^{1/2} {}_n F_{n-1}\left[\begin{array}{c} \dfrac{1}{2n}, \dfrac{3}{2n}, \ldots, \dfrac{2n-1}{n} \\ \dfrac{3}{2(n-2)}, \dfrac{5}{2(n-2)}, \ldots, \dfrac{2n-3}{2(n-2)}, \dfrac{1}{2} \end{array}; \eta^2(u)\right],$$

$$(16b_2) \qquad \begin{aligned} x_2(u) = &\frac{1}{2\lambda_2}\left(\frac{u}{\lambda_2}\right)^{(n-1)/2} \\ &\times {}_n F_{n-1}\left[\begin{array}{c} \dfrac{n+1}{2n}, \dfrac{n+3}{2n}, \ldots, \dfrac{3n-1}{2n} \\ \dfrac{n+1}{2(n-2)}, \dfrac{n+3}{2(n-2)}, \ldots, \dfrac{3n-5}{2(n-2)}, \dfrac{3}{2} \end{array}; \eta^2(u)\right] \end{aligned}$$

if n is odd.

Since $0 < u < a$, condition (13) ensures the convergence of the generalized hypergeometric series [11] occurring in (15), (16).

3. Computation of $H_\pm(a, X)$. The inverse function $x_\pm(u)$ being expressed as a convergent power series in $\eta(u)$, it can be differentiated term by term in Eq. (3) to obtain for the integrand a uniformly convergent series of functions which, consequently, may be integrated term by term. The integrals to be computed are of the type

$$(17) \qquad \int \frac{u^{k(n-2)/2}}{\sqrt{a-u}} \, du,$$

and no difficulty arises except for the length of the calculation. Finally, the result may be written in the form of a convergent series as follows:

$$
\begin{aligned}
(18) \qquad H_\pm(a, X) = \frac{1}{2\sqrt{\lambda_2}} \sum_{k=0}^{\infty} & \frac{\left[-(\pm 1)^n \left(\frac{a}{\lambda_2} \right)^{(n-2)/2} \lambda_n/\lambda_2 \right]^k}{k!} \\
& \times \frac{\Gamma(\sigma_k - k)}{\Gamma(\sigma_k)} B_{P_n(X)/a} \left[\sigma_k, \frac{1}{2} \right],
\end{aligned}
$$

where

$$(19) \qquad \sigma_k = \frac{(n-2)k+1}{2},$$

and where

$$(20) \qquad B_\alpha[\beta, \gamma] = \frac{\alpha^\beta}{\beta} {}_2F_1[\beta, 1-\gamma, \beta+1; \alpha], \qquad \alpha < 1,$$

is the incomplete Beta function [6].

When $P_n(X) = a$, Eq. (18) still holds, the incomplete Beta function being replaced by the complete Beta function [6]

$$(21) \qquad B_1[\sigma_k, \tfrac{1}{2}] = \frac{\Gamma(\sigma_k)\Gamma(\tfrac{1}{2})}{\Gamma(\sigma_k + \tfrac{1}{2})}.$$

We then obtain the complete hyperelliptic integral $J(a)$ given in reference [8].

From a computational point of view, Eq. (18) is particularly interesting when λ_n is small, so that only a few terms are needed to get a good approximation of $H_\pm(a, X)$.

4. Application to Algebraic Equations. The link between solutions of algebraic equations and generalized hypergeometric series and functions has been widely studied more than sixty years ago by Mellin [10], Birkeland [3], Bellardinelli [2], Appell and Kampé de Fériet [1]. Our formulae are somewhat different from theirs, and we have not tried to prove the exact equivalence. By comparison of the two results, one may obtain some new transformation formulae on generalized hypergeometric functions (see Appendix).

From (13), (15) and (16) we obtain the following proposition:

Among the real solutions of the algebraic equation

$$x^n + \rho x^2 + \sigma = 0, \qquad n > 2,$$

the two smallest ones in absolute value can be expressed as hypergeometric series provided that

$$Q = -\frac{\sigma}{\rho} > 0 \quad \text{and} \quad R = \frac{n-2}{2\sigma}\left(\frac{Qn}{n-2}\right)^{n/2} \in (-1, +1).$$

These two solutions x_- and x_+ are

$$x_+ = -x_- = Q^{1/2}{}_{n/2}F_{n/2-1}\left[\begin{matrix} \dfrac{1}{n}, \dfrac{3}{n}, \ldots, \dfrac{n-1}{n} \\ \dfrac{3}{n-2}, \dfrac{5}{n-2}, \ldots, \dfrac{n-1}{n-2} \end{matrix}; R\right]$$

if n is even, and

$$x_+ = x_1 - \frac{1}{2\rho}x_2, \qquad x_- = -x_1 - \frac{1}{2\rho}x_2$$

with

$$x_1 = Q^{1/2}{}_nF_{n-1}\left[\begin{matrix} \dfrac{1}{2n}, \dfrac{3}{2n}, \ldots, \dfrac{2n-1}{2n} \\ \dfrac{3}{2(n-2)}, \dfrac{5}{2(n-2)}, \ldots, \dfrac{2n-3}{2(n-2)}, \dfrac{1}{2} \end{matrix}; R^2\right],$$

$$x_2 = Q^{(n-1)/2}{}_nF_{n-1}\left[\begin{matrix} \dfrac{n+1}{2n}, \dfrac{n+3}{2n}, \ldots, \dfrac{3n-1}{2n} \\ \dfrac{n+1}{2(n-2)}, \dfrac{n+3}{2(n-2)}, \ldots, \dfrac{3n-5}{2(n-2)}, \dfrac{3}{2} \end{matrix}; R^2\right]$$

if n is odd.

Note that when n is even and $\sigma > 0$, the proposition is still valid by analytic continuation for $R \in (-\infty, +1)$ [9].

When n is even, setting $n = 2m$, we have the following corollary:

The smallest real solution, in absolute value, of the algebraic equation

$$x^m + \rho x + \sigma = 0, \qquad m \geq 2,$$

is expressible as a squared generalized hypergeometric series, provided that

$$Q = -\frac{\sigma}{\rho} > 0 \quad \text{and} \quad R = \frac{m-1}{\sigma}\left(\frac{mQ}{m-1}\right)^m \in (-1, +1).$$

Its expression is

$$x = Q\left[{}_mF_{m-1}\left[\begin{matrix} \dfrac{1}{2m}, \dfrac{3}{2m}, \ldots, \dfrac{2m-1}{2m} \\ \dfrac{3}{2(m-1)}, \dfrac{5}{2(m-1)}, \ldots, \dfrac{2m-1}{2(m-1)} \end{matrix}; R\right]\right]^2 .$$

5. Application to Mechanics.

From a mechanical point of view, the integral $J(a)$ of Eq. (1) is nothing but the half-period of a one-dimensional oscillator with total energy $E = a/2$, moving in the potential $U(x) = P_n(x)/2$.

The condition (13) required for the hypergeometric series to converge is merely the condition for the motion to be bounded [4], except for the case where P_n has no local maximum (n even and $\lambda_n > 0$). In that case it can be proved [7] that the hypergeometric series representing $J(a)$ may be analytically continued [9].

The incomplete integral $H(a, X)$ of Eq. (2) represents the time t to go from the origin to the current point on the X-axis [5]. For P_n given by Eq. (5), the time t is expressed by the series (18) and may be written in the form

$$(22) \qquad t = X \sum_{p=0}^{\infty} \gamma_p X^p,$$

where the coefficients γ_p are themselves convergent series depending on a, λ_2, and λ_n.

This means that we are able, at least in principle, to give the solution $X(t)$ of the motion. But in fact, it is not so interesting to have the solution as a power series of t, especially when it is periodic (one would have to know too many terms of the series). Nevertheless, Eq. (22) may be useful for computation, because the position X at a given time t is not necessarily the most important thing we wish to know. It may be more interesting to know at which time t a given point X of the trajectory is reached.

6. Conclusion. Incomplete hyperelliptic integrals are not so easy to handle as complete ones, and their applications are more limited. But their link with elapsed time in a mechanical anharmonic system gives them some computational importance. Accessorily, they have led us to meet again with old studies on algebraic equations and generalized hypergeometric series.

Appendix. *Link with Appell's and Kampé de Fériet's Works.* For the algebraic equation

$$(A\text{-}1) \qquad z^{n+1} - z + a = 0,$$

the following results are given in [3] and [1].

Among the $n + 1$ solutions of Eq. (A-1), denoted by z_j $(j = 1, 2, \ldots, n + 1)$, n are given by

$$(A\text{-}2) \qquad \varepsilon_j z_j = F_0(a^n) + \varepsilon_j A_1 a F_1(a^n) + \cdots + \varepsilon_j^{n-1} A_{n-1} a^{n-1} F_{n-1}(a^n)$$

with

$$(A\text{-}3) \qquad F_k(a^n) = {}_n F_{n-1} \left[\begin{array}{c} \dfrac{k-1}{n} + \dfrac{1}{n+1}, \dfrac{k-1}{n} + \dfrac{2}{n+1}, \ldots, \dfrac{k-1}{n} + \dfrac{n}{n+1} \\ \dfrac{k+1}{n}, \dfrac{k+2}{n}, \ldots, \dfrac{\widehat{n}}{n}, \ldots, \dfrac{k+n}{n} \end{array} ; \dfrac{(n+1)^{n+1}}{n^n} a^n \right]$$

and

$$(A\text{-}4) \qquad A_k = -\frac{1}{n} \frac{\left[\frac{k-1}{n} + 1, k - 1 \right]}{k!}.$$

ε_j are the nth roots of unity. The hat on $\dfrac{\widehat{n}}{n}$ indicates that the argument 1 must be ruled out and the brackets in (A-4) denote the Pochhammer symbol

$$(A\text{-}5) \qquad [a, b] = a(a + 1) \cdots (a + b - 1).$$

The sum of the $(n + 1)$ roots of (A-1) being zero, the last one is given by

$$(A\text{-}6) \qquad z_{n+1} = -\sum_{j=1}^{n} z_j = -nA_1 a F_1(a^n) = a F_1(a^n).$$

The series (A-3) converge only for $|a| < n/(n+1)^{(n+1)/n}$, but transformation formulae for generalized hypergeometric functions allow the solutions of Eq. (A-1) to be expressed by hypergeometric series for every a [1].

Applying these results to the case $n = 5$, it may be proved that Eq. (A-1) has three real solutions, the smallest one in absolute value being

$$(A\text{-}7) \qquad z_5 = a \, {}_4F_3 \left[\begin{matrix} \dfrac{1}{5}, \dfrac{2}{5}, \dfrac{3}{5}, \dfrac{4}{5} \\ \dfrac{2}{4}, \dfrac{3}{4}, \dfrac{5}{4} \end{matrix} ; \ \dfrac{5^5}{4^4} a^4 \right].$$

The propositions established in Section 4 allow us to express z_5 as a squared generalized hypergeometric series,

$$(A\text{-}8) \qquad z_5 = a \left[{}_5F_4 \left[\begin{matrix} \dfrac{1}{10}, \dfrac{3}{10}, \dfrac{5}{10}, \dfrac{7}{10}, \dfrac{9}{10} \\ \dfrac{3}{8}, \dfrac{5}{8}, \dfrac{7}{8}, \dfrac{9}{8} \end{matrix} ; \ \dfrac{5^5}{4^4} a^4 \right] \right]^2.$$

It is not easy to prove directly the exact equivalence of (A-7) and (A-8). The convergence condition $|a| < 4/5^{5/4}$ is the same in both cases and may be removed by analytic continuation.

Laboratoire de Physique Appliquée
Université de Pau et des Pays de l'Adour
F 64000 Pau, France

1. P. APPELL & J. KAMPÉ DE FÉRIET, *Fonctions Hypergéométriques et Hypersphériques, Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

2. G. BELLARDINELLI, "Sulla risoluzione delle equazioni algebriche mediante le funzioni ipergeometriche," *Rend. Reale Acad. Naz. Lincei Roma*, 5a seria, v. 30, 1921, pp. 208–217.

3. R. BIRKELAND, "Résolution de l'équation algébrique trinôme par des fonctions hypergéométriques supérieures," *C. R. Acad. Sci. Paris*, v. 171, 1920, pp. 778–783.

4. J. P. CODACCIONI & R. CABOZ, "Anharmonic oscillators and generalized hypergeometric functions," *J. Math. Phys.*, v. 25, 1984, pp. 2436–2438.

5. J. P. CODACCIONI & R. CABOZ, "Anharmonic oscillators revisited," *Internat. J. Non-Linear Mech.*, v. 20, 1985, pp. 291–295.

6. A. ERDÉLYI, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

7. J. F. LOISEAU, *Contribution à l'Étude des Solutions Exactes et Approchées de l'Oscillateur Anharmonique Monômial en Mécanique Classique*, Thèse d'Etat, Université de Pau et des Pays de l'Adour, Pau, 1986.

8. J. F. LOISEAU, J. P. CODACCIONI & R. CABOZ, "Hyperelliptic integrals and multiple hypergeometric series," *Math. Comp.*, v. 50, 1988, pp. 501–512.

9. Y. L. LUKE, *The Special Functions and Their Approximations*, Academic Press, New York, 1969.

10. HJ. MELLIN "Zur Theorie der trinomischen Gleichungen," *Ann. Acad. Sci. Fenn.*, Helsinfors v. 7, no. 7, 1915.

11. H. M. SRIVASTAVA & P. W. KARLSSON, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Ltd. (John Wiley & Sons), New York, 1985.

12. E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, New York, 1927.