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# The Weighted Particle Method for Convection-Diffusion Equations Part 2: The Anisotropic Case

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**Abstract.** This paper is devoted to the presentation and the analysis of a new particle method for convection-diffusion equations. The method has been presented in detail in the first part of this paper for an isotropic diffusion operator. This part is concerned with the extension of the method to anisotropic diffusion operators. The consistency and the accuracy of the method require much more complex conditions on the cutoff functions than in the isotropic case. After detailing these conditions, we give several examples of cutoff functions which can be used for practical computations. A detailed error analysis is then performed.

**1. Presentation of the Method.** The purpose of this paper is to present and analyze a particle approximation of the following convection-diffusion equation:

$$\frac{\partial f}{\partial t} + \operatorname{div}(\mathbf{a}f) + a_0 f = \nu D(t) f, \tag{1}$$

which can be considered as a model equation for numerous physical problems, such as the incompressible Navier-Stokes equation or the Fokker-Planck equation of the kinetic theory of plasmas. In this equation, $x$ belongs to $\mathbb{R}^n$ and $t$ is positive. $\mathbf{a}(x,t)$ is a given vector field and $a_0$ a given scalar function. $D(t)f$ denotes an anisotropic diffusion operator, which, in its most general form, can be written as

$$D(t)f = \sum_{i,j} \frac{\partial}{\partial x_i}\left(L_{ij}(x,t)\frac{\partial f}{\partial x_j}\right) \tag{2}$$

with $\mathbf{L}(x,t)$ an $n \times n$ positive symmetric matrix, with possible degeneracies. $\nu$ is the viscosity parameter, which throughout this paper will be considered as being smaller than 1.

In the first part of this paper [1], we proposed a particle approximation for a convection-diffusion equation of type (1), when the diffusion matrix $\mathbf{L}$ is scalar. Let us recall that the derivation of this approximation is mainly divided into two steps: the first step is the definition of an integral operator $Q^\varepsilon(t)$ of the form

$$Q^\varepsilon(t)\cdot f(x) = \int \sigma^\varepsilon(x,y,t)(f(y)-f(x))\,dy, \tag{3}$$

where $\sigma^\varepsilon(x,y,t)$ is intended to provide an approximation of the diffusion operator $D(t)$ when $\varepsilon$ goes to 0. In the second step, we introduce the particle approximation $f_h(x,t)$ of the solution $f(x,t)$ according to

$$f_h(x,t) = \sum_k \omega_k(t) f_k(t)\delta(x - x_k(t)), \tag{4}$$

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where $x_k(t)$, $\omega_k(t)$ and $f_k(t)$ are the particle positions, volumes and strengths. The particle approximation $\overline{Q}_h^\varepsilon(t)$ of the diffusion operator $D(t)$ is obtained by numerical quadrature of the integral operator $Q^\varepsilon(t)$ using the particles as quadrature points. This approximation reads as follows:

$$\overline{Q}_h^\varepsilon(t) \cdot f_k(t) = \sum_l \sigma^\varepsilon(x_k(t), x_l(t), t)(f_l(t) - f_k(t))\omega_l(t).$$

Then the particle approximation of the convection-diffusion equation consists in letting the positions of the particles evolve according to the convection field $\mathbf{a}(x,t)$. The variation of the volumes is monitored by $\operatorname{div} \mathbf{a}$, while the variation of the strengths accounts for $a_0$ and for $\overline{Q}_h^\varepsilon(t)$. Namely,

$$(5) \qquad \begin{cases} \dfrac{dx_k}{dt} = \mathbf{a}(x_k(t), t), \\ \dfrac{d\omega_k}{dt} = \operatorname{div} \mathbf{a}(x_k(t), t) \cdot \omega_k(t), \\ \dfrac{df_k}{dt} + (a_0 + \operatorname{div} \mathbf{a})(x_k(t), t) \cdot f_k(t) = \overline{Q}_h^\varepsilon(t) \cdot f_k(t). \end{cases}$$

The method is completely specified once the approximation $Q^\varepsilon(t)$ of the diffusion operator $D(t)$ is defined. In Part 1, such an approximation is proposed for an isotropic diffusion operator (that is for scalar matrices $\mathbf{L}$). For the simplest case of the Laplacian operator $\Delta$, we introduce a cutoff function $\eta_\varepsilon(x)$, which is defined by

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right),$$

where the function $\eta(x)$ has the following moment properties:

$$\int x^\alpha \eta(x)\, dx = \begin{cases} 2 & \text{if the multi-index } \alpha = 2e_i, \\ 0 & \text{otherwise, for } 1 \le |\alpha| \le r+1 \end{cases}$$

(we denote by $e_i$ the $i$th vector of the canonical basis of $\mathbb{R}^n$). Then we define $\sigma^\varepsilon(x,y)$ by

$$(6) \qquad \sigma^\varepsilon(x,y) = \frac{1}{\varepsilon^2}\eta_\varepsilon(x-y).$$

Taylor's formula shows that $Q^\varepsilon(t) \cdot f$ is an approximation of $\Delta f$ up to the order $r$ [1].

The main difficulty in the anisotropic case is the derivation of a suitable integral operator $Q^\varepsilon(t)$. A first method was proposed in [2] and will be discussed in Section 3. However, its algorithmic complexity is too large for practical use. Our method relies on a direct extension of formula (6) to the anisotropic case. We propose the following choice of $\sigma^\varepsilon(x,y,t)$:

$$(7) \qquad \sigma^\varepsilon(x,y,t) = \frac{1}{\varepsilon^2}\sum_{i,j=1}^{n} M_{ij}(x,y,t)\psi_{ij}^\varepsilon(y-x),$$

where

$$\psi_{ij}^\varepsilon(x) = \frac{1}{\varepsilon^n}\psi_{ij}\left(\frac{x}{\varepsilon}\right)$$

is a matrix cutoff function, and $\mathbf{M} = (M_{ij}(x,y,t))$ is a matrix to be determined as a function of $\mathbf{L}(x,t)$.

In Section 3 we show that (7) actually provides an extension of (6) to the anisotropic case. The need for a different cutoff function for each component of the matrix $\mathbf{L}$ arises from the need to approximate different second-order crossed derivatives. That the matrix $\mathbf{M}$ may be different from $\mathbf{L}$ is an important feature of this method which will be detailed in Section 3.

The outline of this paper is as follows: we discuss the consistency conditions for $Q^\varepsilon$ in the next section. Then, practical examples are given in Section 3. The error estimates are stated and proved in Section 4. Since the proofs are very similar as in the isotropic case, they will only be outlined. We refer to [1] for a detailed bibliography on the particle approximation of convection-diffusion equations.

## 2. Derivation of the Integral Operator; Consistency Conditions.

2.1. *Introduction and Notations.* In this section, we will investigate sufficient conditions on $\psi$ and $\mathbf{M}$ which ensure that $Df$ and $Q^\varepsilon f$ are close up to the order $\varepsilon^r$. Let us introduce some notations. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are multi-indices in $\mathbb{N}^n$, we define:

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \qquad \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n),$$
$$\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$
$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

We denote by $(e_1, e_2, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$, and by $\mathscr{S}_n(\mathbb{R})$ the space of $n \times n$ real symmetric matrices. We will also use the Sobolev spaces $W^{k,\infty}(\mathbb{R}^n)$ provided with their usual norms:

$$\|g\|_{k,\infty} = \sup_{0 \le p \le k} |g|_{p,\infty}, \qquad |g|_{p,\infty} = \operatorname*{sup\,ess}_{|\alpha| = p,\ x \in \mathbb{R}^n} |\partial^\alpha g(x)|.$$

We now introduce the diffusion operator $D$ according to formula (2); in the remainder of this section, the time will be kept fixed and will be omitted in the formulae. We suppose that the matrix $\mathbf{L}$ belongs to $W^{s,\infty}(\mathbb{R}^n)$, with $s$ to be specified later. We investigate the integral operator $Q^\varepsilon$, given by formula (3), where $\sigma^\varepsilon(x, y, t)$ is determined by (7). We assume that $\mathbf{M}(x, y, t)$ and $\psi(x)$ are functions with values in $\mathscr{S}_n(\mathbb{R})$, the regularity of which will be specified later, and that they satisfy the additional hypotheses:

$$\mathbf{M}(x, y, t) = \mathbf{M}(y, x, t) \quad \forall x, y \in \mathbb{R}^n, \forall t > 0, \quad \psi_{ij} \text{ is even.}$$

With these hypotheses, $\sigma^\varepsilon(x, y, t)$ is invariant under the interchange of $x$ and $y$, which ensures the conservation property of the operator $Q^\varepsilon$:

$$\int Q^\varepsilon(t) \cdot f(x)\, dx = \iint \sigma^\varepsilon(x, y, t)(f(y) - f(x))\, dy\, dx = 0. \tag{8}$$

We now introduce the matrix $\mathbf{m}(x, t) = \mathbf{M}(x, x, t)$, and we note that, owing to the symmetry of $\mathbf{M}$, we have

$$\left.\frac{\partial M_{ij}}{\partial y_l}\right|_{(x,y)=(x,x)} = \frac{1}{2} \frac{\partial m_{ij}}{\partial x_l}(x) \quad \forall i, j, l \in \mathbb{N},\ \forall x, y \in \mathbb{R}. \tag{9}$$

Let us finally introduce the moments of the cutoff functions $\psi_{ij}$: for any $i, j$ in $[1, n]$, and for any multi-index $\alpha$ in $\mathbb{N}^n$, we define

$$Z_{i,j}^\alpha = \int \psi_{ij}(x) x^\alpha\, dx. \tag{10}$$

2.2. *Consistency and Accuracy of the Approximation of $D$ by $Q^\varepsilon$.*

HYPOTHESIS H. We assume that there exists an integer $r \geq 2$ such that:

(i) $Z^\alpha_{i,j} = 0$ for $1 \leq |\alpha| \leq r+1$ and $|\alpha| \neq 2$;

(ii) for any integer $k, l$ in $[1, n]$, we have

$$\sum_{i,j=1}^n m_{ij}(x) Z_{ij}^{e_k+e_l} = 2L_{kl}(x).$$

PROPOSITION 1. *Let* $\mathbf{m}$, $\mathbf{M}$ *and* $\psi$ *be given as in the general hypotheses of Section* 1. *In addition, assume that* $\mathbf{M} \in W^{r+1,\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, $\mathbf{m} \in W^{r+1,\infty}(\mathbb{R}^n)$, *and* $(1+|x|^{r+2})\psi(x) \in L^1(\mathbb{R}^n)$. *If Hypothesis* H *is satisfied, there exists a positive constant* $C = C(\mathbf{M}, \psi)$ *such that*

$$\|Dg - Q^\varepsilon g\|_{0,\infty} \leq C\varepsilon^r \|g\|_{r+2,\infty}.$$

*Remark.* The choice of an even $\psi$ makes the order $r$ of the approximation necessarily even (see Hypothesis H(ii)). The choice of $\psi$'s that are not even would lead to similar results, but would complicate the expression of $Q^\varepsilon$ to maintain the conservation property (8).

*Proof.* We first apply Taylor's expansion formula to the difference $g(y) - g(x)$ in formula (3). This leads to

$$Q^\varepsilon g = \sum_{|\alpha|=1}^{r+1} \frac{\partial^\alpha g(x)}{\alpha!} \int_{\mathbb{R}^n} \sigma^\varepsilon(x,y)(y-x)^\alpha \, dy + S^\varepsilon g, \tag{11}$$

where the remainder $S^\varepsilon g$ is given by

(12)

$$S^\varepsilon g = \sum_{|\alpha|=r+2} \frac{r+2}{\alpha!} \iint_{[0,1]\times\mathbb{R}^n} \sigma^\varepsilon(x,y)(y-x)^\alpha (1-u)^{r+1} \partial^\alpha g(x+u(y-x)) \, du \, dy.$$

Then we apply Taylor's formula again for the computation of the integrals involving $\sigma^\varepsilon$, by expanding $\mathbf{M}(x,y)$ in powers of $(y-x)$:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sigma^\varepsilon(x,y)(y-x)^\alpha \, dy \\
&\quad = \frac{1}{\varepsilon^2} \sum_{i,j=1}^n \int_{\mathbb{R}^n} M_{ij}(x,y) \psi^\varepsilon_{ij}(y-x)(y-x)^\alpha \, dy \\
&\quad = \frac{1}{\varepsilon^2} \sum_{i,j=1}^n \sum_{|\beta|=0}^{r+1-|\alpha|} \frac{1}{\beta!} \partial_y^\beta M_{ij}(x,x) \int_{\mathbb{R}^n} \psi^\varepsilon_{ij}(y-x)(y-x)^{\alpha+\beta} \, dy + T^\varepsilon_\alpha,
\end{aligned} \tag{13}$$

where the remainder $T^\varepsilon_\alpha$ is given by

$$\begin{aligned}
T^\varepsilon_\alpha = {}& \frac{1}{\varepsilon^2} \sum_{i,j=1}^n \sum_{|\beta|=r+2-|\alpha|} \frac{r+2-|\alpha|}{\beta!} \\
&\times \iint_{[0,1]\times\mathbb{R}^n} (1-v)^{r+1-|\alpha|} \partial_y^\beta M_{ij}(x, x+v(y-x)) \\
&\qquad \times \psi^\varepsilon_{ij}(y-x)(y-x)^{\alpha+\beta} \, dv \, dy.
\end{aligned} \tag{14}$$

Now, by substituting (13) in (11) we get the following decomposition:

$$Q^\varepsilon g = \sum_{m=1}^{r+1} Q_\varepsilon^m g + R^\varepsilon g,$$

where $Q_\varepsilon^m g$ is a differential operator of order $m$ and $R^\varepsilon$ is a remainder, obtained as the sum of the remainders (12) and (14):

$$R^\varepsilon g = S^\varepsilon g + \sum_{|\alpha|=1}^{r+1} \frac{1}{\alpha!}\partial^\alpha g(x) T_\alpha^\varepsilon. \tag{15}$$

Moreover, we note that

$$\int \psi_{ij}^\varepsilon(y-x)(y-x)^\alpha\, dy = \varepsilon^{|\alpha|} Z_{ij}^\alpha,$$

so that the coefficients of $Q_\varepsilon^m g$ are given by means of an expansion into powers of $\varepsilon$ according to

$$Q_\varepsilon^m g(x) = \sum_{|\alpha|=m} \frac{\partial^\alpha g(x)}{\alpha!} \times \left[\sum_{p=0}^{r+1-m} \varepsilon^{p+m-2} \sum_{i,j} \sum_{|\beta|=p} \left(\frac{1}{\beta!} Z_{ij}^{\alpha+\beta} \partial_y^\beta M_{ij}(x,x)\right)\right].$$

We note that for $3 \le m \le r+1$ the length of the multi-index $\alpha+\beta$ which appears in the moment $Z_{ij}^{\alpha+\beta}$ is certainly greater than 3, so that, using Hypothesis H(i), we immediately conclude that

$$Q_\varepsilon^m = 0 \quad \text{for } 3 \le m \le r+1.$$

Similarly, the only terms which do not vanish in $Q_\varepsilon^1$ and $Q_\varepsilon^2$ are of degree 0 with respect to $\varepsilon$, and they can be written as

$$Q_\varepsilon^1 g = \sum_{k=1}^n \frac{\partial g}{\partial x_k} \left[\sum_{i,j=1}^n \sum_{l=1}^n \frac{\partial M_{ij}}{\partial y_l}(x,x) Z_{ij}^{e_k+e_l}\right],$$

$$Q_\varepsilon^2 g = \sum_{k,l=1}^n \frac{1}{2} \frac{\partial^2 g}{\partial x_k \partial x_l} \left[\sum_{i,j=1}^n M_{ij}(x,x) Z_{ij}^{e_k+e_l}\right].$$

Then, using (9), we can write

$$Q_\varepsilon^1 g = \sum_{k,l=1}^n \frac{1}{2} \frac{\partial g}{\partial x_k} \frac{\partial}{\partial x_l} \left[\sum_{i,j=1}^n m_{ij}(x) Z_{ij}^{e_k+e_l}\right],$$

$$Q_\varepsilon^2 g = \sum_{k,l=1}^n \frac{1}{2} \frac{\partial^2 g}{\partial x_k \partial x_l} \left[\sum_{i,j=1}^n m_{ij}(x) Z_{ij}^{e_k+e_l}\right].$$

It is clear that Hypothesis H(ii) leads to

$$Q_\varepsilon^1 g = \sum_{k,l=1}^n \frac{\partial g}{\partial x_k} \frac{\partial L_{kl}}{\partial x_l}, \qquad Q_\varepsilon^2 g = \sum_{k,l=1}^n \frac{\partial^2 g}{\partial x_k \partial x_l} L_{kl}.$$

Thus, $Q^1_\varepsilon g + Q^2_\varepsilon g = Dg$ and $Q^\varepsilon g - Dg = R^\varepsilon g$. Now, going back to (15), (14) and (12), and using the regularity hypotheses on $\mathbf{M}$, $\psi$, and $g$, we easily find that

$$\|R^\varepsilon g\|_{0,\infty} \leq C\varepsilon^r \|g\|_{r+2,\infty},$$

where $C$ actually depends on $\|\mathbf{M}\|_{r+1,\infty}$ and $\|(1+|x|^{r+2})\psi\|_{0,1}$. This completes the proof. □

### 3. Examples.

3.1. *First Method*: $\psi$ *is a Hessian Matrix.*

3.1.1. *General setting.* This method is the direct extension of [1, Section 5.2]. We let $\varsigma$ be an even compactly supported (or vanishing at infinity) function from $\mathbb{R}^n$ into $\mathbb{R}$, such that

$$\int_{\mathbb{R}^n} x^\alpha \varsigma(x)\,dx = \begin{cases} 0 & \text{for } 1 \leq |\alpha| \leq r-1, \\ 1 & \text{for } \alpha = 0, \end{cases} \tag{16}$$

and we let

$$\psi_{ij}(x) = \frac{\partial^2 \varsigma}{\partial x_i \partial x_j}(x) \quad \text{and} \quad \mathbf{m}(x) = \mathbf{L}(x). \tag{17}$$

It follows that Hypothesis H(i) is satisfied and that for $|\alpha| = 2$ we have

$$Z^\alpha_{ij} = \begin{cases} 0 & \text{if } \alpha \neq e_i + e_j, \\ 1 & \text{if } i \neq j \text{ and } \alpha = e_i + e_j, \\ 2 & \text{if } i = j \text{ and } \alpha = 2e_i, \end{cases}$$

which immediately leads to Hypothesis H(ii).

3.1.2. $\varsigma$ *is a tensor product of one-dimensional functions.* This general setting includes the case of

$$\varsigma(x) = \prod_{j=1}^n \bar{\varsigma}_j(x_j), \tag{18}$$

where $\bar{\varsigma}_j$ is an even compactly supported function from $\mathbb{R}$ into $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x_j^{\alpha_j} \bar{\varsigma}_j(x_j)\,dx_j = \begin{cases} 0 & \text{for } 1 \leq \alpha_j \leq r-1, \\ 1 & \text{for } \alpha_j = 0. \end{cases}$$

Now, an immediate extension of (18) is obtained by letting

$$\begin{aligned} \psi_{ii}(x) &= \eta(x_i)\bar{\eta}(\widehat{x_i}), \\ \psi_{ij}(x) &= \xi(x_i, x_j)\bar{\xi}(\widehat{x_{ij}}) \quad \text{if } i \neq j, \end{aligned}$$

with

$$\begin{aligned} \widehat{x_i} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \\ \widehat{x_{ij}} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \end{aligned}$$

and with $\eta, \bar{\eta}, \xi, \bar{\xi}$ satisfying the following moment conditions:

$$\int_{-\infty}^{+\infty} y^\alpha \eta(y)\, dy = \begin{cases} 0 & \text{for } 0 \le \alpha \le r+1,\ \alpha \ne 2, \\ 2 & \text{for } \alpha = 2, \end{cases} \tag{19}$$

$$\int_{\mathbf{R}^{n-1}} \hat{y}^{\hat{\alpha}} \bar{\eta}(\hat{y})\, d\hat{y} = \begin{cases} 0 & \text{for } 1 \le |\hat{\alpha}| \le r-1, \\ 1 & \text{for } \hat{\alpha} = 0, \end{cases} \tag{20}$$

$$\int_{\mathbf{R}^2} z_1^{\alpha_1} z_2^{\alpha_2} \xi(z_1, z_2)\, dz_1\, dz_2 = \begin{cases} 0 & \text{for } 1 \le \alpha_1 + \alpha_2 \le r+1,\ (\alpha_1, \alpha_2) \ne (1,1), \\ 1 & \text{for } (\alpha_1, \alpha_2) = (1,1), \end{cases} \tag{21}$$

$$\int_{\mathbf{R}^{n-2}} \hat{z}^{\hat{\alpha}} \bar{\xi}(\hat{z})\, d\hat{z} = \begin{cases} 0 & \text{for } 1 \le |\hat{\alpha}| \le r-1, \\ 1 & \text{for } \hat{\alpha} = 0. \end{cases} \tag{22}$$

The moment conditions (19) to (22) are sufficient for $\psi$ to satisfy Hypothesis H, even if $\psi$ is not a Hessian matrix. This method may be used when one wishes to work with a very poorly regular cutoff, which is difficult with formulation (17).

3.1.3. *$\varsigma$ is spherically symmetric.* The general setting also applies to $\varsigma(x) = \bar{\varsigma}(\rho)$, where $\rho$ stands for the Euclidean norm of $x$, and $\bar{\varsigma}$ is a regular cutoff. Then,

$$\psi_{ii}(x) = x_i^2 \frac{1}{\rho} \frac{d}{d\rho}\left(\frac{1}{\rho} \frac{d\bar{\varsigma}}{d\rho}(\rho)\right) + \frac{1}{\rho} \frac{d\bar{\varsigma}}{d\rho}(\rho), \tag{23}$$

$$\psi_{ij}(x) = x_i x_j \frac{1}{\rho} \frac{d}{d\rho}\left(\frac{1}{\rho} \frac{d\bar{\varsigma}}{d\rho}(\rho)\right) \quad \text{if } i \ne j. \tag{24}$$

The moment conditions on $\bar{\varsigma}$ are easily deduced from (16). Formulae (23) and (24) may be quite complicated for practical use. In the next subsection we investigate simpler cutoff functions, which are directly inspired by (24).

3.2. *Second Method: $\psi_{ij} = x_i x_j \theta(x)$.*

3.2.1. *General setting.* Throughout this subsection, we will analyze cutoff functions of the type

$$\psi_{ij}(x) = x_i x_j \theta(x), \tag{25}$$

where $\theta$ is a real even function from $\mathbf{R}^n$ to $\mathbf{R}$ such that

$$\int x^\alpha \theta(x)\, dx = 0 \quad \text{for } 3 \le |\alpha| \le r+3 \text{ and } |\alpha| \ne 4.$$

We define the matrix $\mathbf{A} = (a_{kl})_{k,l=1}^n$ by

$$a_{kl} = \int x_k^2 x_l^2 \theta(x)\, dx, \qquad k, l \in [1, n].$$

LEMMA 1. *Given $\psi$ according to formula* (25), *there exists a matrix $\mathbf{m}(x)$ such that Hypothesis* H *is satisfied if and only if*

(i) *For any $k, l$, $k \ne l$, we have $a_{k,l} \ne 0$;*

(ii) *The matrix $\mathbf{A}$ is an invertible symmetric matrix.*

*Then, $\mathbf{m}(x)$ is defined by*

$$m_{kl}(x) = a_{kl}^{-1} L_{kl}(x) \quad \textit{for } k, l \in [1, n],\ k \ne l; \tag{26}$$

$$\sum_{i=1}^n a_{ki} m_{ii}(x) = 2 L_{kk}(x) \quad \textit{for } k \in [1, n]. \tag{27}$$

*Proof.* It is easy to check Hypothesis H(i). The $Z$ coefficients (given by (10)) are equal to

$$\text{if } k \neq l: \quad Z_{ij}^{e_k+e_l} = \begin{cases} 0 & \text{if } e_i + e_j \neq e_k + e_l, \\ a_{kl} & \text{if } e_i + e_j = e_k + e_l, \end{cases}$$
$$\text{if } k = l: \quad Z_{ij}^{2e_k} = \begin{cases} 0 & \text{if } i \neq j, \\ a_{ik} & \text{if } i = j. \end{cases}$$

It follows that Hypothesis H(ii) is equivalent to Eqs. (26) and (27). This completes the proof of Lemma 1. □

3.2.2. *$\theta$ is a tensor product of one-dimensional functions.* We apply this general setting to

$$\theta(x) = \prod_{i=1}^{n} \eta_i(x_i),$$

where the $\eta_i$'s are even functions from $\mathbf{R}$ into $\mathbf{R}$, normalized to 1. The matrix $\mathbf{A}$ is given by

$$a_{kl} = \int_{\mathbf{R}} x_k^2 \eta_k(x_k)\, dx_k \int_{\mathbf{R}} x_l^2 \eta_l(x_l)\, dx_l \quad \text{for } k \neq l,$$
$$a_{kk} = \int_{\mathbf{R}} x_k^4 \eta_k(x_k)\, dx_k.$$

Conditions (i) and (ii) of Lemma 1 are equivalent to

$$b_i \neq 0 \quad \forall i = 1, \ldots, n, \quad \left(\prod_{i=1}^{n} c_i\right)\left(\sum_{i=1}^{n} \frac{b_i}{c_i} + 1\right) \neq 0,$$

where

$$b_i = \int_{\mathbf{R}} x_i^2 \eta_i(x_i)\, dx_i, \qquad c_i = \frac{a_{ii} - b_i^2}{b_i}.$$

The inversion of $\mathbf{A}$ is impossible by analytical means in the general case; so we assume that $\theta$ is a tensor power:

$$\theta(x) = \prod_{i=1}^{n} \eta(x_i).$$

Thus, $a_{kl} \equiv \alpha$ for all $k \neq l$, and $a_{kk} \equiv \beta$ for all $k$. The inversion of $\mathbf{A}$ leads to

$$m_{kl}(x) = \alpha^{-1} L_{kl}(x) \quad \text{for } k, l \in [1, n], \ k \neq l;$$
$$m_{kk}(x) = \frac{2}{\beta - \alpha} L_{kk}(x) - \frac{2\alpha}{(\beta - \alpha)(\beta + (n-1)\alpha)} \operatorname{Tr} \mathbf{L}(x),$$

where Tr stands for the trace of a matrix.

With this choice of cutoff, the approximation is of order 2 at most, since, for instance, the following 4th order moment is necessarily nonzero:

$$\int_{\mathbf{R}^n} x_1^2 x_2^2 \psi_{33}(x)\, dx = \prod_{j=1}^{3} \int_{\mathbf{R}} x_j^2 \eta_j(x_j)\, dx_j \neq 0.$$

An interesting extension consists of $\theta$ being invariant under permutations of its arguments. Indeed, the matrix $\mathbf{A}$ and the solvability conditions for $\mathbf{m}(x)$ are the same as for the tensor power, but the accuracy of the approximation is no longer limited to the order 2.

3.2.3. *$\theta$ is spherically symmetric.* We now suppose that $\theta(x) = \bar{\theta}(|x|)$, with $\bar{\theta}$ being a function from $\mathbf{R}^+$ into $\mathbf{R}$. Now the matrix $\mathbf{A}$ is written as

$$a_{kl} = \alpha = \int_0^\infty \rho^{4+(n-1)}\bar{\theta}(\rho)\,d\rho \int_{S^{n-1}} \omega_1^2\omega_2^2\,d\omega \quad \text{for } k \neq l,$$
$$a_{kk} = \beta = \int_0^\infty \rho^{4+(n-1)}\bar{\theta}(\rho)\,d\rho \int_{S^{n-1}} \omega_1^4\,d\omega,$$

where $S^{n-1}$ stands for the $(n-1)$-dimensional sphere. We denote

$$\bar{\alpha} = \int_{S^{n-1}} \omega_1^2\omega_2^2\,d\omega \quad \text{and} \quad \bar{\beta} = \int_{S^{n-1}} \omega_1^4\,d\omega.$$

$\bar{\alpha}$ and $\bar{\beta}$ are not independent. Indeed, we have

$$\int_{S^{n-1}} (\omega_1^2 + \cdots + \omega_n^2)^2\,d\omega = \operatorname{meas}(S^{n-1}) = n\bar{\beta} + n(n-1)\bar{\alpha}. \tag{28}$$

Moreover, using spherical coordinates, we have

$$\operatorname{meas}(S^{n-1}) = 2\pi \prod_{j=1}^{n-2} I_j$$

and

$$\bar{\beta} = 2\pi \left(\prod_{j=1}^{n-3} I_j\right) \int_{-\pi/2}^{\pi/2} \cos^{n-2}\phi \sin^4\phi\,d\phi,$$

where $I_j$ stands for

$$I_j = \int_{-\pi/2}^{\pi/2} \cos^j\phi\,d\phi$$

and satisfies $I_j/I_{j-2} = (j-1)/j$. Thus, we have

$$\begin{aligned}\frac{\bar{\beta}}{\operatorname{meas}(S^{n-1})} &= \frac{1}{I_{n-2}} \int_{-\pi/2}^{\pi/2} \sin^4\phi \cos^{n-2}\phi\,d\phi \\ &= 1 - 2\frac{I_n}{I_{n-2}} + \frac{I_{n+2}}{I_{n-2}} = \frac{3}{n(n+2)},\end{aligned}$$

and from (28) we deduce

$$\frac{\bar{\alpha}}{\operatorname{meas}(S^{n-1})} = \frac{1}{n(n+2)}.$$

Thus, introducing

$$\gamma = \frac{\operatorname{meas}(S^{n-1})}{n(n+2)} \int_0^\infty \rho^{4+(n-1)}\bar{\theta}(\rho)\,d\rho, \tag{29}$$

we have $a_{kl} = \gamma$ if $k \neq l$ and $a_{kk} = 3\gamma$ for all $k$. Therefore, $\mathbf{m}$ is given by

$$m_{kl} = \gamma^{-1}L_{kl} \quad \text{if } k \neq l, \qquad m_{kk} = \gamma^{-1}L_{kk} - \frac{\gamma^{-1}}{n+2}\operatorname{Tr}(\mathbf{L}).$$

If $\theta$ is normalized so that $\gamma = 1$, the solution can be written in a more compact form as

$$\mathbf{m} = \mathbf{L} - \frac{1}{n+2}\operatorname{Tr}(\mathbf{L})\mathbf{Id}_n, \tag{30}$$

where $\mathbf{Id}_n$ stands for the identity matrix in $\mathbb{R}^n$. To obtain an order $r = 2q$ approximation, it is sufficient to require

$$\int_0^\infty \rho^{2k} \rho^{2+(n-1)} \bar{\theta}(\rho)\, d\rho = 0, \qquad 2 \le k \le q. \tag{31}$$

Examples of cutoff functions $\theta$ satisfying these conditions are

$$\bar{\theta}(\rho) = e^{-\beta\rho^2}(a_0 + a_1\rho^2 + \cdots + a_{q-1}\rho^{2(q-1)}), \qquad \beta > 0. \tag{32}$$

Conditions (29) and (31) lead to an invertible linear system for the coefficients $a_i$.

From a practical point of view, we believe that the present example provides the most economical method. Indeed, thanks to (30), the matrix $\mathbf{m}$ is directly given in terms of $\mathbf{L}$, with minor computations. Besides, it is fairly easy to design highly accurate cutoffs $\theta$. Moreover, since $\theta$ is a one-dimensional function of the radius $r$, it can be easily stored in the memory of the computer, rather than recomputed each time. Finally, super-Gaussians like (32) can be replaced by rational functions or $B$-splines.

3.3. *Connections with Previous Work.* We first show how the present approximation is connected with the first part of this paper [1]. We investigate scalar matrices. Thus, we let $\mathbf{L}(x) = \lambda(x)\mathbf{Id}_n$ and $\mathbf{M}(x, y) = \mu(x, y)\mathbf{Id}_n$. Then, formula (7) for $\sigma^\varepsilon$ leads to

$$\sigma^\varepsilon(x, y) = \frac{1}{\varepsilon^2}\mu(x, y)\sum_{i=1}^n \psi_{ii}^\varepsilon(y - x).$$

Thus the cutoff $\eta$, used in Part 1, is related to $\psi$ by

$$\eta(x) = \sum_{i=1}^n \psi_{ii}(x).$$

Since the matrix $\mathbf{m}$ is diagonal, the off-diagonal elements of the matrix $\psi$ need not be defined. Now, Hypothesis H(ii) immediately leads to

$$\int x_k x_l \eta(x)\, dx = 2\delta_{kl} \quad \text{for any } k, l \text{ in } [1, n],\ \mu(x, x) = \lambda(x),$$

which are exactly the consistency conditions found in Part 1.

We now mention the method developed in [2] for the anisotropic case. In this method, the function $\sigma^\varepsilon$ is not given by the general formula (7) but by

$$\sigma^\varepsilon(x, y) = \sum_{i,j=1}^n \int L_{ij}(z)\frac{\partial\varsigma_\varepsilon}{\partial x_i}(x - z)\frac{\partial\varsigma_\varepsilon}{\partial x_j}(z - y)\, dz, \tag{33}$$

where the cutoff $\varsigma^\varepsilon$ satisfies the moment conditions (16). In [2], it is proved that a particle method based on the approximation (33) is $L^2$-stable, consistent, and convergent. However, it leads to a double integral operator $Q^\varepsilon$, which is algorithmically more costly than the simple integral operators obtained by formula (7). This is the reason why we have rejected this method and preferred to investigate the $\sigma$'s given by (7). However, the analysis below (Section 5) would apply to (33) with very minor changes.

3.4. *Miscellaneous Remarks.* The matrix $\mathbf{M}(x, y)$ can easily be defined from the knowledge of the matrix $\mathbf{m}(x)$. Indeed, any type of equiweighted mean value of

$\mathbf{m}(x)$ and $\mathbf{m}(y)$ is convenient. We suggest the use of

$$\mathbf{M}(x,y) = \mathbf{m}\left(\frac{x+y}{2}\right) \quad \text{or} \quad \mathbf{M}(x,y) = \frac{1}{2}(m(x)+m(y)).$$

None of the examples presented above guarantee the positivity of the cross section $\sigma^\varepsilon$. More precisely, for $\psi$ given by (25), with a spherically symmetric $\theta$, we obtain

$$\sigma^\varepsilon(x,y) = \frac{1}{\varepsilon^{n+4}}\bar{\theta}\left(\frac{|y-x|}{\varepsilon}\right)\sum_{i,j=1}^{n} M_{ij}(x,y)(y-x)_i(y-x)_j.$$

Thus, $\sigma^\varepsilon$ is positive if and only if $\bar{\theta}$ is positive and $\mathbf{M}$ is a positive symmetric matrix. The former condition implies that the approximation is limited to the order 2, as in the isotropic case (see Part 1, Section 4). The latter condition is satisfied if $\mathbf{m}(x)$ is a positive matrix for all $x$. If we denote by $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ the ordered set of eigenvalues of $\mathbf{L}$, this condition leads to (see formula (30)):

$$\lambda_1 \ge \frac{1}{n+2}\sum_{i=1}^{n}\lambda_i.$$

This means that the eigenvalues of $\mathbf{L}$ have to be close enough. Clearly, this is not a natural condition for the continuous problem. Furthermore, other choices would lead to different conditions. Thus, we will not make any assumption on the positivity of $\sigma^\varepsilon$.

The uniform ellipticity of $\mathbf{L}$ is never required for the error estimate, since we do not need the regularizing properties of the diffusion operator (see Section 4). Thus, the method also applies to degenerate operators, or can be used to numerically investigate the singular limit $\nu \to 0$. A particularly interesting case is that of a completely degenerate diffusion operator in one particular direction. This case arises, for instance, in the Fokker-Planck equation of the kinetic theory of plasmas [3]. Let us suppose that $\mathbb{R}^n$ is split into $\mathbb{R}^p \times \mathbb{R}^{(n-p)}$, and that $\mathbf{L}$ is block decomposed according to this splitting:

$$\mathbf{L} = \begin{pmatrix} \mathbf{Id}_p & 0 \\ 0 & 0 \end{pmatrix}.$$

Then with (30), $\mathbf{m}$ is found to be

$$\mathbf{m} = \begin{pmatrix} \left(1-\frac{p}{n+2}\right)\mathbf{Id}_p & 0 \\ 0 & -\frac{p}{n+2}\mathbf{Id}_{n-p} \end{pmatrix}.$$

Thus, negative values of $\sigma$ should be expected when $y-x$ belongs to the degenerate direction; this can be interpreted as a sort of antidiffusion. It has to be numerically determined if this feature introduces some numerical instabilities.

## 4. The Error Estimate.

4.1. *Introduction.* We recall that our purpose is the numerical solution, by means of particle methods, of the convection-diffusion equation (1), with a prescribed initial data $f_0(x)$. For that purpose, an initial distribution of particles is defined, by specifying the initial positions $x_k^0$, the initial volumes $\omega_k^0$ and the initial strengths $f_k^0$, with $k$ in $\mathbb{Z}^n$. The initial discretization is performed on a regular grid, of mesh size $\mathbf{h} = (h_1, \dots, h_n)$, so that we let

$$x_k^0 = \sum_{i=1}^{n} k_i h_i e_i, \quad \omega_k^0 = h_1 \cdot \dots \cdot h_n, \quad f_k^0 = f_0(x_k^0), \quad h = (h_1^2 + h_2^2 + \dots + h_n^2)^{1/2}.$$

Now the whole set of positions, volumes and strengths is evolved according to the system (5). The aim of this section is to prove that the particle solution defined by (4) is an approximation in the sense of measures of the solution $f(x,t)$. Moreover, from a practical point of view, it is useful to define a smoothed particle approximation. This is done by introducing a cutoff $\varsigma^\varepsilon$ satisfying the moment conditions (16) (possibly to another order $r'$), and by letting

$$f_h^\varepsilon(x,t) = \sum_k \omega_k(t) f_k(t) \varsigma^\varepsilon(x - x_k(t)). \tag{34}$$

In this section, we will prove that $f_h^\varepsilon$ is an approximation of $f$ in the space $L^\infty$.

The proof is very similar to that of Part 1 [1]. We begin by investigating the approximation of the solution of the convection-diffusion equation (1) by the solution of the integro-differential equation obtained by replacing $D$ with $Q^\varepsilon$, subject to the same initial condition $f_0$. Then, we study the discrete system (5) and prove that $f_k(t)$ is an approximation of $f^\varepsilon(x_k(t), t)$. Finally, the $L^\infty$ estimate between $f$ and $f_h^\varepsilon$ is proved.

4.2. *Properties of the Integro-Differential Equation.* The aim of this subsection is to study the approximation of the solution of the convection-diffusion equation (1) by the solution of the integro-differential equation

$$\frac{\partial f^\varepsilon}{\partial t} + \operatorname{div}(\mathbf{a} f^\varepsilon) + a_0 f = \nu Q^\varepsilon(t) \cdot f^\varepsilon \tag{35}$$

with the same initial condition $f_0$. First, a stability estimate will be proved which consists of a uniform $W^{m,\infty}$ bound on the solution $f^\varepsilon$ with respect to the parameter $(\nu/\varepsilon^2)$. Then the error estimate of $f - f^\varepsilon$ will be shown.

PROPOSITION 2. *Assume that there exists an integer $m \geq 1$ such that*

$$\mathbf{a} \in L^\infty([0,T], W^{m+1,\infty}(\mathbf{R}^n)), \qquad a_0 \in L^\infty([0,T], W^{m,\infty}(\mathbf{R}^n)),$$
$$\mathbf{M} \in L^\infty([0,T], W^{m,\infty}(\mathbf{R}^n \times \mathbf{R}^n)), \qquad \psi \in L^1(\mathbf{R}^n).$$

*Then, for any constant $C_{\text{stab}}$ and any pair $(\nu, \varepsilon)$ such that*

$$\frac{\nu}{\varepsilon^2} \leq C_{\text{stab}}, \tag{36}$$

*and for any initial data $f_0$ in the space $W^{m,\infty}(\mathbf{R}^n)$, there exists a constant $C = C(T, a_0, \mathbf{a}, \mathbf{M}, \psi, C_{\text{stab}})$ such that the solution $f^\varepsilon$ of Eq.* (35), *with initial data $f_0$, satisfies $f^\varepsilon \in L^\infty([0,T], W^{m,\infty}(\mathbf{R}^n))$ with*

$$\|f^\varepsilon(t)\|_{m,\infty} \leq C \|f_0\|_{m,\infty} \quad \forall t \in [0,T].$$

*Proof.* The use of Proposition 3 of Part 1 [1, Section 2] leads to

$$\|f^\varepsilon(t)\|_{m,\infty} \leq \Phi(\nu/\varepsilon^2) \|f_0\|_{m,\infty},$$

where the function $\Phi$ blows up as $(\nu/\varepsilon^2)$ goes to infinity. So, as long as the parameter $(\nu/\varepsilon^2)$ remains bounded by some constant $C_{\text{stab}}$, the stability estimate holds. □

*Remark.* In Part 1 [1], for the case of the Laplace operator, we have found additional assumptions on the cutoff $\eta$ which make the cross section $\sigma^\varepsilon$ positive. Under these assumptions, it is possible to remove the constraint $(\nu/\varepsilon^2) \leq C_{\text{stab}}$. As we explained in Section 3, we have not found similar assumptions in the anisotropic case, for general, $x$-dependent matrices $\mathbf{L}$. However, it seems that such a constraint should naturally appear in any method of this type and is not restrictive when applied to problems with a small diffusion. The constraint (36) implies that $f^\varepsilon$ is not an approximation of $f$ in the usual sense, since $\varepsilon$ cannot tend to zero if $\nu$ remains fixed. But rather, the solution $f_\nu$ of the convection-diffusion equation (1) parametrized by $\nu$, and the solution $\bar{f}_\nu$ of the integro-differential equation (35) (with $\varepsilon$ being any function of $\nu$ satisfying the constraint (36)), are asymptotically close as $\nu$ goes to zero. Thus, the approximation will improve as $\nu$ goes to zero. This is the converse of classical methods, such as finite element methods, which lead to a better approximation when the diffusion coefficient is sufficiently large. Such a behavior is consistent, since particle methods are intended to provide accurate methods for convection dominated problems. Moreover, $\nu$ going to zero and $\varepsilon$ remaining fixed is compatible with the constraint (36). So, this method allows the study of the singular limit $\nu \to 0$, without any stability problem. This is not the case for most methods. Besides, as pointed out in Part 1 [1], the time discretization of the differential system (5), by an explicit first-order Euler scheme, introduces another stability constraint, which is written $(\nu\Delta t/\varepsilon^2) \leq C$. Thus, in any case, a bound on $(\nu/\varepsilon^2)$ is required to guarantee the stability of the time discretization. For all these reasons, we believe that, though limited by the stability condition (36), the present method should be useful in many cases of practical interest.

PROPOSITION 3. *Assume that there exists an integer $r \geq 2$ such that*

$$\mathbf{a} \in L^\infty([0,T], W^{r+3,\infty}(\mathbb{R}^n)), \qquad a_0 \in L^\infty([0,T], W^{r+2,\infty}(\mathbb{R}^n)),$$
$$\mathbf{L} \in L^\infty([0,T], W^{r+2,\infty}(\mathbb{R}^n)), \qquad \mathbf{M} \in L^\infty([0,T], W^{r+2,\infty}(\mathbb{R}^n \times \mathbb{R}^n)),$$

*and $(1+|x|^{r+2})\psi(x) \in L^1(\mathbb{R}^n)$. Assume that Hypothesis* H *is satisfied. Then, for any positive constant $C_{\text{stab}}$, and for any pair $(\nu, \varepsilon)$ satisfying the stability constraint* (36), *there exists a positive constant $C = C(T, a_0, \mathbf{a}, \mathbf{M}, \psi, \mathbf{L}, C_{\text{stab}})$ such that $f^\varepsilon$ and $f$, respectively solutions of Eqs.* (35) *and* (1) *with the same initial data $f_0$ in $W^{r+2,\infty}(\mathbb{R}^n)$, satisfy*

$$\|(f - f^\varepsilon)(t)\|_{0,\infty} \leq C\varepsilon^r \nu \|f_0\|_{r+2,\infty} \quad \forall t \in [0,T].$$

*Proof.* We let $g^\varepsilon = f - f^\varepsilon$. We have

$$\frac{\partial g^\varepsilon}{\partial t} + \operatorname{div}(\mathbf{a} g^\varepsilon) + a_0 g^\varepsilon - \nu Q^\varepsilon(t) \cdot g^\varepsilon = \nu(D(t)f - Q^\varepsilon(t)f).$$

By virtue of Proposition 2 and condition (36), there exists $C$ such that

$$\|g^\varepsilon(t)\|_{0,\infty} \leq CT\nu \operatorname*{Max}_{t\in[0,T]} \|(Df - Q^\varepsilon f)(t)\|_{0,\infty}.$$

Then, Proposition 1 gives

$$\|(Df - Q^\varepsilon f)(t)\|_{0,\infty} \leq C\varepsilon^r \|f(t)\|_{r+2,\infty}.$$

Finally, classical estimates on diffusion equations (see appendix) give

$$\|f(t)\|_{r+2,\infty} \leq \|f_0\|_{r+2,\infty}.$$

This proves the result. $\square$

4.3. *Properties of the Discrete System* (5). We now turn to the estimate of $f_k(t) - f^\varepsilon(x_k(t), t)$. For that purpose we introduce the space $l^\infty(\mathbb{Z}^n)$, which consists of all sequences $\bar{g} = (g_k)_{k\in\mathbb{Z}^n}$ such that

$$\|\bar{g}\|_\infty = \operatorname*{Sup}_{k\in\mathbb{Z}^n} |g_k| < \infty.$$

We also introduce the particle approximation of the integral operator $Q_h^\varepsilon(t)$ acting on continuous functions $g$:

$$Q_h^\varepsilon(t)g(x) = \sum_{l\in\mathbb{Z}^n} \omega_l(t)\sigma^\varepsilon(x, x_l(t), t)(g(x_l(t)) - g(x)).$$

We also define the analogue of this operator acting on a sequence $\bar{g}$:

$$(\overline{Q}_h^\varepsilon(t)\bar{g})_k = \sum_{l\in\mathbb{Z}^n} \omega_l(t)\sigma^\varepsilon(x_k(t), x_l(t), t)(g_l - g_k).$$

It is then clear that if $g$ and $\bar{g}$ are related by $g_l = g(x_l(t))$, we immediately get

$$(\overline{Q}_h^\varepsilon(t)\bar{g})_k = (Q_h^\varepsilon(t)g)(x_k(t)).$$

The discrete counterpart of the conservation property follows from the symmetry of $\sigma^\varepsilon$. It is written as

$$\sum_{k\in\mathbb{Z}^n} (\overline{Q}_h^\varepsilon g)_k\omega_k(t) = \sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n} \omega_k(t)\omega_l(t)\sigma^\varepsilon(x_k(t), x_l(t), t)(g_l - g_k) = 0.$$

We first prove that $Q_h^\varepsilon(t)g$ is an approximation of $Q^\varepsilon(t)g$ for any sufficiently regular function $g$. Then the estimate on $f_k(t) - f(x_k(t), t)$ is given.

PROPOSITION 4. *Assume that there exists an integer $m \geq n$ such that the hypotheses of Proposition 2 are satisfied. Moreover, assume that $\psi \in W^{m+1}(\mathbb{R}^n)$ and is compactly supported. Then, there exists a constant $C = C(\mathbf{a}, T, \mathbf{M}, \psi)$ such that for any function $g$ in $W^{m,\infty}(\mathbb{R}^n)$, and for any $t$ in $[0, T]$, we have*

$$\|Q^\varepsilon(t)g - Q_h^\varepsilon(t)g\|_{0,\infty} \leq C\frac{h^m}{\varepsilon^{m+1}}\|g\|_{m,\infty}.$$

*Proof.* We can write

$$(Q^\varepsilon(t) - Q_h^\varepsilon(t))g(x) = \frac{1}{\varepsilon^2}\left(\int F(x, y, t)\,dy - \sum_{l\in\mathbb{Z}^n} \omega_l(t)F(x, x_l(t), t)\right),$$

where $F$ is defined by

$$F(x, y, t) = \sum_{i,j=1}^{n} M_{ij}(x, y, t)\psi_{ij}^\varepsilon(y - x)(g(y) - g(x)).$$

Thanks to a classical quadrature estimate [1, Proposition 4], we easily get

$$|(Q^\varepsilon(t) - Q_h^\varepsilon(t))g(x)| \leq \frac{C}{\varepsilon^2}h^m\|F(x, \cdot)\|_{m,1}, \tag{37}$$

where $C = C(\mathbf{a}, T)$. Now, it is an easy matter to see that for $|\alpha| \leq m-1$ we have

$$\|\partial_y^\alpha F\|_{0,1} \leq \frac{C(\alpha)}{\varepsilon^{|\alpha|}} \|\mathbf{M}\|_{m-1,\infty} \|\psi\|_{m-1,1} \|g\|_{m-1,\infty}. \tag{38}$$

In the case of an $m$th derivative, we can spare a power of $\varepsilon$ by expanding the difference $g(y) - g(x)$ to the first order in $y - x$. This technical point is carried out in detail in [1, Proposition 4] and we do not develop it here. For $|\alpha| = m$, this leads to

$$\|\partial_y^\alpha F\|_{0,1} \leq \frac{C(\alpha)}{\varepsilon^{m-1}} \|\mathbf{M}\|_{m,\infty} \|g\|_{m,\infty} \|\psi\|_{m,1}. \tag{39}$$

Then, (39) and (38) together with (37) establish the result. □

PROPOSITION 5. *Assume that there exists an integer $m \geq n$ such that the hypotheses of Proposition 4 are satisfied. The initial condition $f_0$ is supposed to be in $W^{m,\infty}(\mathbb{R}^n)$. Then, for any constant $C_{\text{stab}}$ and for any pair $(\nu, \varepsilon)$ satisfying the constraint* (36), *there exists a positive constant $C = C(\mathbf{a}, a_0, T, \mathbf{L}, \mathbf{M}, \psi, C_{\text{stab}})$ such that the solution $f^\varepsilon$ of Eq.* (35), *and the solution $\bar{f}$ of the scheme* (5), *satisfy*

$$\operatorname*{Max}_{k \in \mathbb{Z}^n} |f_k(t) - f^\varepsilon(x_k(t), t)| \leq C\nu \frac{h^m}{\varepsilon^{m+1}} \|f_0\|_{m,\infty}.$$

*Proof.* We define $e_k(t) = f^\varepsilon(x_k(t), t) - f_k(t)$. It satisfies the following system of differential equations:

$$\frac{de_k}{dt} + \sum_{l \in \mathbb{Z}^n} \beta_{kl} e_l(t) = \psi_k(t)$$

with

$$\begin{aligned}
\beta_{kk}(t) &= (\operatorname{div} \mathbf{a} + a_0)(x_k(t), t) \\
&\quad + \frac{\nu}{\varepsilon^2} \sum_{l \neq k} \sum_{i,j=1}^n \omega_l(t) M_{ij}(x_k(t), x_l(t), t) \psi_{ij}^\varepsilon(x_l(t) - x_k(t)), \\
\beta_{kl}(t) &= -\frac{\nu}{\varepsilon^2} \sum_{i,j=1}^n \omega_l(t) M_{ij}(x_k(t), x_l(t), t) \psi_{ij}^\varepsilon(x_l(t) - x_k(t)) \quad \text{if } k \neq l, \\
\psi_k(t) &= \nu((Q^\varepsilon(t) - Q_h^\varepsilon(t)) \cdot f^\varepsilon)(x_k(t), t).
\end{aligned}$$

Then, by the same arguments as those developed in [1, Proposition 5], there exists a constant $C = C(T, a_0, \mathbf{a})$ such that $\omega_l(t) \leq Ch^n$ and that the number of particles in a ball of radius $\varepsilon$ is bounded by $C(\varepsilon/h)^n$ as long as $t$ is in $[0, T]$. It follows that

$$|\beta_{kl}| \leq C \frac{\nu h^n}{\varepsilon^{n+2}} \|\mathbf{M}\|_{0,\infty} \|\psi\|_{0,\infty} \quad \text{if } k \neq l.$$

Thus,

$$\sum_{l \neq k} |\beta_{kl}| \leq C(a_0, \mathbf{a}, T) \frac{\nu}{\varepsilon^2} \|\mathbf{M}\|_{0,\infty} \|\psi\|_{0,\infty} \leq C(a_0, \mathbf{a}, T, \mathbf{M}, \psi, C_{\text{stab}}),$$

and similarly

$$|\beta_{kk}| \leq C(a_0, \mathbf{a}, T, \mathbf{M}, \psi, C_{\text{stab}}).$$

Therefore, since $e_k(0) = 0$, Gronwall's lemma gives

$$\|\bar{e}(t)\|_\infty \leq \int_0^T e^{C(t-\tau)} \|\bar{\psi}(\tau)\|_\infty \, d\tau \leq C \operatorname*{Max}_{t \in [0,T]} \|\bar{\psi}(t)\|_\infty. \tag{40}$$

We apply Propositions 2 and 4 and get

$$|\psi_k(t)| \le C\nu \frac{h^m}{\varepsilon^{m+1}} \|f^\varepsilon(t)\|_{m,\infty} \le C\nu \frac{h^m}{\varepsilon^{m+1}} \|f_0\|_{m,\infty},$$

which, with (40), implies the stated estimate. □

4.4. *The Error Estimate for the Smoothed Particle Approximation.* We now state the final theorem concerning the approximation of the solution $f$ by the smoothed particle approximation $f_h^\varepsilon$ defined by (34). The proof simply consists in collecting the conclusions of the preceding propositions, and is left to the reader.

THEOREM. *Let $m, m' \ge n$, $r \ge 2$, $r' \ge 0$ be four integers, and define $s = \mathrm{Max}(r+2, m, r', m')$. Assume that*

$$\mathbf{a} \in L^\infty([0,T], W^{s+1,\infty}(\mathbb{R}^n)), \qquad a_0 \in L^\infty([0,T], W^{s,\infty}(\mathbb{R}^n)),$$
$$\mathbf{L} \in L^\infty([0,T], W^{s,\infty}(\mathbb{R}^n)), \qquad \mathbf{M} \in L^\infty([0,T], W^{s,\infty}(\mathbb{R}^n \times \mathbb{R}^n)).$$

*Assume that $\psi(x) \in W^{m,1}(\mathbb{R}^n)$, that $\varsigma(x) \in W^{m',1}(\mathbb{R}^n)$ and that $\psi$ and $\varsigma$ have compact supports. Assume further that* $\mathbf{M}$ *and $\psi$ satisfy Hypothesis* H, *and that $\varsigma$ satisfies the moment condition* (16) *up to the order $r'$. Then, for any positive constant $C_{\mathrm{stab}}$, and for any pair $(\nu, \varepsilon)$ such that*

$$\frac{\nu}{\varepsilon^2} \le C_{\mathrm{stab}},$$

*there exists a constant $C = C(a_0, \mathbf{a}, T, \mathbf{L}, \mathbf{M}, \psi, \varsigma, C_{\mathrm{stab}})$ such that for any $t$ in $[0,T]$, the solution $f$ of the convection-diffusion equation* (1) *and the function $f_h^\varepsilon$ given by* (34) *satisfy*

$$\|f(t) - f_h^\varepsilon(t)\|_{0,\infty} \le C\left[\nu\left(\varepsilon^r + \frac{h^m}{\varepsilon^{m+1}}\right) + \left(\varepsilon^{r'} + \frac{h^{m'}}{\varepsilon^{m'}}\right)\right].$$

**5. Conclusion.** This paper was devoted to a new particle approximation of convection-diffusion equations, when the diffusion coefficient is small. In Part 1 [1], we developed this method in the case of an isotropic diffusion operator. The present part was concerned with its extension to anisotropic operators. We have seen that this extension requires that particular attention be given to maintaining the consistency of the method, particularly in the choice of the cutoff functions. We investigated practical examples of cutoff functions which fulfill the requirements of consistency and accuracy, and proposed one of them for practical use. The convergence of the method is proved. Its stability is subject to the requirement that the diffusion coefficient be not too large compared to the square of the smoothing length. Therefore, this method improves as the diffusion goes to zero and consequently should only be used for diffusive perturbations of convective problems. The numerical validation of the method will be the next step in the present study.

## Appendix. $W^{m,\infty}$ Estimates for the Convection-Diffusion Equation.

PROPOSITION. *Assume that* $\mathbf{L}$ *is a positive symmetric matrix, and that $\nu \le 1$. Assume further that there is an integer $m \ge 1$ such that*

$$\mathbf{a} \in L^\infty([0,T], W^{m+1,\infty}(\mathbb{R}^n)), \qquad a_0 \in L^\infty([0,T], W^{m,\infty}(\mathbb{R}^n)),$$
$$\mathbf{L} \in L^\infty([0,T], W^{m,\infty}(\mathbb{R}^n)), \qquad f_0 \in W^{m,\infty}(\mathbb{R}^n).$$

*Then for any positive $T$, there exists a constant $C = C(T, \mathbf{a}, a_0, \mathbf{L})$ such that for any $t$ in $[0,T]$ and any $\nu$ in $[0,1]$, the solution of the convection-diffusion equation (1), with initial data $f_0$, satisfies*

$$\|f(t)\|_{m,\infty} \leq C(T, \mathbf{a}, a_0, \mathbf{L})\|f_0\|_{m,\infty}.$$

*Proof* (B. Perthame, private communication). We introduce the perturbed equation

$$\frac{\partial f}{\partial t} + \operatorname{div}(\mathbf{a}f) + a_0 f = \nu D^\varepsilon(t) \cdot f, \tag{41}$$

where

$$D^\varepsilon(t) = D(t) + \varepsilon\Delta = \sum_{i,j} \frac{\partial}{\partial x_i}\left(L^\varepsilon_{ij}\frac{\partial}{\partial x_j}\right),$$
$$\mathbf{L}^\varepsilon = \mathbf{L} + \varepsilon\, \mathbf{Id}_n.$$

The matrix $\mathbf{L}^\varepsilon$ is positive definite. Thus, the classical strong maximum principle holds for Eq. (41). Since the estimate is independent of $\varepsilon$, the same estimate is true in the limit $\varepsilon = 0$. Similar arguments are valid for the derivatives. □

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