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The Weighted Particle Method for Convection-Diffusion Equations
Part 1: The Case of an Isotropic Viscosity

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Abstract. The aim of this paper is to present and study a particle method for convection-diffusion equations based on the approximation of diffusion operators by integral operators and the use of a particle method to solve integro-differential equations described previously by the second author. The first part of the paper is concerned with isotropic diffusion operators, whereas the second part will consider the general case of a nonconstant matrix of diffusion. In the former case, the approximation of the diffusion operator is much simpler than in the general case. Furthermore, we get two possibilities of approximations, depending on whether or not the integral operator is positive.

1. Introduction. The particle method was first introduced to compute the flow of homogeneous incompressible and inviscid fluids (see Leonard [2]). In such a method the fluid is represented by pointwise vortices which travel with the fluid velocity. In the case of a viscous fluid, the particle method must take into account the diffusion effects. If the fluid is only slightly viscous, most of the classical methods become unstable and lead to unreliable results; it then seems useful to construct particle methods which are capable to treat diffusion terms. Applications of such methods can be found in fluid dynamics (e.g., the incompressible Navier-Stokes equation) and in the kinetic theory of plasma physics (e.g., the Fokker-Planck equation).

The random walk method gave a first answer to this problem. This method, which has numerous variants, is based on the introduction of a Monte-Carlo technique to add a probabilistic part to the motion of the particles. This method, introduced by Chorin [3], has been applied in various cases by Roberts [4] and Spalart [5] for example (see also the book of Duderstadt and Martin [6] and the references therein). The method which was proposed by Cottet, Huberson and Mas-Gallic in [7], [8] and [9] is based on a viscous splitting of the equation as studied by Beale and Majda in [10] and on the use of the Green kernel to obtain an exact treatment of the diffusion equation. This method is valid in the case of a small viscosity coefficient and relies on the use of the Gaussian function, which may lead to an exaggerated cost; the one proposed here presents several advantages with respect to the splitting method. The first advantage is the possibility of using other functions than the Gaussian, rational fractions for example; other advantages are its conservation property and the possibility of considering nonsmall or nonconstant viscous coefficients. Furthermore, the proof of convergence of the method

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is simpler in the present case. An alternate method inspired by the work of Gingold and Monaghan [11] was studied in [12] and, by the choice of a suitable cutoff function (see Section 5), can be viewed as a particular case of the present method. In this paper, we give a detailed presentation together with some extensions of a method first sketched in [13]. Finally, let us mention that another deterministic particle method, which can be viewed as a compromise between the Monte-Carlo method and the viscous splitting method, can be found in [21].

The basic idea of the present method is very different from the one used in the random walk method. In addition to its position and its volume, a third degree of freedom, called the strength, is associated with each particle. As usual, the time evolution of both the position and the volume of the particle is governed by the convective part of the equation, while the time evolution of the strength is governed by the diffusion part of the equation. A comparison of this method with the random walk method has been made by Choquin and Lucquin-Desreux in [14]. The acceleration technique introduced by Beale in [15] is also used. Other numerical experiments have been done, see [16], [17] and [18].

In Part 1 of this paper we shall restrict ourselves to the case of a scalar diffusion operator, whereas in Part 2 the general case of a diffusion matrix will be treated, and different kinds of approximations will be studied. We must mention that in [19] a first attempt to discretize a matrix of diffusion was studied. Some comments on this method are given in Part 2.

From now on, given a vector field $\mathbf{a}\colon (x,t) \in \mathbb{R}^n \times \mathbb{R} \to \mathbf{a}(x,t) \in \mathbb{R}^n$, a function $a_0\colon (x,t) \in \mathbb{R}^n \times \mathbb{R} \to a_0(x,t) \in \mathbb{R}$ and a viscosity coefficient $b\colon (x,t) \in \mathbb{R}^n \times \mathbb{R} \to b(x,t) \in \mathbb{R}_+$, we shall focus our attention on the resolution of the following equation

$$\frac{\partial f}{\partial t} + \operatorname{div}(\mathbf{a} f) + a_0 f - \nu \operatorname{div}(b \operatorname{grad} f) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$f(\cdot, 0) = f_0 \quad \text{on } \mathbb{R}^n.$$

We assume that $\mathbf{a}$, a_0 and b are sufficiently smooth. We denote by D the diffusion operator

$$D(t)f = \operatorname{div}(b(\cdot,t) \operatorname{grad} f), \tag{1.2}$$

$$b \in L^\infty(Q_T) \tag{1.3}$$

with $Q_T = \mathbb{R}^n \times (0,T)$.

The method consists in first replacing the diffusion operator by an integral operator and then solving the integro-differential equation by a now classical particle method. Let $\varepsilon > 0$ be a real number and $\sigma^\varepsilon\colon (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T] \to \sigma^\varepsilon(x,y,t)$ be a function depending on the parameter ε such that

$$\sigma^\varepsilon \in L^\infty((0,T) \times \mathbb{R}^n_x; L^1(\mathbb{R}^n_y)) \cap L^\infty((0,T) \times \mathbb{R}^n_y; L^1(\mathbb{R}^n_x));$$

the integral operator is defined for $f \in L^\infty(\mathbb{R}^n)$ by

$$Q^\varepsilon(t)f(x) = \int_{\mathbb{R}^n} (\sigma^\varepsilon(x,y,t)f(y) - \sigma^\varepsilon(y,x,t)f(x))\, dy. \tag{1.4}$$

If σ^ε satisfies some moment conditions, the operator $Q^\varepsilon(t)$ is an approximation of $D(t)$, and the solution of

$$\text{(1.5)} \qquad \begin{cases} \dfrac{\partial f}{\partial t} + \operatorname{div}(\mathbf{a}f) + a_0 f - \nu Q^\varepsilon(t) f = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ f(\cdot, 0) = f_0 \quad \text{on } \mathbb{R}^n \end{cases}$$

is an approximation to the solution of (1.1).

Now, Eq. (1.5) is solved by a particle method, which means that the exact solution is approximated by a combination of Dirac measures, the particles, the positions of which evolve in time. Let a quadrature rule be given by a set of indices $\mathcal{J} \subset \mathbb{Z}^n$, points $x_k^0 \in \mathbb{R}^n$ and weights $\omega_k^0 > 0$, for $k \in \mathcal{J}$. The points x_k^0 are the locations of the particles and the weights ω_k^0 are the volumes of the particles.

From now on we assume that $\mathbf{a} \in (L^\infty(0,T;\operatorname{Lip}(\mathbb{R}^n)))^n$. Then, we define travelling particles which follow the integral curves of the vector field $\mathbf{a}$ by setting $x_k(t) = X(t; x_k^0, 0)$, which is the solution of

$$\text{(1.6)} \qquad \begin{cases} \dfrac{dX}{dt}(t) = \mathbf{a}(X(t), t), \\ X(0) = x_k^0. \end{cases}$$

We denote by $J(t;\xi,s)$ the Jacobian determinant of the change of variable $\xi \to X(t;\xi,s)$, and we set $\omega_k(t) = J(t; x_k^0, 0)\omega_k^0$. Then, using the location and the volume of the particles respectively as nodes and weights, we obtain a quadrature formula

$$\int_{\mathbb{R}^n} g(x)\, dx \simeq \sum_{k \in \mathcal{J}} \omega_k(t) g(x_k(t)).$$

From the definition of $Q^\varepsilon(t)$ and the above quadrature formula, we derive the definition of a discrete version $Q_h^\varepsilon(t)$ of the operator Q^ε,

$$\text{(1.7)} \qquad Q_h^\varepsilon(t) g = \sum_{l \in \mathcal{J}} \omega_l(t)(\sigma^\varepsilon(x, x_l(t), t) g(x_l(t)) - \sigma^\varepsilon(x_l(t), x, t) g(x)).$$

This operator maps $C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and the particle method consists in looking for a measure f_h of the form

$$\text{(1.8)} \qquad f_h(t) = \sum_{k \in \mathcal{J}} \omega_k(t) f_k(t) \delta(x - x_k(t)).$$

The coefficients $f_k(t)$ are solutions of ordinary differential equations obtained thanks to an analogue of (1.7). The method consists in solving the following system, which gives the positions, the volumes and the strengths of the particles:

$$\text{(1.9)} \qquad \begin{cases} \dfrac{dx_k}{dt}(t) = \mathbf{a}(x_k(t), t), \\ \dfrac{d\omega_k}{dt}(t) = \operatorname{div} \mathbf{a}(x_k(t), t)\omega_k(t), \\ \dfrac{df_k}{dt}(t) + (\operatorname{div} \mathbf{a}(x_k(t), t) + a_0(x_k(t), t)) f_k(t) \\ \quad = \nu\varepsilon^{-2} \displaystyle\sum_{l \in \mathcal{J}} \omega_l(t)\{\sigma^\varepsilon(x_k(t), x_l(t), t) f_l(t) - \sigma^\varepsilon(x_l(t), x_k(t), t) f_k(t)\}, \\ x_k(0) = x_k^0, \quad \omega_k(0) = \omega_k^0, \quad f_k(0) = f_0(x_k^0). \end{cases}$$

The equations satisfied by the positions and the volumes are classical, which is not the case for the ordinary differential equation which gives the strengths.

When computing the flow of a slightly viscous incompressible fluid, a great advantage of this method is that it enables computations for high Reynolds numbers. On the other hand, in the general case, in order to derive L^∞ estimates which do not depend on ε, we shall need to impose a lower bound on ε which depends on the viscosity ν. In fact, for any fixed positive number C_s, if the following inequality is satisfied by the viscosity ν and the scaling parameter ε (see the beginning of Section 2)

$$\nu \le C_s \varepsilon^2, \tag{1.10}$$

the integro-differential equation (1.5) is stable in L^∞ with respect to the initial condition, and so is the numerical scheme. This condition ensures that the numerical viscosity ε^2 is not too small compared to the physical viscosity ν. As shown in Part 2, this method may be generalized to the case of a nonscalar diffusion operator.

On the other hand, if the viscosity is constant ($b = 1$), choosing a nonnegative kernel σ assures the stability of the method without assuming that condition (1.10) is verified. In that particular case, the integral operator is positive and the integro-differential equation, as well as the numerical scheme, possesses a maximum principle property. In that case, we obtain L^∞ estimates which do not depend on the norm of the kernel, and thus which do not depend on ε. Choosing a positive cutoff function may also be interesting in the study of stationary solutions, since the solution of Eq. (1.5) and the regularized solution of the scheme have the same asymptotic time behavior as the solution of (1.1).

The paper is organized as follows. In Section 2, by means of Taylor expansions, we prove the convergence of the integral operator Q^ε towards the diffusion operator D. We study the convergence of the solution of problem (1.5) towards the solution of (1.1). For this, we need stability results for both Eqs. (1.1) and (1.5). Section 3 is devoted to the study of the particle method. We establish the stability and the convergence of the method in L^∞. In Section 4, the case of a constant viscosity is emphasized, and we assume that the kernel is nonnegative. First we prove that the integral operator is positive and establish a stability property of the integro-differential equation; we derive L^∞ estimates and we prove the L^∞ stability of the scheme with the same kind of arguments. Finally, in Section 5, some examples of kernels are given and some comments are made on the time-discretized algorithm.

Let us now introduce some notations. By $\mathrm{Lip}(\mathbb{R}^n)$ we denote the space of Lipschitz continuous functions on $\mathbb{R}^n$ and by $C_0^0(\mathbb{R}^n)$ the space of compactly supported continuous functions. As usual, $W^{k,\infty}(\mathbb{R}^n)$ is the classical Sobolev space provided with the classical seminorm and norm

$$|g|_{k,\infty} = \sup_{|\alpha|=k, x\in\mathbb{R}^n} \mathrm{ess}\, |\partial^\alpha g(x)|, \qquad \|g\|_{k,\infty} = \sup_{0\le p\le k} |g|_{p,\infty}.$$

By $C, C', \ldots$ we shall denote positive constants which do not depend on the discretization parameters to be introduced. We shall use the following standard notation: for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}, \quad |\alpha| = \sum_{i=1}^n \alpha_i, \quad \alpha! = \prod_{i=1}^n \alpha_i!, \qquad \alpha + \beta = (\alpha_i + \beta_i)_{1 \le i \le n}.$$

The canonical basis of $\mathbb{R}^n$ will be denoted by $(e_1, \dots, e_n)$.

Let $\mathcal{J} \subset \mathbb{Z}^n$; the space $l^\infty(\mathcal{J})$ is the space of bounded sequences provided with its usual norm: for $\overline{g} = (g_l)_{l \in \mathcal{J}}$ in $l^\infty(\mathcal{J})$,

$$\|\overline{g}\|_\infty = \sup_{l \in \mathcal{J}} |g_l| < +\infty. \tag{1.11}$$

By S^{n-1} we denote the unit sphere of $\mathbb{R}^n$ and by $\operatorname{meas} S^{n-1}$ the total mass of its measure.

From now on we shall assume that the kernel σ is symmetric. This condition is not required for the subsequent analysis, at least in Sections 2 and 3, but the interest in considering nonsymmetric kernels seems to be rather academic.

2. Approximation of the Convection-Diffusion Equation by an Integro-Differential Equation. If the kernel σ^ε satisfies some moment conditions, we shall prove that the integro-differential equation is an approximation of the convection diffusion equation. This kind of result is to be compared with the classical plasma physics approximation of the Boltzmann equation by the Fokker-Planck equation. This approximation is called small angles collision approximation and occurs in the case of collisional plasmas in which the collisions are elastic.

Let us fix a constant $C_s > 0$; we shall assume in the remainder of this section that the parameter ε satisfies the stability inequality (1.10). Moreover, we assume that the kernel σ^ε has the following form:

$$\sigma^\varepsilon(x, y, t) = \frac{1}{\varepsilon^2} \mu(x, y, t) \eta_\varepsilon(x - y), \tag{2.1}$$

where

$$\begin{cases} \mu \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n \times [0, T]), \\ \mu(x, y, t) = \mu(y, x, t) \quad \text{for any } x, y \in \mathbb{R}^n, t \in (0, T), \end{cases} \tag{2.2}$$

$$\begin{cases} \eta_\varepsilon(x) = \dfrac{1}{\varepsilon^n} \eta\left(\dfrac{x}{\varepsilon}\right), \\ \eta \in L^1(\mathbb{R}^n), \\ \eta(-x) = \eta(x) \quad \text{for any } x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

The time dependence of the functions will be omitted in the next proposition, since t is only a parameter.

PROPOSITION 1. *Assume that the function η satisfies the moment conditions*

$$\int_{\mathbb{R}^n} x^\alpha \eta(x)\, dx = \begin{cases} 0, & \forall \alpha \in \mathbb{N}^n, \alpha \ne 2e_i, 1 \le |\alpha| \le r + 1, \\ 2, & \textit{if } \alpha = 2e_i, i \in \{1, \dots, n\}, \end{cases} \tag{2.4}$$

$$\int_{\mathbb{R}^n} |x|^{r+2} |\eta(x)|\, dx < +\infty. \tag{2.5}$$

Assume also that $\mu \in L^\infty(\mathbb{R}^n_x, W^{r+1,\infty}(\mathbb{R}^n_y))$ *and is such that*

$$\mu(x,x) = b(x) \tag{2.6}$$

for any $x \in \mathbb{R}^n$. *Then there exists a constant* $C > 0$ *such that*

$$\|Dg - Q^\varepsilon g\|_{0,\infty} \le C\varepsilon^r \|g\|_{r+2,\infty}, \tag{2.7}$$

for any function $g \in W^{r+2,\infty}(\mathbb{R}^n)$.

Proof. Using a Taylor expansion of g with integral remainder, and applying it in (1.4), we obtain

$$\begin{aligned} Q^\varepsilon g(x) = \varepsilon^{-2} \Bigg\{ &\sum_{|\alpha|=1}^{r+1} \frac{1}{\alpha!} \partial^\alpha g(x) \int_{\mathbb{R}^n} (y-x)^\alpha \sigma^\varepsilon(x,y)\, dy \\ &+ (r+2) \sum_{|\alpha|=r+2} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{r+1} \\ &\times \int_{\mathbb{R}^n} \partial^\alpha g(x+\theta(y-x))(y-x)^\alpha \sigma^\varepsilon(x,y)\, dy\, d\theta \Bigg\}. \end{aligned} \tag{2.8}$$

Expanding μ by Taylor's formula and substituting it in (2.8), we can write

$$Q^\varepsilon g(x) = \sum_{m=1}^{r+1} Q^\varepsilon_m g(x) + R^\varepsilon g(x), \tag{2.9}$$

where Q^ε_m is a differential operator of order m and R^ε is the remainder. Setting

$$Z^\varepsilon_\alpha = \int_{\mathbb{R}^n} (y-x)^\alpha \eta_\varepsilon(x-y)\, dy,$$

we have

$$Q^\varepsilon_m g(x) = \varepsilon^{-2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \partial^\alpha g(x) \left\{ \mu(x,x) Z^\varepsilon_\alpha + \sum_{|\beta|=1}^{r+1-|\alpha|} \frac{1}{\beta!} \partial^\beta_y \mu(x,x) Z^\varepsilon_{\alpha+\beta} \right\}$$

and

$$\begin{aligned} R^\varepsilon g(x) = \varepsilon^{-2} \Bigg\{ &\sum_{|\alpha|=1}^{r+1} \frac{r+2-|\alpha|}{\alpha!} \partial^\alpha g(x) \sum_{|\beta|=r+2-|\alpha|} \frac{1}{\beta!} \\ &\times \int_0^1 \int_{\mathbb{R}^n} (1-\tau)^{r+1-|\alpha|} \eta_\varepsilon(x-y) \partial^\beta_y \mu(x, x+\tau(y-x))(y-x)^{\alpha+\beta}\, dy\, d\tau \\ &+ (r+2) \sum_{|\alpha|=r+2} \frac{1}{\alpha!} \int_0^1 \int_{\mathbb{R}^n} (1-\theta)^{r+1} \partial^\alpha g(x+\theta(y-x)) \\ &\times \eta_\varepsilon(x-y)\mu(x,y)(y-x)^\alpha\, dy\, d\theta \Bigg\}. \end{aligned}$$

The moment conditions (2.4) give

$$Z^\varepsilon_\alpha = \begin{cases} 0, & \forall \alpha \in \mathbb{N}^n, \alpha \ne 2e_i, 1 \le |\alpha| \le r+1, \\ 2\varepsilon^2, & \text{if } \alpha = 2e_i, i \in \{1,\dots,n\}. \end{cases}$$

As a consequence of the symmetry of μ and relation (2.6), we have

$$\frac{\partial \mu}{\partial y_i}(x,x) = \frac{1}{2}\frac{\partial b}{\partial x_i}(x).$$

Finally, thanks to the conditions (2.4), we find

$$Q^\varepsilon_m g(x) = \begin{cases} 0, & \text{if } m=0 \text{ or } 3 \le m \le r+1, \\ b(x)\Delta g(x), & \text{if } m = 2, \\ \nabla g(x)\cdot\nabla b(x), & \text{if } m=1, \end{cases} \tag{2.10}$$

and

$$|R^\varepsilon g(x)| \le C\varepsilon^r \|\eta\|_{0,1}\|\mu\|_{L^\infty(\mathbb{R}^n_x, W^{r+1,\infty}(\mathbb{R}^n_y))}\|g\|_{r+2,\infty}. \tag{2.11}$$

Combining (2.9), (2.10) and (2.11) leads to the desired result. □

Remarks. 1. From the definition of R^ε it is easy to check that arguing as previously, the following inequality is true for any function $g \in W^{r+2,p}(\mathbb{R}^n)$:

$$\|R^\varepsilon g\|_{0,p} \le C\varepsilon^r \|\eta\|_{0,1}\|\mu\|_{L^\infty(\mathbb{R}^n_x, W^{r+1,\infty}(\mathbb{R}^n_y))}\|g\|_{r+2,p}.$$

Then concluding that inequality (2.7) also holds in any L^p is straightforward,

$$\|Dg - Q^\varepsilon g\|_{0,p} \le C\varepsilon^r \|g\|_{r+2,p}. \tag{2.12}$$

2. If the function η is not even or the function μ is not symmetric (one has to consider the operator Q^ε in the form (1.4)), an analogous result can be obtained, assuming that $\mu \in L^\infty(\mathbb{R}^n_x, W^{r+2,\infty}(\mathbb{R}^n_y)) \cap L^\infty(\mathbb{R}^n_y, W^{r+2,\infty}(\mathbb{R}^n_x))$ and that

$$\frac{\partial^2 \mu}{\partial y_i^2}(x,x) = \frac{\partial^2 \mu}{\partial x_i^2}(x,x). \tag{2.13}$$

A Taylor expansion of the function μ up to order $r+2$ must be used to derive this result. □

The following classical result is mainly based on the maximum principle property of parabolic equations. Since we are interested in slightly viscous problems, from now on we shall assume that $\nu \le 1$.

PROPOSITION 2. *Assume that* $\mathbf{a} \in (L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n)))^n$, $b \in L^\infty(Q_T)$ *and* $a_0 \in L^\infty(Q_T)$. *If* $f_0 \in L^\infty(\mathbb{R}^n)$, *problem* (1.1) *has a unique solution* f *in* $L^\infty(Q_T)$ *and*

$$\|f(\cdot,t)\|_{0,\infty} \le \exp(T\|\operatorname{div}\mathbf{a} + a_0\|_{0,\infty})\|f_0\|_{0,\infty}. \tag{2.14}$$

If moreover $\mathbf{a} \in (L^\infty(0,T;W^{m+1,\infty}(\mathbb{R}^n)))^n$, b, $a_0 \in L^\infty(0,T;W^{m,\infty}(\mathbb{R}^n))$ *and* $f_0 \in W^{m,\infty}(\mathbb{R}^n)$, *the solution belongs to* $L^\infty(0,T;W^{m,\infty}(\mathbb{R}^n))$ *and there exists a constant* $C = C(T, a_0, \mathbf{a}, \nabla b) > 0$ *such that*

$$\|f(\cdot,t)\|_{m,\infty} \le C\|f_0\|_{m,\infty}, \qquad 0 \le t \le T. \tag{2.15}$$

Let us point out that in inequality (2.15) the constant may depend on ν if b is not constant. Nevertheless, the assumption that $\nu \le 1$, ensures that the growth of the solution remains bounded independently of ν.

We now present a stability result for integro-differential equations which will be useful when comparing the solutions of problem (1.1) and (1.5). Although this result is classical, we need a precise bound, and so give a proof.

Consider the following problem

$$\begin{cases} \dfrac{\partial f}{\partial t} + \operatorname{div}(\mathbf{a}f) + a_0 f - \nu Q(t)f = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ f(\cdot,0) = f_0 \quad \text{on } \mathbb{R}^n, \end{cases} \tag{2.16}$$

where

$$Q(t)f(x) = \int_{\mathbb{R}^n} (\sigma(x,y,t)f(y) - \sigma(y,x,t)f(x))\,dy. \tag{2.17}$$

We assume that

$$\int_{\mathbb{R}^n} \sigma(x,y,t)\,dy = \int_{\mathbb{R}^n} \sigma(y,x,t)\,dy, \tag{2.18}$$

which ensures that $1 \in \operatorname{Ker} Q$ and

$$\sigma \in L^\infty(\mathbb{R}^n_y \times (0,T); L^1(\mathbb{R}^n_x)) \cap L^\infty(\mathbb{R}^n_x \times (0,T); L^1(\mathbb{R}^n_y)). \tag{2.19}$$

We set

$$K(t)f(x) = \int_{\mathbb{R}^n} \sigma(x,y,t)f(y)\,dy \tag{2.20}$$

and

$$\lambda(x,t) = \int_{\mathbb{R}^n} \sigma(y,x,t)\,dy, \tag{2.21}$$

so that $Q(t)f = K(t)f - \lambda f$. The operators K and Q map L^∞ into itself and we have

$$\|K(t)f\|_{0,\infty} \le \|K\|\,\|f\|_{0,\infty}, \tag{2.22}$$

where

$$\|K\| = \sup_{\substack{x\in\mathbb{R}^n \\ t\in(0,T)}} \int_{\mathbb{R}^n} |\sigma(x,y,t)|\,dy. \tag{2.23}$$

The following proposition states a stability result which depends obviously on the norm of the kernel.

PROPOSITION 3. *Assume that* $\mathbf{a} \in (L^\infty(0,T; W^{1,\infty}(\mathbb{R}^n)))^n$ *and* $a_0 \in L^\infty(Q_T)$. *If* $f_0 \in L^\infty(\mathbb{R}^n)$, *problem* (2.16) *has a unique solution* f *in* $L^\infty(Q_T)$ *and*

$$\|f(\cdot,t)\|_{0,\infty} \le \|f_0\|_{0,\infty} \exp(T\{\|a_0 + \operatorname{div}\mathbf{a} + \nu\lambda\|_{0,\infty} + 2\nu\|K\|\}). \tag{2.24}$$

Let m *be an integer. Assume now that* $\sigma = \sigma^\varepsilon$ *is given by* (2.1) *with* μ *and* η *satisfying the hypotheses of Proposition* 1,

$$\eta \in W^{m,1}(\mathbb{R}^n) \quad \textit{and} \quad \mu \in L^\infty((0,T); W^{m,\infty}(\mathbb{R}^n_x \times \mathbb{R}^n_y)).$$

Assume that $\mathbf{a} \in L^\infty(0,T; W^{m+1,\infty}(\mathbb{R}^n))$, $a_0 \in L^\infty(0,T; W^{m,\infty}(\mathbb{R}^n))$. *If* $f_0 \in W^{m,\infty}(\mathbb{R}^n)$, *the solution* f^ε *of problem* (1.5) *belongs to* $L^\infty(0,T; W^{m,\infty}(\mathbb{R}^n))$ *and there exists a constant* $C = C(T, a_0, \mathbf{a}, \mu, \eta) > 0$ *such that*

$$\|f(\cdot,t)\|_{m,\infty} \le C\nu\varepsilon^{-2}\|f_0\|_{m,\infty}. \tag{2.25}$$

Proof. We derive inequality (2.24) by a fixed point iteration technique. Let us first remark that setting

$$\alpha = \|a_0 + \operatorname{div}\mathbf{a} + \nu\lambda\|_{0,\infty} + \nu\|K\|,$$

the function $g = fe^{-\alpha t}$ is a solution of

$$\frac{\partial g}{\partial t} + \mathbf{a} \cdot \nabla g + b_0 g - \nu K g = 0, \tag{2.26}$$

where $b_0 = a_0 + \operatorname{div} \mathbf{a} + \nu\lambda + \alpha$. We point out that

$$b_0 = a_0 + \operatorname{div} \mathbf{a} + \nu\lambda + \|a_0 + \operatorname{div} \mathbf{a} + \nu\lambda\|_{0,\infty} + \nu\|K\| \geq \nu\|K\| \geq 0.$$

Let $f \in L^\infty(Q_T)$, and let Φ be the mapping defined by $g = \Phi f$, where g is the solution of

$$\begin{cases} \dfrac{\partial g}{\partial t} + \mathbf{a} \cdot \nabla g + b_0 g = \nu K f, \\ g(\cdot, 0) = f_0. \end{cases}$$

The mapping Φ is explicitly given by

$$\begin{aligned} (\Phi f)(x,t) &= f_0(X(0;x,t)) \exp\left(-\int_0^t b_0(X(\tau;x,t),\tau)\,d\tau\right) \\ &\quad + \nu \int_0^t (Kf)(X(\tau;x,t),\tau) \exp\left(-\int_\tau^t b_0(X(\sigma;x,t),\sigma)\,d\sigma\right) d\tau, \end{aligned} \tag{2.27}$$

where the functions $X(\tau;x,t)$ have been defined by (1.6). Φ maps $L^\infty(Q_T)$ into itself, and

$$\begin{aligned} &|(\Phi f - \Phi g)(x,t)| \\ &\quad \leq \nu\|f - g\|_{0,\infty}\|K\| \int_0^t \exp\left(-\int_\tau^t b_0(X(\sigma;x,t),\sigma)\,d\sigma\right) d\tau. \end{aligned}$$

Since $b_0 \geq \nu\|K\|$, we have

$$\nu\|K\| \int_0^t \exp\left(-\int_\tau^t b_0(X(\sigma;x,t),\sigma)\,d\sigma\right) d\tau < 1,$$

which proves that Φ is a contraction. Then, Φ has a unique fixed point g^* which is the solution of (2.26) in $L^\infty(Q_T)$ and the iterative sequence defined by

$$g^{k+1} = \Phi g^k, \qquad g^0 = 0, \tag{2.28}$$

converges to g^*. We verify easily that for any $g \in L^\infty(Q_T)$

$$\|\Phi g\|_{0,\infty} \leq \|f_0\|_{0,\infty} + (1 - \exp(-\nu\|K\|T))\|g\|_{0,\infty}. \tag{2.29}$$

Setting $l = 1 - \exp(-\nu\|K\|T)$ yields

$$\|g^{k+1}\|_{0,\infty} \leq \|f_0\|_{0,\infty} + l\|g^k\|_{0,\infty} \leq \frac{1 - l^{k+1}}{1 - l}\|f_0\|_{0,\infty},$$

and finally

$$\|g^*\|_{0,\infty} \leq \exp(\nu\|K\|T)\|f_0\|_{0,\infty}. \tag{2.30}$$

By construction, the function

$$f^* = g^* \exp(t(\|a_0 + \operatorname{div} \mathbf{a} + \nu\lambda\|_{0,\infty} + \nu\|K\|))$$

is the solution of problem (2.16), and inequality (2.30) leads to (2.24). Now, let us consider the case where the kernel σ is defined by (2.1):

$$\sigma(x,y,t) = \varepsilon^{-2}\mu(x,y,t)\eta_\varepsilon(x - y).$$

We have

$$\|K\| \le C\varepsilon^{-2}, \tag{2.31}$$

where the constant C depends on μ and η. Let $\alpha \in \mathbb{N}^n$, $|\alpha| \ge 1$. Integrating by parts, we easily verify that we can write

$$\partial^\alpha(Q(t)f) = \partial^\alpha(K(t)f) - \partial^\alpha(\lambda f) = Q(t)(\partial^\alpha f) + R_\alpha(t)f,$$

where the operator R_α is a linear integro-differential operator such that

$$\begin{aligned}\|R_\alpha(t)f\|_{0,\infty} &\le C\varepsilon^{-2}\|\mu\|_{|\alpha|,\infty}\|\eta_\varepsilon\|_{0,1}\|f\|_{|\alpha|-1,\infty} \\ &\le C'\varepsilon^{-2}\|f\|_{|\alpha|-1,\infty},\end{aligned}$$

where the constant C' depends on μ and η. Formally differentiating the equation, we get that the function $\partial^\alpha f$ is the solution of Eq. (2.16) with a right-hand side term which contains the derivatives of f of lower order. The proof of the existence of a solution in $L^\infty(0,T;W^{m,\infty}(\mathbb{R}^n))$ and of the estimate (2.25) then follows by induction. □

If, moreover, ν and ε satisfy inequality (1.10), we have

$$\|f(\cdot,t)\|_{m,\infty} \le CC_s\|f_0\|_{m,\infty}. \tag{2.32}$$

Let f and f^ε be the respective solutions of (1.1) and (1.5). The following result holds.

THEOREM 1. *Assume that* $\mathbf{a} \in (L^\infty(0,T;W^{r+3,\infty}(\mathbb{R}^n)))^n$, *and that* $b, a_0 \in L^\infty(0,T;W^{r+2,\infty}(\mathbb{R}^n))$. *Assume also that* σ^ε *satisfies the hypotheses of Proposition 1 with* $r \ge 2$, *and that condition* (1.10) *is satisfied. Then there exists a constant* $C = C(T, C_s, a_0, \mathbf{a}, b, \eta, \mu) > 0$ *such that for any function* $f_0 \in W^{r+2,\infty}(\mathbb{R}^n)$

$$\|(f - f^\varepsilon)(\cdot,t)\|_{0,\infty} \le C\nu\varepsilon^r\|f_0\|_{r+2,\infty}. \tag{2.33}$$

Proof. Setting $g = f - f^\varepsilon$, we have

$$\begin{cases} \dfrac{\partial g}{\partial t} + \operatorname{div}(\mathbf{a}, g) + a_0 g - \nu Q^\varepsilon(t)g = \nu(D - Q^\varepsilon(t))f, \\ g(\cdot,0) = 0. \end{cases} \tag{2.34}$$

Applying Proposition 3, we get

$$\|g(\cdot,t)\|_{0,\infty} \le C\nu\int_0^t \|(D - Q^\varepsilon(\tau))f\|_{0,\infty}\,d\tau, \tag{2.35}$$

where, in consequence of (2.32), the constant C depends on C_s. Then, Proposition 1 yields

$$\|g(\cdot,t)\|_{0,\infty} \le C\nu\varepsilon^r\|f\|_{L^\infty(0,T;W^{r+2,\infty}(\mathbb{R}^n))}, \tag{2.36}$$

where C depends again on $T, C_s, \mathbf{a}, a_0, \mu$ and η, and the theorem follows in view of Proposition 2. □

3. Particle Approximation. Now, we come to the discretization method and we fix an initial distribution of particles. We choose a set of indices $\mathcal{J} \subset \mathbb{Z}^n$, a set of points $x_k^0 \in \mathbb{R}^n$ and a set of real numbers $\omega_k^0 > 0$, $k \in \mathcal{J}$. As mentioned in the introduction, we define the evolution in time of the particles by setting

$$x_k(t) = X(t; x_k^0, 0), \qquad \omega_k(t) = \omega_k^0 J(t; x_k^0, 0), \tag{3.1}$$

where the functions X are defined by (1.6).

In order to avoid technical difficulties, we suppose that the initial distribution of particles is uniform and that η is compactly supported. That is, given a parameter $h > 0$, we have

$$\mathcal{J} = \mathbb{Z}^n, \quad x_k^0 = kh, \quad \omega_k^0 = h^n. \tag{3.2}$$

While our analysis could be carried through in a more general case, this assumption leads to simpler proofs. Furthermore, in the case of the whole space, it has been proved (see, for example, P. A. Raviart [20]) that the order of the corresponding quadrature rule is only limited by the smoothness of the functions. Now we prove the convergence of the discrete operator towards the continuous operator.

PROPOSITION 4. *Let $m \geq n$ be an integer. Assume that*

$$\mathbf{a} \in (L^\infty(0,T; W^{m+1,\infty}(\mathbb{R}^n)))^n \quad \text{and} \quad a_0 \in L^\infty(0,T; W^{m,\infty}(\mathbb{R}^n)).$$

If $\eta \in W^{m,1}(\mathbb{R}^n)$ and if $\mu \in L^\infty((0,T); W^{m,\infty}(\mathbb{R}^n \times \mathbb{R}^n))$, there exists a constant $C = C(T, \mathbf{a}, a_0, \mu, \eta) > 0$ such that for any function $g \in W^{m,\infty}(\mathbb{R}^n)$ and for any $t \in [0,T]$

$$\|Q^\varepsilon(t)g - Q_h^\varepsilon(t)g\|_{0,\infty} \leq C \frac{h^m}{\varepsilon^{m+1}} \|g\|_{m,\infty}. \tag{3.3}$$

Proof. We have

$$\begin{aligned}(Q^\varepsilon(t)g &- Q_h^\varepsilon(t)g)(x) \\ &= \varepsilon^{-2}\Big\{ \int_{\mathbb{R}^n} \eta_\varepsilon(x-y)\mu(x,y,t)(g(y)-g(x))\,dy \\ &\qquad - \sum_{k\in\mathcal{J}} \omega_k(t)\eta_\varepsilon(x - x_k(t))\mu(x, x_k(t), t)(g(x_k(t)) - g(x))\Big\},\end{aligned}$$

and we recall the following result of [19]: for any function $\varphi \in W^{m,1}(\mathbb{R}^n)$, we have

$$\left| \int_{\mathbb{R}^n} \varphi(x)\,dx - \sum_{k\in\mathcal{J}} \omega_k(t)\varphi(x_k(t)) \right| \leq C h^m \|\varphi\|_{m,1}, \tag{3.4}$$

where the constant C depends on T and $\mathbf{a}$. Applying this inequality to the function

$$y \to \psi(x,y,t) = \eta_\varepsilon(x-y)\mu(x,y,t)(g(y)-g(x)),$$

we get

$$|(Q^\varepsilon(t)g - Q_h^\varepsilon(t)g)(x)| \leq C\varepsilon^{-2}h^m \|\psi(x,\cdot,t)\|_{m,1}. \tag{3.5}$$

Let us compute the derivatives of ψ with respect to y: with $\alpha \in \mathbb{N}^n$, we have

$$\begin{aligned}\partial_y^\alpha \psi(x,y) = &\sum_{\beta+\gamma<\alpha} \frac{\alpha!}{\beta!\gamma!(\alpha-\beta-\gamma)!}(-1)^{|\beta|}\partial^\beta \eta_\varepsilon(x-y)\partial_y^\gamma \mu(x,y,t)\partial^{\alpha-\beta-\gamma}g(y) \\ &+ \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(-1)^{|\beta|}\partial^\beta\eta_\varepsilon(x-y)\partial_y^\gamma\mu(x,y,t)(g(y)-g(x)),\end{aligned}$$

where the sums are respectively over the set of $\beta, \gamma \in \mathbb{N}^n$ such that $\beta_i + \gamma_i < \alpha_i$ for any i, $1 \leq i \leq n$, and over the set of β, γ such that $\beta_i + \gamma_i = \alpha_i$, for any $1 \leq i \leq n$. We note that

$$\begin{aligned}\int_{\mathbb{R}^n} &|\partial^\beta \eta_\varepsilon(x-y)\partial_y^\gamma\mu(x,y,t)(g(y)-g(x))|\,dy \\ &\leq |g|_{1,\infty}|\partial_y^\gamma \mu(x,\cdot,t)|_{0,\infty} \int_{\mathbb{R}^n} |y-x|\,|\partial^\beta\eta_\varepsilon(x-y)|\,dy.\end{aligned}$$

Using the change of variable $y = \varepsilon z + x$, we verify that

$$\int_{\mathbf{R}^n} |y-x|\,|\partial^\beta \eta_\varepsilon(x-y)|\,dy = \varepsilon^{1-|\beta|}\int_{\mathbf{R}^n} |z|\,|\partial^\beta \eta(z)|\,dz \le C\varepsilon^{1-|\beta|}|\eta|_{|\beta|,1}.$$

We obtain

$$|\psi(x,\cdot,t)|_{s,1} \le C\sum_{p+q=0}^{s-1} \varepsilon^{-p}|\eta|_{p,1}|\mu(x,\cdot,t)|_{q,\infty}|g|_{s-p-q,\infty} + C\sum_{p+q=s} \varepsilon^{1-p}|g|_{1,\infty}|\mu(x,\cdot,t)|_{q,\infty}|\eta|_{p,1}.$$

Finally,

$$|\psi(x,\cdot,t)|_{s,1} \le C\varepsilon^{1-s}\|g\|_{s,\infty}$$

and

$$\|\psi(x,\cdot,t)\|_{m,1} \le C\varepsilon^{1-m}\|g\|_{m,\infty},$$

where the constants C depend on T, μ and η. Combining this inequality with inequality (3.5) leads to the desired result. □

Remark. If $\eta \in W^{m,1}(\mathbf{R}^n)$ and

$$\mu \in L^\infty((0,T)\times\mathbf{R}^n_x; W^{m,\infty}(\mathbf{R}^n_y)) \cap L^\infty((0,T)\times\mathbf{R}^n_y; W^{m,\infty}(\mathbf{R}^n_x))$$

but either η is not even, or μ is not symmetric, we have

$$\|Q^\varepsilon(t)g - Q^\varepsilon_h(t)g\|_{0,\infty} \le C\frac{h^m}{\varepsilon^{m+2}}\|g\|_{m,\infty}. \tag{3.6}$$

The proof of this result is straightforward. □

We approximate the solution of (1.5) by the measure f_h,

$$f_h(t) = \sum_{k\in\mathcal{J}} \omega_k(t) f_k(t)\delta(x - x_k(t)),$$

where the coefficients f_k are solutions of the ordinary differential equation

$$\begin{aligned}\frac{df_k}{dt}(t) &+ (\operatorname{div}\mathbf{a}(x_k(t),t) + a_0(x_k(t),t))f_k(t)\\ &= \nu\varepsilon^{-2}\sum_{l\in\mathcal{J}} \omega_l(t)\eta_\varepsilon(x_k(t)-x_l(t))\mu(x_l(t),x_k(t),t)(f_l(t)-f_k(t)).\end{aligned} \tag{3.7}$$

We have the following bound for the local error.

PROPOSITION 5. *Let $m \ge n$ be an integer. Assume that $\mathbf{a}$, a_0, μ and η satisfy the hypotheses of Proposition 4. There exists a constant $C = C(T, \mathbf{a}, a_0, \mu, \eta) > 0$, such that for any function $f_0 \in W^{m,\infty}(\mathbf{R}^n)$ and any $t \in [0,T]$*

$$\sup_{k\in\mathcal{J}} |f^\varepsilon(x_k(t),t) - f_k(t)| \le C\nu\frac{h^m}{\varepsilon^{m+1}}\|f_0\|_{m,\infty}. \tag{3.8}$$

Proof. We set

$$\bar{e}(t) = (e_k(t))_{k\in\mathcal{J}}, \qquad e_k(t) = f^\varepsilon(x_k(t),t) - f_k(t), \tag{3.9}$$

and we have

$$\frac{de_k}{dt}(t) + \sum_{l \in \mathcal{J}} \beta_{kl}(t) e_l(t) = \nu \psi_k(t), \tag{3.10}$$

$$e_k(0) = 0, \tag{3.11}$$

where

$$\begin{aligned} \psi_k(t) &= ((Q^\varepsilon(t) - Q_h^\varepsilon(t)) f^\varepsilon)(x_k(t), t), \\ \beta_{kl}(t) &= -\nu\varepsilon^{-2}\omega_l(t)\eta_\varepsilon(x_k(t) - x_l(t))\mu(x_k(t), x_l(t), t) \quad \text{for } l \neq k, \\ \beta_{kk}(t) &= \operatorname{div} \mathbf{a}(x_k(t), t) + a_0(x_k(t), t) \\ &\quad + \nu\varepsilon^{-2} \sum_{l \neq k} \omega_l(t)\eta_\varepsilon(x_k(t) - x_l(t))\mu(x_k(t), x_l(t), t). \end{aligned}$$

There exist two constants $C > 0$ and $C' > 0$, independent of h, such that for any $t \in [0, T]$ and any k, l (see for example [18]),

$$C^{-1} h^n \leq \omega_k(t) \leq C h^n, \tag{3.12}$$

$$C' h \leq |x_k(t) - x_l(t)| \leq C'' h. \tag{3.13}$$

Then, for $k \neq l$,

$$\begin{aligned} |\beta_{kl}(t)| &\leq \nu\varepsilon^{-2}\omega_l(t)|\eta_\varepsilon(x_k(t) - x_l(t))\mu(x_k(t), x_l(t), t)| \\ &\leq C C_s h^n \varepsilon^{-n}, \end{aligned}$$

which proves that the coefficients β_{kl} are bounded for h and ε fixed. Let us now bound the diagonal coefficients. Since the support of η is bounded, the number of particles to take into account in the sum appearing in $\beta_{kk}(t)$ is bounded by the number of particles in a ball of radius $C\varepsilon$. Thanks to (3.13), this number is bounded by $C(\varepsilon/h)^n$, and we get

$$\begin{aligned} |\beta_{kk}(t)| &\leq \|\operatorname{div} \mathbf{a}\|_{0,\infty} + \|a_0\|_{0,\infty} + \nu\varepsilon^{-2} \sum_{l \neq k} \omega_l(t)|\eta_\varepsilon(x_k(t) - x_l(t))\mu(x_k(t), x_l(t), t)| \\ &\leq \|\operatorname{div} \mathbf{a}\|_{0,\infty} + \|a_0\|_{0,\infty} + C\nu\varepsilon^{-2} h^n \varepsilon^{-n} \left(\frac{\varepsilon}{h}\right)^n \leq C(1 + C_s). \end{aligned}$$

Then, setting $\overline{\psi}(t) = (\psi_k(t))_{k \in \mathcal{J}}$, there exists a constant $C > 0$, depending on C_s but neither on h, nor on ν or ε, such that

$$\frac{d}{dt}\|\overline{e}(t)\|_\infty \leq C\|\overline{e}(t)\|_\infty + \nu\|\overline{\psi}(t)\|_\infty.$$

Applying Gronwall's lemma, we obtain

$$\|\overline{e}(t)\|_\infty \leq \nu \int_0^t e^{C(t-\tau)} \|\overline{\psi}(\tau)\|_\infty \, d\tau. \tag{3.14}$$

The proof is now complete, since $\|\overline{\psi}(t)\|_\infty$ is estimated by means of Propositions 4 and 3. □

Remark. Since inequality (1.10) is satisfied, it is sufficient to require that $h \leq c_0\varepsilon$ for the local error to be bounded by

$$\|\overline{e}(t)\|_\infty \leq C C_s c_0^n \varepsilon.$$

This proves the convergence of the local error with only one particle per cell of width ε, for example. □

In order to obtain an approximation of the exact solution f in the sense of functions, we define a regularized version of f_h. Let ς be such that

$$\int_{\mathbb{R}^n} \varsigma(x)\,dx = 1.$$

For any real number $\varepsilon > 0$, we set $\varsigma_\varepsilon(x) = \varepsilon^{-n}\varsigma(\varepsilon^{-1}x)$. We define

$$f_h^\varepsilon(x,t) = \sum_{k\in\mathcal{J}} \omega_k(t) f_k(t) \varsigma_\varepsilon(x - x_k(t)). \tag{3.15}$$

Let us now assume that the function ς satisfies the moment conditions

$$\int_{\mathbb{R}^n} x^\alpha \varsigma(x)\,dx = 0, \qquad \alpha\in\mathbb{N}^n, 1 \le |\alpha| \le r'-1, \tag{3.16}$$

$$\int_{\mathbb{R}^n} |x|^{r'} |\varsigma(x)|\,dx < +\infty \tag{3.17}$$

for an integer $r' \ge 2$. We assume also that ς is compactly supported. We then have

THEOREM 2. *Let $m \ge n$, $m' \ge n$, $r \ge 2$ and $r' \ge 0$ be integers, and $s = \max(r', r+2, m, m')$. Assume that $\mathbf{a} \in (L^\infty(0,T; W^{s+1,\infty}(\mathbb{R}^n)))^n$ and $a_0 \in L^\infty(0,T; W^{s,\infty}(\mathbb{R}^n))$. Assume that condition* (1.10) *is satisfied. If $\eta \in W^{m,1}(\mathbb{R}^n)$ and satisfies the moment conditions* (2.4) *and* (2.5), *if*

$$\mu \in L^\infty((0,T); W^{m,\infty}(\mathbb{R}^n\times\mathbb{R}^n)) \cap L^\infty((0,T)\times\mathbb{R}^n_x; W^{r+1,\infty}(\mathbb{R}^n_y)),$$

and if $\varsigma \in W^{m',1}(\mathbb{R}^n)$ and verifies conditions (3.16)–(3.17), *there exists a constant $C = C(T, \mathbf{a}, a_0, \varsigma, \mu, \eta, C_s) > 0$ such that for any function $f_0 \in W^{s,\infty}(\mathbb{R}^n)$ and any $t\in[0,T]$*

$$\|(f - f_h^\varepsilon)(\cdot,t)\|_{0,\infty} \le C\left(\varepsilon^{r'} + \frac{h^{m'}}{\varepsilon^{m'}} + \nu\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right)\right)\|f_0\|_{s,\infty}. \tag{3.18}$$

Proof. We write

$$(f - f_h^\varepsilon)(\cdot,t) = (f - f^\varepsilon)(\cdot,t) + (f^\varepsilon - \pi_h^\varepsilon(t)f^\varepsilon)(\cdot,t) + (\pi_h^\varepsilon(t)f^\varepsilon - f_h^\varepsilon)(\cdot,t),$$
$$\pi_h^\varepsilon(t)f^\varepsilon(x,t) = \sum_{k\in\mathcal{J}} \omega_k(t) f^\varepsilon(x_k(t),t)\varsigma_\varepsilon(x - x_k(t)).$$

We apply successively Theorem 3 of [19], Proposition 3, Theorem 1 and Proposition 2 to obtain bounds on the first two terms; it thus remains to bound $(\pi_h^\varepsilon(t)f^\varepsilon - f_h^\varepsilon)(\cdot,t)$. Setting again $e_k(t) = f^\varepsilon(x_k(t),t) - f_k(t)$, we have

$$(\pi_h^\varepsilon(t)f^\varepsilon - f_h^\varepsilon)(x,t) = \sum_{k\in\mathcal{J}} \omega_k(t) e_k(t) \varsigma_\varepsilon(x - x_k(t)). \tag{3.19}$$

Since the support of ς is bounded, arguing as in the proof of the previous proposition, we get

$$\|(\pi_h^\varepsilon(t)f^\varepsilon - f_h^\varepsilon)(\cdot,t)\|_{0,\infty} \le C\|\bar{e}(t)\|_\infty. \tag{3.20}$$

Finally, Proposition 5 provides a bound on $\|\bar{e}(t)\|_\infty$, and combining this with the bounds obtained for the previous terms yields the announced result. □

Remarks. 1. The assumption that the supports of the functions ς and η are bounded is not necessary but leads to simpler proofs. In fact, it would be sufficient to assume that both functions are rapidly decreasing at infinity.

2. If $f_0 \in L^1(\mathbb{R}^n)$, any solution of (1.1) satisfies the L^1 conservation relation

$$\frac{d}{dt}\int_{\mathbb{R}^n} f(x,t)\,dx + \int_{\mathbb{R}^n} a_0(x,t)f(x,t)\,dx = 0. \tag{3.21}$$

It is easy to check that any solution of (1.5) satisfies the same relation. Exchanging the roles of k and l, for any sequence $\overline{\varphi} = (\varphi_k)_{k\in\mathcal{J}}$ in $l^\infty(\mathcal{J})$, we have

$$\sum_{k,l\in\mathcal{J}}\sum \omega_k(t)\omega_l(t)\eta_\varepsilon(x_k(t)-x_l(t))\mu(x_k(t),x_l(t),t)(\varphi_l-\varphi_k) = 0;$$

thus, any solution of the ordinary differential equation (3.7) satisfies the following discrete analogue of (3.21)

$$\frac{d}{dt}\left(\sum_{k\in\mathcal{J}}\omega_k(t)f_k(t)\right) + \sum_{k\in\mathcal{J}}\omega_k(t)a_0(x_k(t),t)f_k(t) = 0, \tag{3.22}$$

and the scheme is conservative.

3. In the case of a nonsymmetric kernel, Eq. (3.7) becomes

$$\begin{aligned}\frac{df_k}{dt}(t) &+ (\operatorname{div}\mathbf{a}(x_k(t),t) + a_0(x_k(t),t))f_k(t)\\ &= \nu\varepsilon^{-2}\sum_{l\in\mathcal{J}}\omega_l(t)\{\eta_\varepsilon(x_k(t)-x_l(t))\mu(x_k(t),x_l(t),t)f_l(t)\\ &\qquad - \eta_\varepsilon(x_l(t)-x_k(t))\mu(x_l(t),x_k(t),t)f_k(t)\}.\end{aligned} \tag{3.23}$$

The convergence of f_h^ε towards f^ε can still be proved, and we get

$$\|(f-f_h^\varepsilon)(\cdot,t)\|_{0,\infty} \le C\left(\varepsilon^{r'} + \frac{h^{m'}}{\varepsilon^{m'}} + \nu\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+2}}\right)\right)\|f_0\|_{s,\infty}. \quad \square$$

4. Particular Case of a Positive Kernel: Uniformly Stable Approximation. In this section we restrict ourselves to the case of the Laplace operator and consider the previous integral operator in the case of a nonnegative kernel σ. We shall prove that a maximum principle property is true for both the integro-differential equation and the particle method. Thus, without assuming that inequality (1.10) is satisfied, we shall obtain L^∞ estimates. This approximation is called uniformly stable precisely because the stability of the method is proved without any assumption on the discretization parameter. In fact, we need not assume in this section that the viscosity is small, although the particle method is well known to be better suited to slightly viscous media. The results, and sometimes the proofs, are very similar to those of the previous sections; thus, some proofs will only be sketched.

We assume that the viscosity coefficient b is constant, equal to 1, and that the function σ^ε is given by

$$\sigma^\varepsilon(x,y,t) = \frac{1}{\varepsilon^2}\eta_\varepsilon(x-y), \tag{4.1}$$

where the function η_ε satisfies the assumptions (2.3). Thus,

$$\sigma^\varepsilon \in L^\infty((0,T)\times \mathbf{R}^n_y; L^1(\mathbf{R}^n_x)) \cap L^\infty((0,T)\times \mathbf{R}^n_x; L^1(\mathbf{R}^n_y))$$

is symmetric and

$$Q^\varepsilon(t)f(x) = \varepsilon^{-2}\int_{\mathbf{R}^n} \eta_\varepsilon(x-y)(f(y)-f(x))\,dy.$$

We first prove the analogue of Proposition 3 in the case of a nonnegative kernel.

PROPOSITION 6. *Assume that η is nonnegative. Assume also that $\mathbf{a} \in (L^\infty(0,T;W^{1,\infty}(\mathbf{R}^n)))^n$ and $a_0 \in L^\infty(Q_T)$. If $f_0 \in L^\infty(\mathbf{R}^n)$, the unique solution f of problem* (2.16) *in $L^\infty(Q_T)$ is bounded as follows:*

$$\|f(\cdot,t)\|_{0,\infty} \le \exp(\alpha t)\|f_0\|_{0,\infty}, \tag{4.2}$$

where

$$\alpha = -\inf\{(a_0 + \operatorname{div}\mathbf{a})(x,t), (x,t)\in Q_T\}. \tag{4.3}$$

Let m be an integer; if $\mathbf{a} \in L^\infty(0,T;W^{m+1,\infty}(\mathbf{R}^n))$, $a_0 \in L^\infty(0,T;W^{m,\infty}(\mathbf{R}^n))$ and $\eta \in W^{m,1}(\mathbf{R}^n)$, then for any initial condition $f_0 \in W^{m,\infty}(\mathbf{R}^n)$ the solution belongs to $L^\infty(0,T;W^{m,\infty}(\mathbf{R}^n))$, and there exists a constant $C = C(T,a_0,\mathbf{a}) > 0$ such that

$$\|f(\cdot,t)\|_{m,\infty} \le C\|f_0\|_{m,\infty}. \tag{4.4}$$

Proof. Proposition 3 assures the existence and the uniqueness of the solution in $L^\infty(Q_T)$; it suffices, then, to establish the estimate. Let us first assume that f_0 is nonnegative and let us return to the fixed point method defined by (2.28). Since σ is ≥ 0, by (2.27) we have that $f \ge 0$ implies $\Phi f \ge 0$. This proves that all terms g^k of the sequence are ≥ 0 and then that the limit (the existence of which is proved by Proposition 3) is also nonnegative. This limit is the solution of (2.16). Thus, if the initial function is ≥ 0, the solution remains ≥ 0 for any time. In the general case of an initial function which does not have a constant sign, we set

$$g^*(\cdot,t) = f^*(\cdot,t) + \|f_0\|_{0,\infty}\exp(\alpha t),$$

where α is defined by (4.3). Then $\alpha + a_0 + \operatorname{div}\mathbf{a} \ge 0$, and g^* is the solution of

$$\begin{cases} \dfrac{\partial g^*}{\partial t} + \operatorname{div}(\mathbf{a}g^*) + a_0 g^* - \nu Q g^* = (\alpha + a_0 + \operatorname{div}\mathbf{a})\|f_0\|_{0,\infty}\exp(\alpha t) \ge 0, \\ g^*(\cdot,0) = f_0 + \|f_0\|_{0,\infty} \ge 0. \end{cases}$$

The function g^* is then nonnegative and

$$f^*(\cdot,t) \ge -\|f_0\|_{0,\infty}\exp(\alpha t). \tag{4.5}$$

Now, consider the function

$$h^*(\cdot,t) = -f^*(\cdot,t) + \|f_0\|_{0,\infty}\exp(\alpha t).$$

h^* is the solution of the same equation with nonnegative data; h^* is then ≥ 0 and

$$f^*(\cdot,t) \le \|f_0\|_{0,\infty}\exp(\alpha t). \tag{4.6}$$

Combining (4.5) and (4.6) leads to (4.2). The estimate (4.4) is obtained by formally differentiating the equation. □

Again, we denote by f and f^ε the solutions of (1.1) and (1.5), respectively. The following result is the analogue of Theorem 1, and its proof, which is very similar to that of Theorem 1, will only be sketched.

THEOREM 3. *Assume that*

$$\mathbf{a} \in (L^\infty(0,T;W^{5,\infty}(\mathbb{R}^n)))^n \quad \textit{and} \quad a_0 \in L^\infty(0,T;W^{4,\infty}(\mathbb{R}^n)).$$

Assume that η is ≥ 0 and satisfies the hypotheses of Proposition 1. There exists a constant $C = C(T,\eta,a_0,\mathbf{a}) > 0$ such that for any function $f_0 \in W^{4,\infty}(\mathbb{R}^n)$

$$\|(f - f^\varepsilon)(\cdot,t)\|_{0,\infty} \leq C\nu\varepsilon^2\|f_0\|_{4,\infty}. \tag{4.7}$$

Proof. We set $g = f - f^\varepsilon$, and we obtain

$$\begin{cases} \dfrac{\partial g}{\partial t} + \operatorname{div}(\mathbf{a}g) + a_0 g - \nu Q^\varepsilon(t)g = \nu(\Delta - Q^\varepsilon(t))f, \\ g(\cdot,0) = 0. \end{cases} \tag{4.8}$$

Applying Proposition 3, we find

$$\|g(\cdot,t)\|_{0,\infty} \leq C\nu \int_0^t \|(\Delta - Q^\varepsilon(\tau))f\|_{0,\infty}\,d\tau,$$

where the constant C depends on T, $\mathbf{a}$ and a_0, and the theorem follows from Propositions 1 and 2. □

Before stating the convergence of the method, let us first recall a stability result proved in [12].

LEMMA. *Assume that we are given continuous functions of t, $(b_{k,l}(t))_{k,l\in\mathbb{Z}^n}$, which satisfy for some constant $C > 0$ and for any $k \in \mathbb{Z}^n$,*

$$\begin{aligned} &\text{(i)} && b_{k,l}(t) \leq 0 \quad \textit{for any } l \in \mathbb{Z}^n,\ l \neq k, \\ &\text{(ii)} && \sum_{l\in\mathbb{Z}^n} b_{k,l}(t) \geq 0, \\ &\text{(iii)} && \sum_{l\in\mathbb{Z}^n} |b_{k,l}(t)| \leq C. \end{aligned} \tag{4.9}$$

Given continuous functions of t, $g^*(t) = (g_k(t))_{k\in\mathbb{Z}^n}$ and $v^{0*} = (v_k^0)_{k\in\mathbb{Z}^n}$ such that for some constant $M > 0$ and some continuous function $G > 0$,

$$\begin{cases} 0 \leq v_k^0 \leq M, \\ 0 \leq g_k(t) \leq G(t) \end{cases} \tag{4.10}$$

for any $k \in \mathbb{Z}^n$ and all $t \in [0,T]$, there exists a unique solution $v^*(t) = (v_k(t))_{k\in\mathbb{Z}^n}$ of the following differential system

$$\begin{cases} \dfrac{dv_k}{dt}(t) + \displaystyle\sum_{l\in\mathbb{Z}^n} b_{k,l}(t)v_l(t) = g_k(t), \\ v_k(0) = v_k^0. \end{cases} \tag{4.11}$$

Moreover, v^* satisfies the inequality

$$0 \leq v_k(t) \leq M + \int_0^T G(s)\,ds. \tag{4.12}$$

We now prove

PROPOSITION 7. *Let $m \geq n$ be an integer. Assume that* $\mathbf{a}$, a_0 *and* η *satisfy the hypotheses of Proposition 4. There exists a constant* $C = C(T, \mathbf{a}, a_0, \eta) > 0$ *such that for any function* $f_0 \in W^{m,\infty}(\mathbb{R}^n)$ *and any* $t \in [0,T]$

$$\sup_{k\in\mathscr{J}} |f^\varepsilon(x_k(t),t) - f_k(t)| \leq C\nu \frac{h^m}{\varepsilon^{m+1}} \|f_0\|_{m,\infty}. \tag{4.13}$$

Proof. Again, we set

$$e(t) = (e_k(t))_{k\in\mathscr{J}}, \qquad e_k(t) = f^\varepsilon(x_k(t),t) - f_k(t), \tag{4.14}$$

and we have

$$\frac{de_k}{dt}(t) + \sum_{l\in\mathscr{J}} \alpha_{kl}(t) e_l(t) = \nu\psi_k(t), \qquad e_k(0) = 0, \tag{4.15}$$

where

$$\begin{aligned}
&\psi_k(t) = ((Q^\varepsilon(t) - Q_h^\varepsilon(t))f^\varepsilon)(x_k(t),t), \\
&\alpha_{kl}(t) = -\nu\varepsilon^{-2}\omega_l(t)\eta_\varepsilon(x_k(t) - x_l(t)) \quad \text{for } l \neq k, \\
&\alpha_{kk}(t) = \operatorname{div}\mathbf{a}(x_k(t),t) + a_0(x_k(t),t) + \nu\varepsilon^{-2}\sum_{l\neq k}\omega_l(t)\eta_\varepsilon(x_k(t) - x_l(t)).
\end{aligned}$$

The boundedness of the norm of $\bar{e}$ will follow from an application of the previous lemma. Actually, the lemma cannot be applied directly, because the sign of the diagonal coefficients $\alpha_{kk}(t)$ is not known. On the other hand, these diagonal coefficients are easily changed by multiplying the function by some appropriate exponential of t. Precisely, setting $\alpha = \|\operatorname{div}\mathbf{a} + a_0\|_{0,\infty}$, we verify that the functions

$$\tilde{e}_k(t) = e_k(t)e^{-\alpha t} + \nu\int_0^t \|\overline{\psi}(\tau)\|_\infty e^{-\alpha(t-\tau)}\,d\tau$$

are solution of a system which differs from the previous one only by the diagonal coefficients. In fact, we have

$$\frac{d\tilde{e}_k}{dt}(t) + \sum_{l\in\mathscr{J}} \beta_{kl}(t)\tilde{e}_l(t) = \nu(\psi_k(t)e^{-\alpha t} + \|\overline{\psi}(t)\|_\infty), \qquad \tilde{e}_k(0) = 0,$$

where

$$\begin{aligned}
&\beta_{kk}(t) = \alpha_{kk}(t) + \alpha \quad \text{for any } k \in \mathscr{J}, \\
&\beta_{kl}(t) = \alpha_{kl}(t) \quad \text{for any } k, l \in \mathscr{J},\ k \neq l.
\end{aligned}$$

Arguing as in the proof of Proposition 5, we check that the coefficients $\beta_{kl}(t)$ satisfy the following inequalities, for any $k, l \in \mathscr{J}$ and $t \in [0,T]$:

$$\begin{aligned}
&\text{(i)} \qquad \beta_{kl}(t) \leq 0 \quad \text{for } l \neq k, \\
&\text{(ii)} \qquad \sum_{l\in\mathscr{J}} \beta_{kl}(t) \geq 0, \\
&\text{(iii)} \qquad \sum_{l\in\mathscr{J}} |\beta_{kl}(t)| \leq C(1 + \nu\varepsilon^{-2}).
\end{aligned} \tag{4.16}$$

Since, moreover, $\psi_k(t)e^{-\alpha t} + \|\overline{\psi}(t)\|_\infty \geq 0$ for any $k \in \mathcal{J}$, we can apply the lemma, which establishes that $\tilde{e}_k(t) \geq 0$ for any k. Thus, it follows that for any $k \in \mathcal{J}$

$$e_k(t) \geq -\nu \int_0^t \|\overline{\psi}(\tau)\|_\infty e^{\alpha\tau}\, d\tau.$$

Considering then

$$e_k(t)e^{-\alpha t} - \nu \int_0^t \|\overline{\psi}(\tau)\|_\infty e^{-\alpha(t-\tau)}\, d\tau$$

leads to

$$e_k(t) \leq \nu \int_0^t \|\overline{\psi}(\tau)\|_\infty e^{\alpha\tau}\, d\tau,$$

for any $k \in \mathcal{J}$. Combining these two results yields

$$\|\overline{e}(t)\|_\infty \leq \nu \int_0^t \|\overline{\psi}(\tau)\|_\infty \exp(\tau\|\operatorname{div}\mathbf{a} + a_0\|_{0,\infty})\, d\tau, \tag{4.17}$$

and the result follows from an application of Propositions 4 and 6. □

Let ς be a cutoff function with integral 1. We define the regularized version of f_h by

$$f_h^\varepsilon(x,t) = \sum_{k\in\mathcal{J}} \omega_k(t) f_k(t) \varsigma_\varepsilon(x - x_k(t)).$$

We assume that η and ς have compact supports, that $\varsigma \in W^{m',1}(\mathbb{R}^n) \cap C_0^0(\mathbb{R}^n)$ for some integer m' and verifies the moment conditions (3.16) and (3.17) for some integer $r' \geq 0$.

THEOREM 4. *Let $m \geq n$ be an integer and $s = \max(r', 4, m, m')$. Assume that $\mathbf{a} \in (L^\infty(0,T;W^{s+1,\infty}(\mathbb{R}^n)))^n$ and $a_0 \in L^\infty(0,T;W^{s,\infty}(\mathbb{R}^n))$. Assume that η satisfies the hypotheses of Theorem 3 and Proposition 7. Then there exists a constant $C = C(T,\mathbf{a},a_0,\varsigma,\eta) > 0$ such that for any function $f_0 \in W^{s,\infty}(\mathbb{R}^n)$ and any $t \in [0,T]$*

$$\|(f - f_h^\varepsilon)(\cdot,t)\|_{0,\infty} \leq C\left(\varepsilon^{r'} + \frac{h^{m'}}{\varepsilon^{m'}} + \nu\left(\varepsilon^2 + \frac{h^m}{\varepsilon^{m+1}}\right)\right)\|f_0\|_{s,\infty}. \tag{4.18}$$

The proof is very similar to that of Theorem 2 and follows from an application of the previously established results. □

5. Further Remarks.

5.1. *Stability of the Time Discretization.* We present a stability analysis of the ordinary differential equation (3.7) which gives the strength of the particles. For simplicity we assume that there is no convection ($\mathbf{a} = \mathbf{0}$) and no deformation ($a_0 = 0$), that the viscosity b is equal to 1 and that the space dimension is one. The positions of the particles are given by $x_k = kh$, the volumes by $\omega_k = h$, and the equation is

$$\frac{df_k}{dt}(t) - \frac{\nu h}{\varepsilon^2}\sum_l \eta_\varepsilon(x_k - x_l)(f_l(t) - f_k(t)) = 0.$$

We choose a time step $\Delta t > 0$ and we denote by f_k^n the approximation of $f_k(n\Delta t)$. Using Euler's scheme, we write

$$f_k^{n+1} = \sum_l a_{kl} f_l^n,$$

where

$$a_{kl} = \frac{\nu \Delta t}{\varepsilon^3} h \eta \left(\frac{k-l}{\varepsilon} h \right) \quad \text{for } k \neq l,$$
$$a_{kk} = 1 - \frac{\nu \Delta t}{\varepsilon^3} h \sum_{l \neq k} \eta \left(\frac{k-l}{\varepsilon} h \right).$$

The scheme is A-stable if

$$\sup_k \sum_l |a_{kl}| \leq 1.$$

Since

$$\sum_l |a_{kl}| = \left| 1 - \frac{\nu \Delta t}{\varepsilon^3} h \sum_{l \neq k} \eta \left(\frac{k-l}{\varepsilon} h \right) \right| + \frac{\nu \Delta t}{\varepsilon^3} h \sum_{l \neq k} \left| \eta \left(\frac{k-l}{\varepsilon} h \right) \right| \geq 1,$$

the stability conditions are

$$\text{(5.1)} \qquad \begin{cases} \eta \geq 0, \\ \dfrac{\nu \Delta t}{\varepsilon^3} h \displaystyle\sum_{l \neq k} \eta \left(\frac{k-l}{\varepsilon} h \right) \leq 1. \end{cases}$$

Since the function η is even and compactly supported, say in $[-d, d]$, we have

$$\sum_{l \neq k} \eta \left(\frac{k-l}{\varepsilon} h \right) = 2 \sum_{k=1}^{+\infty} \eta \left(\frac{kh}{\varepsilon} \right) \leq 2d \frac{\varepsilon}{h} \|\eta\|_{0,\infty},$$

and the scheme is stable if $\eta \geq 0$ and

$$\text{(5.2)} \qquad 2\nu \Delta t d \|\eta\|_{0,\infty} \leq \varepsilon^2.$$

If the function η is not compactly supported, it is sufficient to bound the sum as follows, for example: we consider the case of the Gaussian function and write

$$\sum_{k=1}^{+\infty} \eta \left(\frac{kh}{\varepsilon} \right) = \frac{1}{\sqrt{4\pi}} \sum_{1}^{+\infty} \exp \left(-\frac{k^2 h^2}{4\varepsilon^2} \right) \leq \frac{\varepsilon}{\sqrt{4\pi} h} \int_0^{+\infty} \exp \left(-\frac{x^2}{4} \right) dx = \frac{\varepsilon}{2h}.$$

The stability condition is then

$$\text{(5.3)} \qquad \nu \Delta t \leq \varepsilon^2.$$

We point out that the parameter h does not appear either in inequality (5.2) or in (5.3). The constraint imposed to Δt is not too strong since it relates the time step to the cell width and becomes less demanding when ν diminishes. The same analysis can be done in higher dimension and in the case where the velocity $\mathbf{a}$ is different from $\mathbf{0}$, provided that $\operatorname{div} \mathbf{a} = 0$ and $a_0 = 0$.

5.2. *Examples of Functions* η. Numerous examples of kernels will be studied in Part 2 of the paper; here we mention some possibilities.

A. *A first example of a spherically symmetric function.* Consider a function $\zeta\colon \mathbb{R}_+ \to \mathbb{R}$ and set

$$\text{(5.4)} \qquad \eta(x) = -2|x|^{-1} \zeta(|x|),$$

where $|x| = (x_1^2+\cdots+x_n^2)^{1/2}$. If for an integer r the function $\bar{\varsigma}$ satisfies the following moment conditions,

$$\int_0^{+\infty} t^{n-1}\bar{\varsigma}(t)\,dt = (\operatorname{meas} S^{n-1})^{-1}, \tag{5.5}$$

$$\int_0^{+\infty} t^{2p+n-1}\bar{\varsigma}(t)\,dt = 0, \qquad 1 \le p \le k = \frac{r}{2}, \tag{5.6}$$

$$\int_0^{+\infty} t^{r+n+1}|\bar{\varsigma}(t)|\,dt < +\infty \tag{5.7}$$

for an even integer r, then the function η satisfies conditions (2.4) and (2.5) for the integer r. This result is easily proved by means of spherical coordinates.

We can also rewrite η as

$$\eta(x) = -2|x|^{-2}\nabla\varsigma(x)\cdot x, \tag{5.8}$$

where $\varsigma(x) = \bar{\varsigma}(|x|)$ for any $x \in \mathbb{R}^n$ and where the function ς satisfies the moment conditions classically imposed on cutoff functions. Let us note that in [12] the case of functions η constructed by (5.8) with nonnegative functions ς was considered. In that case, $r = 2$, and condition (5.6) disappears.

B. *Second derivative of a cutoff function.* Consider again a function $\varsigma\colon \mathbb{R}^n \to \mathbb{R}$ which is at least twice continuously differentiable and which satisfies the moment conditions (3.16) and (3.17) for some integer $r \ge 2$, and set

$$\eta(x) = \Delta\varsigma(x). \tag{5.9}$$

Then the function η satisfies the conditions (2.4) and (2.5) for r, and the integral of η is equal to 0. Furthermore, in the case of a constant function b, say $b = 1$, the natural choice of μ is $\mu = 1$, and the operator Q^ε is reduced to

$$Q^\varepsilon f = \varepsilon^{-2}\eta_\varepsilon * f. \tag{5.10}$$

When constructing the particle method, one has to pay attention to the conservation property of the scheme, and the resulting approximate operator is

$$Q_h^\varepsilon(t)g = \varepsilon^{-2}\sum_{l\in\mathscr{J}} \omega_l(t)\eta_\varepsilon(x - x_l(t))(g(x_l(t)) - g(x)) \tag{5.11}$$

instead of

$$Q_h^\varepsilon(t)g = \varepsilon^{-2}\sum_{l\in\mathscr{J}} \omega_l(t)\eta_\varepsilon(x - x_l(t))g(x_l(t)).$$

Let us also mention that the integer r is even because of the symmetry of the function and that there is no hope to obtain a nonnegative function η.

C. *Another example of radially nonsymmetric functions.* Another obvious choice is to take the function η in the form of a tensor product of one-dimensional functions,

$$\eta(x) = \prod_{1\le i\le n} \bar{\varsigma}_i(x_i), \tag{5.12}$$

where each function $\bar{\varsigma}_i$ satisfies the conditions

$$\int_{\mathbf{R}} \bar{\varsigma}_i(t)\, dt = 1,$$
$$\int_{\mathbf{R}} t^2 \bar{\varsigma}_i(t)\, dt = 2,$$
$$\int_{\mathbf{R}} t^p \bar{\varsigma}_i(t)\, dt = 0, \qquad p = 1 \text{ or } 3 \leq p \leq r+1,$$
$$\int_{\mathbf{R}} t^{r+2} |\bar{\varsigma}_i(t)|\, dt < +\infty.$$

Let us notice that in that case the integer r is obviously equal to 2, because

$$\int_{\mathbf{R}} x_i^2 x_j^2 \eta(x)\, dx = 4. \quad \square$$

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