MATHEMATICS OF COMPUTATION
VOLUME 53, NUMBER 188
OCTOBER 1989, PAGES 657-664

The Validity of Shapiro's Cyclic Inequality

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Abstract. A cyclic sum $S_N(\mathbf{x}) = \sum x_i/(x_{i+1} + x_{i+2})$ is formed with N components of a vector $\mathbf{x}$, where in the sum $x_{N+1} = x_1$, $x_{N+2} = x_2$, and where all denominators are positive and all numerators are nonnegative. It is known that there exist vectors $\mathbf{x}$ for which $S_N(\mathbf{x}) < N/2$ if $N \geq 14$ and even, and if $N \geq 24$. It has been proved that the inequality $S_N(\mathbf{x}) \geq N/2$ holds for $N \leq 13$. Although it has been conjectured repeatedly that the inequality also holds for odd N between 15 and 23, this has apparently never been proved. Here we will confirm that the inequality indeed holds for all odd $N \leq 23$. This settles the question for all N.

1. Introduction. The problem suggested by H. S. Shapiro in 1954 [12] has attracted wide interest; the history of the problem up to 1970 is described vividly by D. S. Mitrinović in his book "Analytic Inequalities" [8, pp. 132ff.]. When the problem was published, it appeared very reasonable to conjecture that $N/2$ is the minimum that the cyclic sum S_N can attain. It came therefore as a surprise that for some N actually $S_N(\mathbf{x}) < N/2$ is possible ([5], reporting a result by Lighthill). This led to the considerable interest in the problem.

It has been proved that $S_N(\mathbf{x}) \geq N/2$ for all admissible vectors $\mathbf{x}$, if $N \leq 13$ [14]. On the other hand, there exist vectors $\mathbf{x}$ such that $S_N < N/2$, if $N \geq 14$ and even, and also for all $N \geq 24$ ([7] contains a slight misprint). The difference in behavior for N even against N odd is explained in [11].

In this investigation it will be shown that $S_N \geq N/2$ for the remaining cases, namely $15 \leq N \leq 23$ and odd. This settles the question of Shapiro's inequality for all N. From a result in [1], it follows that only the case $N = 23$ need to be investigated: if the inequality $S_N(\mathbf{x}) \geq N/2$ holds for $N = 23$, it automatically holds for all lower odd N.

Unfortunately, the only feasible method to show that $S_{23} \geq 23/2$ appears to be based on the discussion and some numerical computation of many different cases. This approach has been used in [9] for $N = 10$, in [6] for $N = 12$, and in [14] for $N = 13$. The largest N where a purely algebraic proof has been successful is $N = 8$ [3].

It is crucial to consider the cases separately depending on which components of $\mathbf{x}$ are zero, and which components are different from zero. The reason for this is clear: S_N is a function of the N variables $x_1, x_2, \ldots, x_N$, where $x_k \geq 0$. At the stationary points of S_N we have $\partial S_N/\partial x_k = 0$ when $x_k > 0$, while at the boundary of the admissible domain where $x_k = 0$ the derivative of S_N need not vanish. Although no two consecutive components of $\mathbf{x}$ are permitted to vanish, the

Received November 8, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26D15, 52A40.
Key words and phrases. Cyclic inequality, cyclic sum, minimization.

number of possibilities nevertheless grows very rapidly with N, and turns out to be over 2500 for $N = 23$. It seems very undesirable to let the computer investigate all these cases.

2. General Description of the Method. The approach, the results, and the notation described in [14] will be used. The number and the positions of the zero components in the vector $\mathbf{x}$ is essential; the string of consecutive nonzero components is called a *segment*. There are three observations that immediately reduce the number of cases to be considered down to 100 cases. First, it is shown in [14] that there is no loss of generality if the segments are rearranged, for instance in order of decreasing segment length. Furthermore, a case with $S_N < N/2$ must necessarily contain a segment of length 6 at least ([14, Section 4]). And last, segments of length 2 need not be considered, because it can be shown that there is always another case that has a lower sum S.

Let us denote by $(c_1, c_2, \ldots, c_l)$ the case where c_1 is the length of the longest segment, down to c_l, the length of the shortest segment. The list of possibilities then starts out with (22), (20,1), (18,3), (18,1,1), (17,4), (16,5), (16,3,1) and ends with $(6,3,3,4*1)$, $(6,3,6*1)$, $(6,8*1)$, namely a 6-segment followed by eight one-segments. It turns out that many additional cases can be eliminated from consideration, if the inequalities to be described below are taken into account, together with the restriction on the pivotal ratio u which is easily obtained for segments of odd length up to length 9.

The remaining cases are then investigated by a comprehensive search in a small region of a two-parameter plane (see Figure 1). The implementation of the search requires only a few lines of programming.

3. The Properties of a Segment. From the remark above it follows that each segment can be analyzed separately, and then segments with the same leading ratio u (see below) are concatenated to find the admissible stationary point. According to [9], there is at most one of them for each case.

Let us therefore analyze a segment of length m in more detail. We take as example m to be odd to enable us to be specific in the signs, where they alternate. Therefore, we set (the zero components are not included in the numbering)

$$\mathbf{x} = x_1 \; 0 \; x_2 \; x_3 \cdots x_m \; x_{m+1} \; 0 \; x_{m+2} \cdots .$$

The sum for the m-segment is

$$S_m = \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_5} + \cdots + \frac{x_{m-1}}{x_m + x_{m+1}} + \frac{x_m}{x_{m+1}} + \frac{x_{m+1}}{x_{m+2}}.$$

A choice of new independent variables

$$y_1 = x_2,\; y_2 = x_3 + x_4, \ldots, y_{m-1} = x_m + x_{m+1},\; y_m = x_{m+1},\; y_{m+1} = x_{m+2}$$

is used with success in [9], [6], and [14], and solving for $\mathbf{x}$,

$$x_{m+2} = y_{m+1},\; x_{m+1} = y_m,\; x_m = y_{m-1} - y_m, \ldots, x_3 = y_2 - y_3 + y_4 \cdots - y_m,\; x_2 = y_1,$$

leads to

$$S_m = \frac{y_1}{y_2} + \frac{y_2 - y_3 \cdots - y_m}{y_3} + \frac{y_3 - y_4 \cdots + y_m}{y_4} \cdots + \frac{y_{m-2} - y_{m-1} + y_m}{y_{m-1}} + \frac{y_{m-1} - y_m}{y_m} + \frac{y_m}{y_{m+1}}$$

or

$$S_m = c_2 + c_3 + \cdots + c_m + c_{m+1},$$

which defines the ratios c.

As in [14, Section 3], we set $r_k = y_k/y_{k+1}$, so that $c_2 = r_1$, $c_{m+1} = r_m$, $y_3c_3 - y_2 = -y_4c_4$, and quite generally, $y_kc_k - y_{k-1} = -y_{k+1}c_{k+1}$, $k = 3, 4, \dots, m$. In terms of the r_k's this can be written as

$$c_{k+1} = r_k(r_{k-1} - c_k), \qquad k = 3, 4, \dots, m. \tag{3.1}$$

For a stationary S_m, namely $\partial S_m/\partial y_k = 0$ for $k = 2, 3, \dots, m+1$, we obtain

$$\begin{array}{rr}
 & -\frac{y_1}{y_2} + \frac{y_2}{y_3} = 0, \\
-\frac{y_2}{y_3} + 0 - \frac{y_4}{y_3} + \frac{y_5}{y_3} \dots & -\frac{y_{m-1}}{y_3} + \frac{y_m}{y_3} + \frac{y_3}{y_4} = 0, \\
+\frac{y_4}{y_3} - \frac{y_3}{y_4} + 0 - \frac{y_5}{y_4} \dots & +\frac{y_{m-1}}{y_4} - \frac{y_m}{y_4} + \frac{y_4}{y_5} = 0, \\
-\frac{y_5}{y_3} + \frac{y_5}{y_4} - \frac{y_4}{y_5} + 0 \dots & -\frac{y_{m-1}}{y_5} + \frac{y_m}{y_5} + \frac{y_5}{y_6} = 0, \\
 & \dots\dots \\
+\frac{y_{m-1}}{y_3} - \frac{y_{m-1}}{y_4} + \frac{y_{m-1}}{y_5} \dots & 0 - \frac{y_m}{y_{m-1}} + \frac{y_{m-1}}{y_m} = 0, \\
-\frac{y_m}{y_3} + \frac{y_m}{y_4} - \frac{y_m}{y_5} \dots & -\frac{y_{m-1}}{y_m} + 0 + \frac{y_m}{y_{m+1}} = 0, \\
 & -\frac{y_m}{y_{m+1}} + \frac{y_{m+1}}{y_{m+2}} = 0.
\end{array}$$

The first and last equation give $r_1 = r_2$, $r_m = r_{m+1}$, and adding all equations gives $u = r_2 = r_m = r_{m+1}$, where r_{m+1} is the leading element of the next segment. This shows, as mentioned in [14, Section 2], that at the stationary point all segments have the same pivotal element u, a fact which is very helpful in the investigation. With the notation above, the remaining equations become

$$\begin{array}{r}
-c_3 - 1 + r_3 = 0, \\
\frac{1}{r_3} - c_4 - 1 + r_4 = 0, \\
-\frac{1}{r_3r_4} + \frac{1}{r_4} - c_5 - 1 + r_5 = 0, \\
\dots\dots, \\
+\frac{1}{r_3r_4\cdots r_{m-2}} - \frac{1}{r_4r_5\cdots r_{m-2}} + \cdots + \frac{1}{r_{m-2}} - c_{m-1} - 1 + r_{m-1} = 0, \\
-\frac{1}{r_3r_4\cdots r_{m-1}} + \frac{1}{r_4r_5\cdots r_{m-1}} \cdots + \frac{1}{r_{m-1}} - c_m - 1 + r_m = 0.
\end{array} \tag{3.2}$$

Next, we leave the first equation as is, add the first equation to the second equation multiplied by r_3, add the second equation to the third equation multiplied by r_4, and so on:

$$c_3 = r_3 - 1,$$
$$c_3 = r_3(r_4 - c_4),$$
$$c_4 = r_4(r_5 - c_5),$$

or in general,

$$c_{k-1} = r_{k-1}(r_k - c_k), \quad k = 4, \ldots, m. \tag{3.3}$$

By returning to Eq. (3.1) it is easy to show as follows that the c_k's are symmetrical within a segment. Since $r_{m-1} = r_3$, $r_{m-2} = r_4, \ldots$ (see [14, Section 3]) and $c_{m+1} = u$, the last equation in (3.1), namely $c_{m+1} = r_m(r_{m-1} - c_m)$, becomes $1 = r_3 - c_m$, and hence, $c_m = c_3$ from Eqs. (3.3). Next, again from Eq. (3.1), $c_m = r_{m-1}(r_{m-2} - c_{m-1})$ or $c_3 = r_3(r_4 - c_{m-1})$ shows that $c_{m-1} = c_4$, and so on. This reduces the number of independent variables by nearly a factor of two.

The equations can now be solved recursively by assuming values for u and r_3, using Eqs. (3.3) and (3.1) in turn:

$$c_3 = r_3 - 1,$$
$$c_4 = r_3(u - c_3),$$
$$r_4 = c_4 + \frac{c_3}{r_3},$$
$$c_5 = r_4(r_3 - c_4),$$
$$r_5 = c_5 + \frac{c_4}{r_4},$$

and so on. The symmetry in c requires that for m odd the condition

$$c_{(m+1)/2} = c_{(m+5)/2}, \tag{3.4a}$$

and for m even,

$$c_{(m+2)/2} = c_{(m+4)/2}, \tag{3.4b}$$

must hold. For a fixed r_3, the values for u are changed to find the values for which this last condition is satisfied. Varying r_3 leads to curves in the $r_3 - u$ plane that have stationary values for S_m and are candidates for stationary values for the cyclic sum S_N.

There are two fortunate circumstances: the recursion formulas are identical for segments of any length, and the search can be restricted to a rather small region, as shown in Figure 1. To show this, we establish several bounds.

4. Some Inequalities. The following inequalities are all based on the fact that the c's and the r's must be positive.

a. Since $c_3 = r_3 - 1$, it follows that

$$r_3 > 1. \tag{4.1}$$

b. Next,

$$c_3 = \frac{y_2 - y_3 + \cdots - y_m}{y_3}$$

can also be written as

$$c_3 = u - 1 + \frac{1}{r_3} - \frac{y_6 c_6}{y_3},$$

and hence

$$u + \frac{1}{r_3} - r_3 - \frac{y_6 c_6}{y_3} = 0, \tag{4.2}$$

or

$$u > r_3 - \frac{1}{r_3}. \tag{4.3}$$

c. Similarly, c_4 can be written from its definition in two ways:

$$c_4 = r_3 - 1 + \frac{y_6 c_6}{y_4} = r_3 - 1 + \frac{1}{r_4} - \frac{y_7 c_7}{y_4}.$$

On the other hand, it follows from the second Eq. (3.2) that

$$c_4 = r_4 + \frac{1}{r_3} - 1, \tag{4.4}$$

and therefore

$$r_3 + \frac{1}{r_4} - r_4 - \frac{1}{r_3} = \frac{y_7 c_7}{y_4} > 0,$$

$$(r_3 - r_4)\left(1 + \frac{1}{r_3 r_4}\right) > 0,$$

and finally

$$r_4 < r_3. \tag{4.5}$$

d. A useful inequality is obtained by the other representation for c_4 above:

$$r_3 - r_4 - \frac{1}{r_3} + \frac{y_6 c_6}{y_4} = 0$$

dividing it by r_3,

$$1 - \frac{r_4}{r_3} - \frac{1}{r_3^2} + \frac{y_6 c_6}{y_3} = 0,$$

and adding it to Eq. (4.2) gives

$$u = r_3 - \frac{1}{r_3} + \frac{1}{r_3^2} - \left(1 - \frac{r_4}{r_3}\right). \tag{4.6}$$

The desired inequality is

$$u < r_3 - \frac{1}{r_3} + \frac{1}{r_3^2}. \tag{4.7}$$

e. Equation (4.6) gives also the result

$$u - r_4 = r_3 - r_4 - \frac{1}{r_3} + \frac{1}{r_3^2} - 1 + \frac{r_4}{r_3} = \left(1 - \frac{1}{r_3}\right)\left(r_3 - \frac{1}{r_3} - r_4\right).$$

If, as we will show next, the second factor is negative, then

$$r_4 > u. \tag{4.8}$$

To this end, we write Eq. (3.1) for $k = 4$ and $k = 5$:

$$\frac{c_5}{r_4} = (r_3 - c_4), \qquad c_6 = r_5(r_4 - c_5);$$

therefore $r_4 > c_5$, or $c_5/r_4 < 1$, so that $r_3 < c_4 + 1$, and then from Eq. (4.4), $r_3 < r_4 + 1/r_3$, as claimed above.

f. Furthermore, by similar considerations, one can show that, after some algebra,

(4.9) $$r_5 > 1 \quad \text{if } r_4 > 1,$$

since $r_5 - 1 = (r_4 - 1/r_4)(1 - 1/r_3) + r_4(r_3 - r_4)$.

Therefore, if $u > 1$, then $r_3 > 1$, $r_4 > 1$, and $r_5 > 1$ follows from Eqs. (4.1) and (4.8). This result then eliminates analytically many cases if the longest segment is a 9-segment.

5. The Curves in the $r_3 - u$ Plane. The recursion formulas and Eq. (3.4) show that an admissible segment of length m is completely determined by r_3 and u. Of particular interest are the values of

$$p_m = \prod_{j=1}^{m} r_j,$$

since from the definition of the r_j's in any particular case the product of all p_k's must equal 1 ([14, Section 3]), and the values of

$$S_m = \sum_{j=2}^{m+1} c_j.$$

The final goal is to show that

$$S_{23} = \sum S_m \geq 23/2$$

in all cases.

As mentioned above, the search in the $r_3 - u$ plane can be restricted to a small region because of the inequalities (4.1), (4.3), and (4.7). Furthermore, segments need only be considered if

(5.1) $$u < 2.2.$$

Otherwise, it follows from Eqs. (4.7), (4.8), and (4.9) that $r_3 > 2.4$, $r_4 > 2.2$, and $r_5 > 1$. A simple computation then shows that $S_7 > 11.8$, exceeding the allowed limit already. All longer segments have an even larger sum. Similar considerations show that $u < 1.4$ must hold, except in four cases.

The search for cases with possibly $S_{23} < 23/2$ can therefore be restricted to the small region shown in Figure 1. The admissible values for an individual segment lie on smooth curves; in Figure 1, the curves for the 8-segment and for the 11-segment are drawn as examples. Segments up to length 9 have just one curve, as can be proved by Descartes's rule of signs, whereas longer segments have one or two curves, with the exception of the 19-segment, which has three curves.

The computation starts with the longest segment in the case being considered. To find a point P on the $r_3 - u$ curve, the r_3 is kept constant and u is changed until Eq. (3.4) is satisfied. The search in r_3 with fixed u is less desirable because of the shape of some curves, like the 11-segment curve. The point P can be ignored, if any of the r's or c's turn out to be negative, or if the sum $S_m \geq 23/2$. For segments which are no longer than length 4, the explicit formulas for p_k and S_k, given in [14],

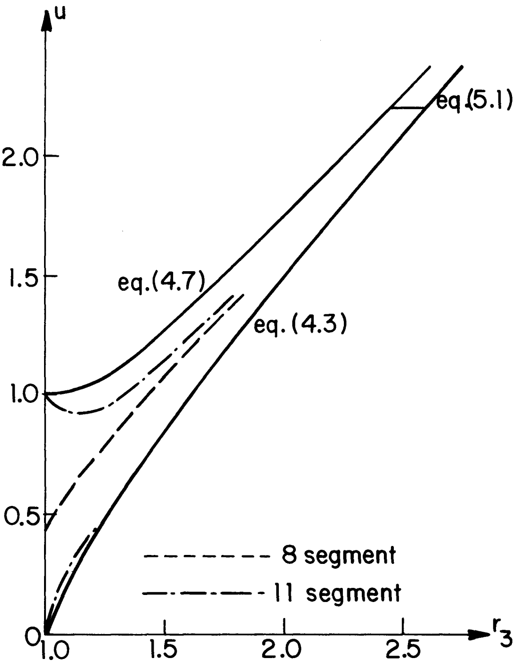

FIGURE 1

Region of admissible solutions, bounded by Eqs. (4.1), (4.3), (4.7), (5.1).

can be computed simultaneously and added to S_m. Only the points where this sum is smaller than 23/2 need to be analyzed further.

Advantage can also be taken of the fact that for 7- and 9-segments, $u \geq .922$, and that $S_5 \geq 3.0$, $S_7 \geq 4.0$, and $S_9 \geq 5.0$.

Among the about twenty cases left with the possibility that $S_{23} < 23/2$, most are resolved by casual inspection of the numerical results. The cases with the smallest sum S_{23} are listed in Table 1, and all other cases have a larger sum, except for the trivial case with all $x_k = 1$.

In order to check the results and the numerical approach, several cases between $N = 14$ and $N = 22$ were computed by the method described above and the same programming implementation, and indeed the values for $S_N < N/2$ were found, for instance, the case (11, 1, 1) led to $S_{16} < 7.989$.

TABLE 1

Case	$(20,1)$	$(18,1,1)$	$(16,1,1,1)$	$(14,4*1)$	$(12,5*1)$
$\min S_{23}$	11.513	11.512	11.513	11.520	11.533

6. A Remark. Since inf $S_N < N/2$ occurs already for $N = 14$, it might be reasonable to expect that for very large N the ratio S_N/N could fall well below the value 1/2. The result in [10] that $S_N/N \geq 0.3307\ldots$ and in [2] that $S_N/N \geq 0.461238\ldots$ for any N were therefore significant. However, in a remarkable paper, Drinfeld [4] proved that $\inf_N(S_N/N) = 0.4945668$. Without the knowledge of Drinfeld's proof, the same result was obtained in [13], including the formulas identical to those in [4]. But this did not constitute a proof, but rather an example of [4], because a definite distribution of the zero-components of $\mathbf{x}$ was assumed. The assumption appeared reasonable, based on previous experience. It would be desirable to prove that for any N this particular distribution of nonzero components always gives the lowest sum S_N, except of course for the case with all components equal to 1. A result of this kind would make the investigation reported here essentially trivial.

It seems astounding that S_N/N, which can be made easily as low as $1/2$ for any $N \geq 3$ by choosing all $x_k = 1$, can never fall below that value by more than about 1%.

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1. P. H. DIANANDA, "On a cyclic sum," *Proc. Glasgow Math. Assoc.*, v. 6, 1963, pp. 11-13.
2. P. H. DIANANDA, "A cyclic inequality and an extension of it. II," *Proc. Edinburgh Math. Soc.* (2), v. 13, 1962, pp. 143-152.
3. D. Ž. DJOKOVIĆ, "Sur une inégalité," *Proc. Glasgow Math. Assoc.*, v. 6, 1963, pp. 1-10.
4. V. G. DRINFELD, "A cyclic inequality," *Math. Notes*, v. 9, 1971, pp. 68-71.
5. C. V. DURELL, "Query," *Math. Gaz.*, v. 40, 1956, p. 266.
6. E. K. GODUNOVA & V. I. LEVIN, "A cyclic sum with 12 terms," *Math. Notes*, v. 19, 1976, pp. 510-517.
7. J. C. LAGARIAS, "The van der Waerden conjecture: Two Soviet solutions," *Notices Amer. Math. Soc.*, v. 29, 1982, pp. 130-133.
8. D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
9. P. NOVOSAD, "Isoperimetric eigenvalue problems in algebra," *Comm. Pure Appl. Math.*, v. 21, 1968, pp. 401-465.
10. R. A. RANKIN, "A cyclic inequality," *Proc. Edinburgh Math. Soc.* (2), v. 12, 1961, pp. 139-147.
11. J. L. SEARCY & B. A. TROESCH, "A cyclic inequality and a related eigenvalue problem," *Pacific J. Math.*, v. 81, 1979, pp. 217-226.
→ H. S. SHAPIRO, "Problem 4603," *Amer. Math. Monthly*, v. 61, 1954, p. 571.
13. B. A. TROESCH, "The shooting method applied to a cyclic inequality," *Math. Comp.*, v. 34, 1980, pp. 175-184.
14. B. A. TROESCH, "On Shapiro's cyclic inequality for $N = 13$," *Math. Comp.*, v. 45, 1985, pp. 199-207.