Long Chains of Nearly Doubled Primes

By Günter Löh

Abstract. A chain of nearly doubled primes is an ordered set $\{a_1, a_2, \ldots, a_\lambda\}$ of prime numbers, interlinked by $a_k = 2a_{k-1} \pm 1$. A search for long chains of this kind has been performed in the range $a_1 < 2^{50}$. Chains of length up to 13 have been found. Shorter chains have been counted in some restricted ranges. Some of these counts are compared with the frequencies predicted by a quantitative version of the prime k-tuples conjecture.

1. Introduction. Let us consider a sequence of nearly doubled integers, recursively defined by $a_k = 2a_{k-1} + \varepsilon$, $\varepsilon \in \{-1, +1\}$, starting with some a_1. If a_1 is a prime, one can ask how many consecutive primes occur in the sequence. We say that the ordered set $\{a_1, a_2, \ldots, a_\lambda\}$ forms a prime chain of length λ, if $a_{\lambda+1}$ is the first composite number of the sequence. The value ε has to be kept fixed throughout the whole prime chain, because a change of sign for $a_1 > 3$ would yield the numbers a_1, $a_2 = 2a_1 + \varepsilon$, $a_3 = 2(2a_1 + \varepsilon) - \varepsilon$, one of which is divisible by 3. A well-known example for the case $\varepsilon = +1$ is the prime chain $\{89, 179, 359, 719, 1439, 2879\}$ of length $\lambda = 6$.

Guy [3, pp. 12-13] pointed out that prime chains appear as a special case when commonly used methods for proving that a number p is prime are applied. These methods require the factorization of either $p - 1$ or $p + 1$. If $p - \varepsilon = 2q$, where q is another prime, the size of the problem is only reduced by one bit. This worst case may occur several times in sequence.

In the case $\varepsilon = +1$, $\lambda \geq 2$, the prime a_1 is a so-called Sophie Germain prime. For these primes a_1, where $a_1 > 2$, the first case of Fermat's last theorem holds (Ribenboim [8, pp. 55-56]).

We can express a_k as $a_k = 2^{k-1}a_1 + \varepsilon(2^{k-1} - 1)$. According to Fermat's little theorem, if $k = a_1 > 2$, then a_1 divides a_k. This means that for $a_1 > 2$ the length λ of a prime chain cannot exceed $a_1 - 1$ (Nederpelt et al. [7]). Actually, most of the prime chains are considerably shorter. It is not known whether arbitrarily long chains exist.

Extending Yates' notation [13, p. 115], we describe the set of primes constituting a prime chain by $S_\varepsilon(\lambda, a_1)$. Thus, for fixed ε, each prime a_1 uniquely determines a chain $S_\varepsilon(\lambda, a_1)$ which contains $\lambda - 1$ shorter chains $S_\varepsilon(\lambda + 1 - k, a_k)$, $k = 2, 3, \ldots, \lambda$, whenever $\lambda > 1$.

We say that a prime chain $S_\varepsilon(\lambda, a_1)$ is complete if it cannot be extended downwards, that is, if the rational number $a_0 = (a_1 - \varepsilon)/2$ is a composite integer or

smaller than 2. Hence, for a given ε, each prime belongs to one and only one complete chain.

It seems reasonable to consider pairs of mutually related prime chains, both of which grow simultaneously. But for most of the possible configurations, one of the chains breaks down prematurely. For instance, looking at the chains $S_{-1}(\lambda_{-1}, a_1)$ and $S_{+1}(\lambda_{+1}, a_1)$, we observe that for $a_1 > 3$ either λ_{-1} or λ_{+1} must be 1, since one of the numbers $2a_1 - 1$, $2a_1 + 1$ must be divisible by 3. On the other hand, we may consider pairs of chains $S_{+1}(\lambda_{+1}, r-1)$ and $S_{-1}(\lambda_{-1}, r+1)$, where $(r-1, r+1)$ is a pair of twin primes. In this case, the length of neither chain is restricted in such a way. Moreover, it is not known how many consecutive twin prime pairs formed by both chains can occur.

For $\varepsilon = +1$, the first remarkable prime chains were presented by Cunningham [2] in 1907. Accordingly, the chains $S_{+1}(\lambda, a_1)$ are also called Cunningham chains. He gave the above example $S_{+1}(6, 89)$ as well as $S_{+1}(6, 63419)$ and $S_{+1}(6, 127139)$. The length $\lambda = 6$ was not surpassed for more than half a century.

In 1965 Lehmer [6] found several chains of length $\lambda = 7$ for both values of ε, specifically, $S_{+1}(7, 1122659)$, $S_{+1}(7, 2164229)$, $S_{+1}(7, 2329469)$, $S_{+1}(7, 10257809)$, $S_{+1}(7, 10309889)$, and, for $\varepsilon = -1$, the chains $S_{-1}(7, 16651)$, and $S_{-1}(7, 165901)$. One additional chain in his paper, with $\varepsilon = -1$ and alleged length $\lambda = 7$ starting with $a_1 = 67651$, must be ruled out since $a_5 = 1082401 = 601 \cdot 1801$. Lehmer also gave some hints on how to efficiently compute such chains. He expected that chains of length $\lambda = 8$ should begin to appear for $a_1 \approx 10^9$.

Fifteen years later, the first chains of length $\lambda = 8$ for both ε were published by Lalout and Meeus. In [5] they reported that Nelson discovered $S_{+1}(8, 19099919)$ and Lalout discovered $S_{-1}(8, 15514861)$. Furthermore, all complete chains of length $\lambda \geq 5$ with $a_1 < 5 \cdot 10^5$, and length $\lambda \geq 6$ with $a_1 < 2 \cdot 10^7$ were given in [5]. In an editorial note by Nelson to that paper, it was stated that chains of length $\lambda = 9$ do exist for both ε, but only $S_{-1}(9, 857095381)$ was explicitly given. It was also reported that for $a_1 < 2 \cdot 10^9$ no chains of length $\lambda = 10$ had been found. It should be mentioned that Sumiyama [11] first published the chain $S_{+1}(9, 85864769)$ already known to Nelson.

In this paper, we have extended the search for long chains as far as $a_1 < 2^{50} \approx 1.12 \cdot 10^{15}$. We have succeeded in finding several prime chains of length λ, $10 \leq \lambda \leq 12$ for both values of ε, and one of length $\lambda = 13$ in the case $\varepsilon = -1$. Moreover, for all λ we determined the frequencies of complete chains up to $a_1 < 2^{26}$. For some larger limits up to $a_1 < 2^{36}$, short chains were discarded in order to reduce the amount of computation. The programs were coded in FORTRAN 77 and ran for about 520 CPU hours on a SIEMENS 7·882 computer.

2. Computations. In all our computations we used powers of 2 as search limits, for a reason to be indicated later. Basically, only complete chains starting with a given prime a_1 were determined. For $a_1 < 2^{26}$ we examined *all* primes as possible first members of a complete chain. For $a_1 > 2^{26}$, however, we chose some minimal length l and restricted ourselves to prime chains with length $\lambda \geq l$. When doing so, one can establish congruence conditions for a_1 involving the first μ primes in order to speed up the search. Such conditions were already explicitly given by Cunningham [2] for $l = 4$, $\mu = 4$ and by Eisner [7] for $l = 7$, $\mu = 7$.

For instance, if we only look for chains of minimal length $l = 12$, in the case $\varepsilon = -1$ the number a_1 must satisfy the following congruences for the first $\mu = 9$ primes:

$$a_1 \equiv 1 \pmod{2 \cdot 3 \cdot 5 \cdot 11 \cdot 13}$$

$$a_1 \equiv 1, 2, 3, \text{ or } 5 \pmod 7$$

$$a_1 \equiv 1, 4, 6, 7, 8, 11, 12, 13, \text{ or } 15 \pmod{17}$$

$$a_1 \equiv 1, 4, 7, 12, 13, 16, \text{ or } 18 \pmod{19}$$

$$a_1 \equiv 1, 2, 3, 4, 5, 7, 9, 10, 13, 14, 17, \text{ or } 19 \pmod{23}.$$

Let q_j denote the jth prime. For $j \geq 2$, let r_j be the least positive power of 2 with $2^{r_j} \equiv 1 \pmod{q_j}$, and set $r_1 = 1$. Then only $\prod_{j=1}^{\mu}(q_j - \min(r_j, l))$ numbers out of a full cycle of $\prod_{j=1}^{\mu} q_j$ numbers remain for further testing. So in the above example, we have a cycle length of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 = 223092870$ numbers, of which only $1 \cdot 1 \cdot 1 \cdot 4 \cdot 1 \cdot 1 \cdot 9 \cdot 7 \cdot 12 = 3024$ have to be considered as a possible a_1.

On these remaining numbers we performed trial division using the primes $q_{\mu+1}$, $q_{\mu+2}, \ldots, q_m$, where m had to be chosen appropriately. The single remainder $a_1 \bmod q_j$ suffices to determine whether any one of the $a_k, k \leq l$, is divisible by q_j. The surviving numbers a_1 and the subsequent members of a possible chain were tested for compositeness by evaluating $13^{a_k-1} \bmod a_k$, for $k = 1, 2, 3, \ldots$, until the first composite number was encountered. Thus, at least l probable primes in succession had to be gathered to obtain a chain in the range being explored. Once such a chain was found, all probable primes were validated by conventional methods. The choice of $m = 300$ brought the total computational effort for trial division and compositeness testing close to a minimum.

We searched for chains of length $\lambda \geq l$, $l = 4, 5, 6$ and $a_1 < 2^{29}, 2^{33}, 2^{36}$, respectively. In these regions we found many prime chains of length λ up to 9 for both ε. In addition, for $\varepsilon = +1$ two chains of length $\lambda = 10$ were found. Looking for longer chains, we continued the search with $l = 10$ and $a_1 < 2^{43}$. In this region we found several chains; the first members of these are listed in Table 1. It should be noted that in the case $\varepsilon = +1$ a chain of length $\lambda = 12$ was found before a chain of length $\lambda = 11$ had occurred. Furthermore, in Table 2 the prime chains with $\lambda \geq l = 12$ and $a_1 < 2^{50}$ are presented.

For the reader's convenience we have compiled Table 3, which shows the first appearance of a complete chain for each ε and all λ up to 12 resp. 13. The chains of length $\lambda \geq 10$ are taken from the previous Tables 1 and 2.

Now, let $(r - 1, r + 1)$ be a pair of twin primes and consider the two chains $S_{+1}(\lambda_{+1}, r - 1)$ and $S_{-1}(\lambda_{-1}, r + 1)$. We ask for the maximal number $\lambda = \min(\lambda_{-1}, \lambda_{+1})$ of consecutive twin prime pairs produced by both chains. For the first appearance of such a twin chain of length λ, we observed the following values of r: $\lambda = 1$, $r = 4$; $\lambda = 2$, $r = 6$; $\lambda = 3$, $r = 211050$; $\lambda = 4$, $r = 253680$; $\lambda = 5$, $r = 41887255410$; $\lambda = 6$, $r = 73768891456260$.

Turning to the frequencies of shorter chains, in Tables 4 and 5 we give the counts $c_\varepsilon(\lambda, 2^\nu)$ of all complete prime chains of length λ for $a_1 < 2^\nu$, $16 \leq \nu \leq 36$, where the table values are restricted to the regions indicated above. The corresponding counts $d_\varepsilon(\lambda, 2^\nu)$ of *all* prime chains, including those which are not complete, can be derived from the values $c_\varepsilon(\lambda, 2^\nu)$, and vice versa.

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TABLE 1

Complete chains with $a_1 < 2^{43}$ and length $\lambda \geq 10$

$\varepsilon = +1$		$\varepsilon = -1$	
a_1	λ	a_1	λ
26089808579	10	205528443121	10
65639153579	10	207526905301	10
372339715439	10	237441500011	10
551652886619	10	428822526001	10
554688278429	12	605727738331	10
570901515029	10	629912195221	10
624758088239	10	1220127683941	10
665043081119	11	1389122693971	11
721438465439	10	2339669157541	11
1243541923469	10	2514864555061	10
1341807015119	10	2605339529101	10
2027840556719	10	3264821602351	10
2452499778299	10	3689229625951	10
2486906850119	10	4704099859471	10
2834226458909	10	4723155667231	10
3495773443139	10	5321563702621	10
3717568843559	10	6131306463151	10
3919801172429	10	6289424815411	10
4228633787069	10	6454116380581	10
4315740820379	10	6860403225391	10
4389398045009	10	7165183292851	11
4709436287819	10	7557661146751	10
5213499728579	11	7603857496951	10
5496251895419	10	7697789715331	10
6424044070289	11	7918316060821	10
7623279613949	10		

TABLE 2

Complete chains with $a_1 < 2^{50}$ and length $\lambda \geq 12$

$\varepsilon = +1$		$\varepsilon = -1$	
a_1	λ	a_1	λ
554688278429	12	216857744866621	12
17287267757759	12	758083947856951	13
175469208383039	12	890299395628501	12
485385457835849	12	952373350504291	12

This interdependence between a table of $c_\varepsilon(\lambda, 2^\nu)$ and a similar table of $d_\varepsilon(\lambda, 2^\nu)$ strongly relies on the fact that the limits for a_1 are given as powers of 2. For this implies that a chain starting with an odd prime a_1 in the interval $2^{\nu-1} < a_1 < 2^\nu$ has its successive members a_k in the intervals $2^{\nu+k-2} < a_k < 2^{\nu+k-1}$, $k \geq 1$. This can easily be seen to be true for *both* values of ε, when the closed form expression for a_k given in the introduction is recalled.

TABLE 3

The smallest a_1 starting a complete chain of length λ

	$\varepsilon = +1$	$\varepsilon = -1$
λ	a_1	a_1
1	13	11
2	3	7
3	41	2
4	509	2131
5	2	1531
6	89	385591
7	1122659	16651
8	19099919	15514861
9	85864769	857095381
10	26089808579	205528443121
11	665043081119	1389122693971
12	554688278429	216857744866621
13	?	758083947856951

TABLE 4

Counts of complete chains $c_\varepsilon(\lambda, 2^\nu)$ with $a_1 < 2^\nu$ and length λ

$\varepsilon = +1$

ν	$\lambda = $ 1	2	3	4	5	6	7	8	9	10
16	5326	608	107	21	4	2	0	0	0	0
17	10109	1077	192	31	5	3	0	0	0	0
18	19242	1887	346	49	9	3	0	0	0	0
19	36795	3387	564	71	14	4	0	0	0	0
20	70205	6185	999	113	25	5	0	0	0	0
21	134307	11297	1711	196	37	11	1	0	0	0
22	257193	20816	3029	325	64	18	3	0	0	0
23	493482	38295	5331	531	110	36	3	0	0	0
24	948147	71045	9516	904	176	51	8	0	0	0
25	1825113	131502	16999	1522	289	77	11	1	0	0
26	3518641	243887	30189	2643	493	105	16	2	0	0
27				4542	846	159	21	2	1	0
28				8009	1397	261	39	3	2	0
29				14039	2342	410	66	5	3	0
30					4032	681	91	8	3	0
31					6965	1102	157	14	3	0
32					12023	1849	254	22	4	0
33					20653	3075	445	38	6	0
34						5244	694	72	10	0
35						8821	1084	118	12	1
36						14970	1766	185	17	2

The values $d_\varepsilon(\lambda, 2^\nu)$ can be derived from those of $c_\varepsilon(\lambda, 2^\nu)$ by summation over certain diagonals using the formula

$$d_\varepsilon(\lambda, 2^\nu) = \sum_{n=0}^{\nu-16} c_\varepsilon(\lambda + n, 2^{\nu-n}) + d_\varepsilon(\lambda + \nu - 15, 2^{15}).$$

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TABLE 5

Counts of complete chains $c_\varepsilon(\lambda, 2^\nu)$ with $a_1 < 2^\nu$ and length λ

$$\varepsilon = -1$$

ν	$\lambda = $ 1	2	3	4	5	6	7	8	9	10
16	5313	595	124	18	5	0	1	0	0	0
17	10131	1039	208	25	7	0	1	0	0	0
18	19282	1854	365	37	14	0	2	0	0	0
19	36816	3359	587	58	24	3	2	0	0	0
20	70183	6206	1008	92	39	4	2	0	0	0
21	134167	11364	1757	164	61	7	3	0	0	0
22	257131	20734	3070	302	84	16	4	0	0	0
23	493390	38284	5445	520	126	22	5	0	0	0
24	947949	71033	9621	867	187	40	8	1	0	0
25	1825061	131520	16880	1484	311	60	12	1	0	0
26	3518643	244165	29930	2651	523	96	18	2	0	0
27				4572	888	149	25	2	0	0
28				7999	1469	239	42	2	0	0
29				13935	2452	382	64	2	0	0
30					4177	618	99	9	1	0
31					7079	1055	165	14	1	0
32					12108	1791	250	24	2	0
33					21143	2977	403	34	5	0
34						5043	667	54	9	0
35						8635	1105	85	14	0
36						14787	1739	152	20	0

TABLE 6

Number of chains with $a_1 < 2^\nu$ and minimal length l

l	ν	Approx.	$\varepsilon = +1$ Count	Ratio	$\varepsilon = -1$ Count	Ratio
2	20	8583	8051	0.938	8088	0.942
	23	50477	48024	0.951	48165	0.954
	26	309743	297544	0.961	297412	0.960
3	20	1499	1246	0.831	1250	0.834
	23	7474	6475	0.866	6603	0.883
	26	39981	35625	0.891	35355	0.884
4	20	252	164	0.651	169	0.671
	23	1016	777	0.765	789	0.777
	26	4665	3699	0.793	3728	0.799
	29	23178	18769	0.810	18784	0.810

The last term of this sum, with first argument $\lambda + \nu - 15 \geq 2$, must be supplied additionally. The required values of $d_\varepsilon(\lambda, 2^{15})$ are as follows: $d_{+1}(\lambda, 2^{15}) = 382, 70, 19, 2, 1$ for $\lambda = 2, 3, 4, 5, 6$, respectively, and $d_{-1}(\lambda, 2^{15}) = 389, 79, 14, 3, 0, 1$ for $\lambda = 2, 3, 4, 5, 6, 7$, respectively. For larger values of λ, all $d_\varepsilon(\lambda, 2^{15})$ are zero.

Since every prime p starts one (not necessarily complete) prime chain with $a_1 = p$, we have, for fixed ν,

$$\sum_{\lambda \geq 1} d_{+1}(\lambda, 2^\nu) = \sum_{\lambda \geq 1} d_{-1}(\lambda, 2^\nu) = \pi(2^\nu),$$

where $\pi(n)$ is the number of all primes not exceeding n. As a check of the tabulated values $c_\varepsilon(\lambda, 2^\nu)$ for $\nu \leq 26$, we verified this relation by summing up the counts $d_\varepsilon(\lambda, 2^\nu)$ derived from Tables 1 and 2. The counts $d_\varepsilon(\lambda, 2^\nu)$ can also be used to compute the number of chains of minimal length l in the range $a_1 < 2^\nu$ by evaluating the sum $\sum_{\lambda \geq l} d_\varepsilon(\lambda, 2^\nu)$. For example, with $l = 2$ and $\varepsilon = +1$ the total number of Sophie Germain primes below 2^ν is obtained. For $\nu = 20, 23, 26$ the actual counts are found in Table 6.

In [12] Sumiyama gave an empirical formula for the distribution of prime chains with $\varepsilon = +1$, but only considering chains for which $a_1 \equiv -1 \pmod{30}$. This congruence is satisfied for all chains with $\lambda \geq 4$ with the exception of $S_{+1}(5,2)$. For the sake of comparison, he counted all chains in question with $a_1 < 3 \cdot 10^7 \approx 1.79 \cdot 2^{24}$. In our notation, he computed $d_{+1}(\lambda, 3 \cdot 10^7)$ for $\lambda \geq 6$, and $d_{+1}(\lambda, 3 \cdot 10^7) - 1$ for $\lambda = 4, 5$, the reduction by 1 being caused by the chain $S_{+1}(5,2)$. Owing to the congruence relation, for $\lambda \leq 3$ Sumiyama's counts are not comparable with ours. Nevertheless, in spite of the different ways of counting and the deviating range limits, we could confirm all his counts.

3. Approximations. As we have said, it is not known whether arbitrarily long chains exist. It is also not known whether infinitely many chains of a given minimal length l exist, even for $l = 2$. However, for any minimal length l and choice of $\varepsilon = \pm 1$, the prime k-tuples conjecture [10, Hypothèse H] does imply that infinitely many chains of this length and type exist. A quantitative form of this conjecture proposed by Bateman and Horn [1] even predicts how many chains should appear, asymptotically, below a limit N.

Specifically, suppose $f_1, f_2, \ldots, f_k$ are linear polynomials in one variable with all coefficients integral and leading coefficients positive, and no two of them differ by a constant factor. Let $Q(N)$ be the number of integers n, $1 \leq n \leq N$, such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes. Then, for large N it is expected that

$$Q(N) \sim C(f_1, f_2, \ldots, f_k) \int_2^N \frac{du}{(\log u)^k},$$

where

$$C(f_1, f_2, \ldots, f_k) = \prod_{j=1}^{\infty} \left\{ \left(1 - \frac{1}{q_j}\right)^{-k} \left(1 - \frac{\omega(q_j)}{q_j}\right) \right\}.$$

Here $\omega(q_j)$ denotes the number of distinct solutions of the congruence

$$f_1(x) f_2(x) \cdots f_k(x) \equiv 0 \pmod{q_j},$$

q_j being the jth prime. It is agreed to exclude the trivial case that $\omega(q_j) = q_j$ for some j, which would imply $C(f_1, f_2, \ldots, f_k) = 0$.

In the case of prime chains of minimal length l the numbers $\omega(q_j)$ are precisely given by $\min(r_j, l)$, the expression introduced above when our sieving procedure was described.

On the other hand, for ordinary twin pairs, triplets, and quadruplets of primes, the Bateman-Horn formula reproduces the well-known prediction of Hardy and Littlewood [4, p. 61], which states that for primes $p \leq N$ the number of twins $(p, p+2)$, of triplets $(p, p+2, p+6)$ or $(p, p+4, p+6)$, and of quadruplets $(p, p+2, p+6, p+8)$ asymptotically equals

$$1.320323632 \int\limits_2^N \frac{du}{(\log u)^2}, \quad 2.858248596 \int\limits_2^N \frac{du}{(\log u)^3}, \quad 4.151180864 \int\limits_2^N \frac{du}{(\log u)^4},$$

respectively. The constants preceding the integrals are taken from [9, p. 65].

When prime chains of minimal length $l = 2, 3, 4$ are considered, the same values of $\omega(q_j)$ as for the respective l-tuples are obtained, with only one exception. In the case of chains with $l = 4$ we have $\omega(q_4) = 3$, while for quadruplets $\omega(q_4) = 4$. Thus the contribution of the factor $(1 - \omega(q_4)/q_4)$ to the constant $C(f_1, f_2, f_3, f_4)$ is $4/7$ for chains, instead of $3/7$ for quadruplets. Summarizing, the approximating functions for chains with $l = 2, 3$ are the same as those given above for twins and triplets, and the number of chains with $l = 4$ is approximated by the third function, multiplied by $4/3$. To evaluate the integrals we shall choose $N = 2^\nu$ for certain values of ν, as we did for the limits of all our counts.

In Table 6 we compare the predicted quantities with the actual counts $\sum_{\lambda \geq l} d_\varepsilon(\lambda, 2^\nu)$ of chains of minimal length $l = 2, 3, 4$ for $\nu = 20, 23, 26$ and both choices of ε. For $l = 4$, $\nu = 29$ is also included. The ratios between the counted and the estimated quantities reflect the behavior of the approximating functions. As the conjecture predicts, for a given l the ratio in question appears to approach 1 for increasing ν. However, for longer chains this convergence becomes noticeably slower. As a matter of fact, meaningful estimates can only be expected if l is small and ν is sufficiently large. Otherwise, the computed approximation will not even produce the correct order of magnitude. Therefore, it appears to be illusive to use the Bateman-Horn formula to guess where a first chain of length $\lambda = 14$ might occur.

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