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A NOTE ON DISCRETE SOLUTIONS OF THE PLATEAU PROBLEM

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ABSTRACT. In this paper we prove theorems for convergence of discrete solutions of the Plateau problem under the assumption that the contour is rectifiable.

1. INTRODUCTION

In [7] the discrete solutions of the *Plateau problem* were defined, and some theorems for its convergence were proved under a very restrictive condition. The purpose of this paper is to show that we can obtain the same conclusions if the contour is rectifiable.

It is well known [2, pp. 107–118] that the Plateau problem can be defined as the following variational problem:

Let $D = \{(u, v) \in \mathbf{R}^2 | u^2 + v^2 < 1\}$ be the unit disk with boundary ∂D and let Γ be a Jordan curve in n-dimensional Euclidean space $\mathbf{R}^n$, $n \geq 2$. Let $C(\overline{D}; \mathbf{R}^n)$ be the space of continuous maps from $\overline{D}$ into $\mathbf{R}^n$, and let $H^1(D; \mathbf{R}^n)$ be the ordinary Sobolev space (for the exact definitions, see [7]). We define the class of maps by

$$X_\Gamma = \{f \in C(\overline{D}; \mathbf{R}^n) \cap H^1(D; \mathbf{R}^n) | f(\partial D) = \Gamma, \ f|_{\partial D} \text{ is monotone}\},$$

where monotone means that, for each $p \in \Gamma$, $(f|_{\partial D})^{-1}(p) \subset \partial D$ is connected. X_Γ may be empty [4, p. 58], but if Γ is rectifiable, then $X_\Gamma \neq \emptyset$ [2, pp. 129–131]. We choose six arbitrary distinct points $z_1, z_2, z_3 \in \partial D$ and $\zeta_1, \zeta_2, \zeta_3 \in \Gamma$, and we define the subset of X_Γ by

$$X_\Gamma^{\text{tp}} = \{f \in X_\Gamma | f(z_i) = \zeta_i, \ i = 1, 2, 3\},$$

where the superscript "tp" stands for "three-point condition". The Plateau problem is to find stationary points of the *energy functional*

$$E(f) = \frac{1}{2} \int\int_D (|f_u|^2 + |f_v|^2) \, du \, dv$$

in X_Γ^{tp}, where $f_u = (\partial f_1/\partial u, \dots, \partial f_n/\partial u)$ and $f_v = (\partial f_1/\partial v, \dots, \partial f_n/\partial v)$. The notation $|\cdot|$ means Euclidean norm.

A solution of the Plateau problem is called a *minimal surface* spanned in Γ even if it is *not* a minimal point of the energy functional. For the existence of

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the minimal surfaces the following theorem is known [2, pp. 101–105; 4, p. 71]:

Theorem A (Douglas-Rado). *Let* $e_\Gamma = \inf\{E(f) : f \in X_\Gamma^{\mathrm{tp}}\}$. *If* $X_\Gamma^{\mathrm{tp}} \neq \emptyset$, *then there exists a map* $x \in X_\Gamma^{\mathrm{tp}}$ *such that* $E(x) = e_\Gamma$.

An x as in Theorem A is called a *Douglas solution.* Evidently, a Douglas solution is a minimal surface.

In §2 we define a (stable) discrete minimal surface using the simplest finite element scheme. In §3 we prove the relative compactness of bounded subsets of discrete maps when the Jordan curve is rectifiable. In [7] a very restrictive condition was assumed to prove the relative compactness, so §3 is the main part of this paper.

2. Definition of the discrete minimal surface

Let $\Omega \subset D$ be a regular triangulation of D with $\overline{\Omega} = \bigcup K_i$, where K_i are triangles. With the triangulation Ω we associate the mesh size of Ω defined by

$$|\Omega| = \max_i \operatorname{diam}(K_i).$$

We assume that there exists a positive constant ω which is independent of the triangulation Ω such that the following inequality holds for each triangle $K_i \subset \Omega$:

$$\text{(H1)} \qquad \operatorname{diam}(K_i)/\rho(K_i) \leq \omega,$$

where $\rho(K_i) = \sup\{\operatorname{diam}(S);\, K_i \supset S : \text{ball}\}$.

Let S_Ω be the set of functions which are continuous on $\overline{\Omega}$ and linear on each triangle K_i. Let $\mathbf{S}_\Omega$ be the set of maps from $\overline{\Omega}$ into $\mathbf{R}^n$ such that each component function belongs to S_Ω. Let $N_\Omega = \{b_i\}_{i=1}^{N+N'}$ be the set of nodal points of Ω where $b_i \in \Omega^\circ$, the interior of Ω, for $1 \leq i \leq N$, and $b_i \in \partial\Omega$ for $N+1 \leq i \leq N+N'$. We number $\{b_{N+1}, \dots, b_{N+N'}\} = N_\Omega \cap \partial D$ in counter-clockwise order. We assume that

$$\text{(H2)} \qquad \Omega \text{ is of } \textit{nonnegative type}.$$

For the definition of the term "nonnegative type", see [1, 7]. This assumption is for the *discrete maximum principle* [7, Lemma 3]. We introduce the admissible class of triangulations of D defined by

$$\Delta^{\mathrm{tp}} = \{\Omega | z_1, z_2, z_3 \in N_\Omega,\ \Omega \text{ satisfies (H1), (H2)}\}.$$

When Ω is given, we define

$$X_{\Gamma,\Omega} = \{f \in \mathbf{S}_\Omega | f(N_\Omega \cap \partial D) \subset \Gamma,\ f|_{\partial D} \text{ is } d\text{-monotone}\},$$

where *d-monotone* means that the order of nodal points on Γ is the same as the order of nodal points on ∂D. Let

$$X_{\Gamma,\Omega}^{\mathrm{tp}} = \{f \in X_{\Gamma,\Omega} | f(z_i) = \zeta_i,\ i = 1, 2, 3\},$$

and let $E_\Omega(f)$ be the energy functional on Ω defined by

$$E_\Omega(f) = \frac{1}{2} \int\int_\Omega (|f_u|^2 + |f_v|^2)\, du\, dv.$$

We extend $f \in \mathbf{S}_\Omega$ to $D - \Omega$ as follows:

If $p \in \partial\Omega$ and $p \notin N_\Omega$, there exists an exterior normal half-line L_p of $\partial\Omega$ on p. For arbitrary $q \in L_p \cap (D - \Omega)$, we define $f(q) = f(p)$. Then the following estimate is valid:

$$E_\Omega(f) \le E(f) \le (1 + C|\Omega|)E_\Omega(f) \quad \text{for any } f \in \mathbf{S}_\Omega,$$

where C is a constant which is independent of Ω and f.

Definition 1. Let $\Omega \in \Delta^{\mathrm{tp}}$.

(D1) $f \in X^{\mathrm{tp}}_{\Gamma,\Omega}$ is a *stable d-minimal surface* if there exists a positive constant δ such that $\|f - g\|_{C(\overline{\Omega};\mathbf{R}^n)} < \delta$ implies $E_\Omega(f) \le E_\Omega(g)$ for $g \in X^{\mathrm{tp}}_{\Gamma,\Omega}$.

(D2) $f \in X^{\mathrm{tp}}_{\Gamma,\Omega}$ is the *d-Douglas solution* if $E_\Omega(f) = \inf\{E_\Omega(g) : g \in X^{\mathrm{tp}}_{\Gamma,\Omega}\}$.

3. Relative compactness

First, we recall a useful lemma [2, pp. 101–102; 4, pp. 67–68]. For any $z \in \mathbf{R}^2$ and any $r > 0$ we define

$$C_{r,z} = \overline{D} \cap \{w \in \mathbf{R}^2 : |w - z| = r\}.$$

For $f \in X^{\mathrm{tp}}_{\Gamma,\Omega}$ we denote by $l(f, C_{r,z})$ the length of the image $f(C_{r,z})$. Let M be a constant with $e_\Gamma < M$.

Lemma 2. *For arbitrary δ, $0 < \delta < 1$, and $f \in X^{\mathrm{tp}}_{\Gamma,\Omega}$ with $E(f) \le M$, there exists ρ, $\delta \le \rho \le \delta^{1/2}$, depending on f and z such that*

$$l(f, C_{\rho,z})^2 \le \lambda(\delta), \tag{3.1}$$

where $\lambda(\delta) = 8\pi M/\log(1/\delta)$.

For $\Omega \in \Delta^{\mathrm{tp}}$ and $f \in X^{\mathrm{tp}}_{\Gamma,\Omega}$ we define

$$L(\Omega, f) = \max\{|f(b_i) - f(b_{i+1})| : b_i \in N_\Omega \cap \partial D,\ i = N+1, \dots, N+N'\},$$

where $b_{N+N'+1} = b_{N+1}$. The following lemma is valid.

Lemma 3. *Let $\Delta^{\mathrm{tp}} \supset \{\Omega_n\}_{n=1}^\infty$ be such that $\lim_{n\to\infty}|\Omega_n| = 0$, and let $f_n \in X^{\mathrm{tp}}_{\Gamma,\Omega_n}$. Suppose that Γ is rectifiable and $E(f_n) \le M$ for any n. Then $\lim_{n\to\infty} L(\Omega_n, f_n) = 0$.*

Proof. The proof is by contradiction. Assume that $\limsup_{n\to\infty} L(\Omega_n, f_n) > 0$. Then there exists a positive constant ε_0 such that, for any $\xi > 0$, there exist a positive integer m and $b_i \in N_{\Omega_m} \cap \partial D$ such that

$$|\Omega_m| < \xi \quad \text{and} \quad |f_m(b_i) - f_m(b_{i+1})| \ge \varepsilon_0. \tag{3.2}$$

For $b_i \in N_{\Omega_m} \cap \partial D$ and $f_m \in X^{\mathrm{tp}}_{\Gamma,\Omega_m}$ as in (3.2), a pair (α_1, α_2) $(\alpha_i \in f_m(N_{\Omega_m} \cap \partial D),\ i = 1, 2)$ is said to be *admissible* if it satisfies the following properties: Γ_1, one of the two connected components of $\Gamma - \{\alpha_1, \alpha_2\}$, contains at least two of $\{\zeta_1, \zeta_2, \zeta_3\}$, and the other connected component Γ_2 contains

$f_m(b_i)$ and $f_m(b_{i+1})$. If $\{\zeta_1, \zeta_2, \zeta_3\} \cap \{f_m(b_i), f_m(b_{i+1})\} \neq \emptyset$, for example in the case of $\zeta_1 = f_m(b_i)$, a pair (α_1, α_2) such that $\alpha_1 = \zeta_1 = f_m(b_i)$ and Γ_1 contains at least one of $\{\zeta_1, \zeta_2, \zeta_3\}$ and Γ_2 contains $f_m(b_{i+1})$ is also said to be admissible.

By a topological argument we can show that there exists a positive constant η depending on $\{\Gamma, \zeta_1, \zeta_2, \zeta_3\}$ and ε_0 such that $|\alpha_1 - \alpha_2| \geq \eta$ for *any* admissible pairs (α_1, α_2) on Γ.

Let $b_k, b_h \in N_{\Omega_m} \cap \partial D$ be such that all of $(f_m(b_{k+p}), f_m(b_{h+q}))$ $(p, q = 0, 1)$ are admissible pairs. For $b_j \in N_{\Omega_m} \cap \partial D$, we denote by $\mathrm{seg}(b_j)$ the segment which connects $f_m(b_j)$ and $f_m(b_{j+1})$.

Lemma 4. *Assume that there exist* $\beta_1 \in \mathrm{seg}(b_k)$ *and* $\beta_2 \in \mathrm{seg}(b_h)$ *such that* $|\beta_1 - \beta_2| < \eta/2$. *Then we have*

$$|f_m(b_k) - f_m(b_{k+1})| + |f_m(b_h) - f_m(b_{h+1})| > \eta. \tag{3.3}$$

Proof. Since $|f_m(b_{k+p}) - f_m(b_{h+q})| \geq \eta$ $(p, q = 0, 1)$, we obtain (3.3) easily. □

Let $l(\Gamma)$ be the length of Γ. Let A be the least integer that satisfies

$$\frac{l(\Gamma) - \varepsilon_0}{\eta} \leq A. \tag{3.4}$$

We take sufficiently small δ, $0 < \delta < 1$, such that

$$(2A - 1)(\lambda(\delta))^{1/2} < \eta/2, \tag{3.5}$$

$$2(A\delta^{1/2} + \gamma(\delta)) < \min\{|z_i - z_j| : i \neq j\}, \tag{3.6}$$

where $\gamma(\delta) = \delta^{2^{A-1}}$. We set $\xi = \gamma(\delta)$ in (3.2), and we choose and fix a positive integer m and $b_i \in N_{\Omega_m} \cap \partial D$ as in (3.2). Then we have

$$|\Omega_m| < \delta^{2^{A-1}}. \tag{3.7}$$

Let $z \in \partial D$ be the center of the shorter arc $\widehat{b_i b_{i+1}}$. By Lemma 2 there exists a positive constant ρ, $\delta \leq \rho \leq \delta^{1/2}$, such that $l(f_m, C_{\rho,z}) \leq \lambda(\delta)^{1/2}$. Let l_1 and r_1 be the left and right endpoint of $C_{\rho,z}$ on ∂D, respectively. Suppose that $l_1 \in \widehat{b_{k_1} b_{k_1+1}}$ and $r_1 \in \widehat{b_{h_1} b_{h_1+1}}$, where $b_{k_1}, b_{h_1} \in N_{\Omega_m} \cap \partial D$. Note that the pair $(f_m(b_{k_1}), f_m(b_{h_1}))$ is not admissible in the extraordinary case like Figure 1.

However, in such a case we can obtain a contradiction and prove this lemma immediately. Hence we may assume without loss of generality that all of the pairs $(f_m(b_{k_1+p}), f_m(b_{h_1+q}))$ $(p, q = 0, 1)$ are admissible because of (3.6). Note that, by (3.7), b_{k_1}, b_{h_1} and b_i are distinct. From (3.1) and (3.5) we have

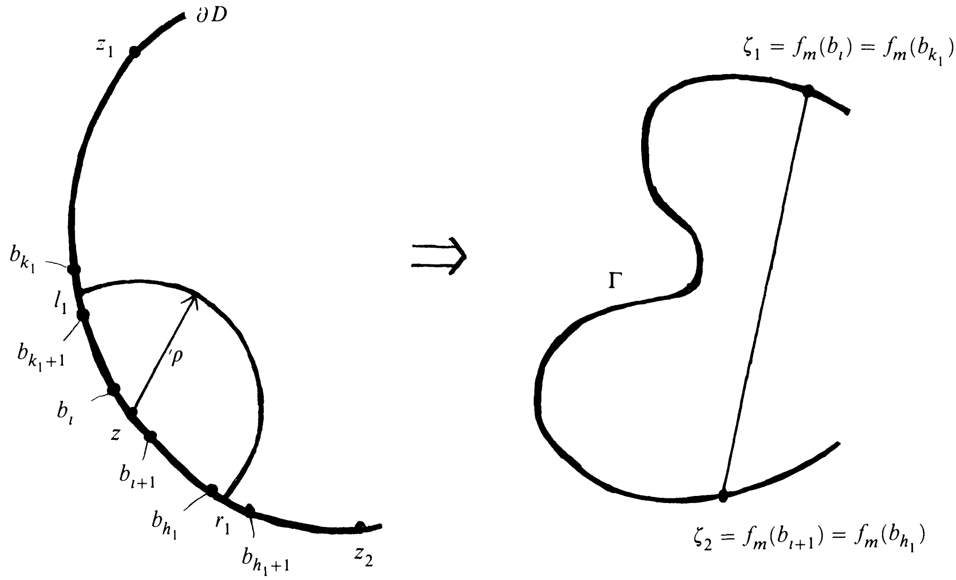

FIGURE 1

$|f_m(l_1) - f_m(r_1)| < (\eta/2)/(2A-1) \le \eta/2$. Thus, from Lemma 4, we obtain

$$|f_m(b_{k_1}) - f_m(b_{k_1+1})| + |f_m(b_{h_1}) - f_m(b_{h_1+1})| > \eta. \tag{3.8}$$

By Lemma 2 there exist positive constants θ_1, $\rho^2 \le \theta_1 \le \rho$, and μ_1, $\rho^2 \le \mu_1 \le \rho$ $(\delta^2 \le \theta_1, \mu_1 \le \delta^{1/2})$, such that

$$l(f_m, C_{\theta_1, l_1}) < \lambda(\rho^2)^{1/2} \le \lambda(\delta)^{1/2} < \frac{\eta}{2(2A-1)}, \quad l(f_m, C_{\mu_1, r_1}) < \frac{\eta}{2(2A-1)}.$$

Let l_2 be the left endpoint of C_{θ_1, l_1} and r_2 the right endpoint of C_{μ_1, r_1}. Let $b_{k_2}, b_{h_2} \in N_{\Omega_m} \cap \partial D$ be nodal points such that l_2 and r_2 are on the arcs $\widehat{b_{k_2} b_{k_2+1}}$ and $\widehat{b_{h_2} b_{h_2+1}}$, respectively. Again, we may assume that all pairs $(f_m(b_{k_2+p}), f_m(b_{h_2+q}))$ $(p, q = 0, 1)$ are admissible because of (3.6). By (3.7), b_{k_j}, b_{h_j} $(j = 1, 2)$ and b_ι are distinct. From (3.1) and (3.5) we have $|f_m(l_2) - f_m(r_2)| < 3(\eta/2)/(2A-1) \le \eta/2$. Thus, by Lemma 4 and (3.8), we obtain

$$\sum_{j=1}^{2} (|f_m(b_{k_j}) - f_m(b_{k_j+1})| + |f_m(b_{h_j}) - f_m(b_{h_j+1})|) > 2\eta.$$

Repeating this procedure A times, we conclude that there exist $2A$ distinct nodal points on ∂D such that

$$\sum_{j=1}^{A} (|f_m(b_{k_j}) - f_m(b_{k_j+1})| + |f_m(b_{h_j}) - f_m(b_{h_j+1})|) > A\eta. \tag{3.9}$$

By (3.4) the right side of (3.9) is greater than $l(\Gamma) - \varepsilon_0$. Thus we obtain

$$\begin{aligned} l(\Gamma) &\geq \sum_{i=N+1}^{N+N'} |f_m(b_i) - f_m(b_{i+1})| \\ &\geq \sum_{j=1}^{A} (|f_m(b_{k_j}) - f_m(b_{k_j+1})| + |f_m(b_{h_j}) - f_m(b_{h_j+1})|) + \varepsilon_0 > l(\Gamma). \end{aligned}$$

This is a contradiction, hence Lemma 3 is proved. □

Corollary 5. *Let $\Delta^{\mathrm{tp}} \supset \{\Omega_n\}_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} |\Omega_n| = 0$, and let $f_n \in X^{\mathrm{tp}}_{\Gamma,\Omega_n}$. Suppose that Γ is rectifiable and $E(f_n) \leq M$ for any n. Then, for any $\varepsilon > 0$, there exist $\delta > 0$ and positive integer n_1 such that*

$$|s - t| < \delta \quad \textit{implies} \quad |f_n(s) - f_n(t)| < \varepsilon,$$

for any $s, t \in \partial D$ and $n \geq n_1$.

Proof. By a topological argument we can show that, for any ε with $0 < \varepsilon < \min\{|\zeta_i - \zeta_j| : i \neq j\}$, there exists $\tau > 0$ such that, if $|\alpha_1 - \alpha_2| < \tau$, $\alpha_1, \alpha_2 \in \Gamma$, then the diameter of the smaller connected component of $\Gamma - \{\alpha_1, \alpha_2\}$ is less than ε.

Suppose that $\varepsilon > 0$ is given and $\tau > 0$ is chosen in the above manner. By Lemma 3 there exists a positive integer n_1 such that $L(\Omega_n, f_n) < \tau/3$ for all $n \geq n_1$. We choose $\delta > 0$ such that $\lambda(\delta)^{1/2} < \tau/3$ and $2\delta^{1/2} < \min\{|z_i - z_j| : i \neq j\}$. By Lemma 2, for any $s \in \partial D$, there exists ρ, $\delta \leq \rho \leq \delta^{1/2}$, depending on s, δ and f_n, such that $l(f_n, C_{\rho,s}) < \tau/3$. Let $l, r \in \partial D$ be the left and right endpoints of $C_{\rho,s}$, and let $b_i, b_j \in N_{\Omega_n} \cap \partial D$ be such that l and r are on the arcs $\widehat{b_i b_{i+1}}$ and $\widehat{b_{j-1} b_j}$, respectively. Since $L(\Omega_n, f_n) < \tau/3$, we obtain

$$|f_n(b_i) - f_n(b_j)| \leq |f_n(b_i) - f_n(l)| + |f_n(l) - f_n(r)| + |f_n(r) - f_n(b_j)| < \tau.$$

Thus we conclude that the diameter of Γ_1, the smaller connected component of $\Gamma - \{f_n(b_i), f_n(b_j)\}$, is less than ε, and, for any $t \in \partial D$ with $|s - t| < \delta$, $f_n(s)$ and $f_n(t)$ are in the convex hull of Γ_1. Hence we obtain $|f_n(s) - f_n(t)| < \varepsilon$. □

Lemma 6. *Let $\Delta^{\mathrm{tp}} \supset \{\Omega_n\}_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} |\Omega_n| = 0$, and let $f_n \in X^{\mathrm{tp}}_{\Gamma,\Omega_n}$. Suppose that Γ is rectifiable and $E(f_n)$ are uniformly bounded. Then there exists a subsequence $\{f_{n_l}\}$ such that $f_{n_l}|_{\partial D}$ converges uniformly to a continuous map $\varphi \in C(\partial D)$ on ∂D. Moreover, $\varphi(\partial D) = \Gamma$ and φ is monotone.*

Proof. The proof is similar to that of the Ascoli-Arzelà theorem. Let $\psi_n = \{(\cos(2\pi i/n), \sin(2\pi i/n)) : i = 0, \dots, n-1\}$, and let $\Psi = \bigcup_{n=1}^{\infty} \psi_n$. Since Ψ is countable, we can number Ψ as $\Psi = \{\gamma_1, \gamma_2, \dots\}$. By the diagonal method we choose a subsequence $\{f_{n_l}\}$ such that, for each j, $f_{n_l}(\gamma_j)$ converges as $n_l \to \infty$.

Suppose that an arbitrary $\varepsilon > 0$ is given. For this ε we choose $\delta > 0$ and a positive integer n_1 as in Corollary 5. Let K be a positive integer such that the length of an edge of the regular K-gon inscribed ∂D is less than δ, that is, $2\sin(\pi/K) < \delta$. Let $\psi_K = \{\xi_1, \dots, \xi_K\}$, and let n_2 be a positive integer such that $|f_{n_i}(\xi_k) - f_{n_j}(\xi_k)| < \varepsilon$, for $n_i, n_j \geq n_2$ and $k = 1, \dots, K$. For arbitrary $s \in \partial D$ there exists $\xi_k \in \psi_K$ such that $|s - \xi_k| < \delta$. Thus, by Corollary 5, we obtain

$$|f_{n_i}(s) - f_{n_j}(s)| \leq |f_{n_i}(s) - f_{n_i}(\xi_k)| + |f_{n_i}(\xi_k) - f_{n_j}(\xi_k)| + |f_{n_j}(\xi_k) - f_{n_j}(s)| < 3\varepsilon,$$

for $n_i, n_j \geq n_0 = \max\{n_1, n_2\}$. Since n_0 is independent of s, $\{f_{n_i}\}$ converges uniformly on ∂D. The last part of the lemma is obvious. □

4. Theorems

Using Lemma 6, we obtain the following theorems. The proofs of the theorems are quite similar to those of the theorems in [7].

Theorem 7. *Suppose that* Γ *is rectifiable. Let* $\Delta^{\text{tp}} \supset \{\Omega_n\}_{n=1}^{\infty}$ *be such that* $\lim_{n\to\infty} |\Omega_n| = 0$, *and let* $\{x_n \in X_{\Gamma,\Omega_n}^{\text{tp}}\}_{n=1}^{\infty}$ *be a sequence of the d-Douglas solutions.*

Then there exists a subsequence $\{x_{n_i}\}$ *which converges to one of the Douglas solutions* $x \in X_\Gamma^{\text{tp}}$ *in the following sense:*

$$\lim_{n_i\to\infty} \|x - x_{n_i}\|_{H^1(D;\mathbf{R}^n)} = 0, \tag{4.1}$$

and if $x \in W^{1,p}(D;\mathbf{R}^n)$, $p > 2$, *then*

$$\lim_{n_i\to\infty} \|x - x_{n_i}\|_{C(\overline{D};\mathbf{R}^n)} = 0. \tag{4.2}$$

If the Douglas solution is unique, then x_n *converges in the sense of* (4.1) *and* (4.2).

A harmonic map $x \in X_\Gamma^{\text{tp}}$ is said to be an *isolated stable minimal surface* if there exists a constant δ such that

$$0 < \|x - y\|_{C(\overline{D};\mathbf{R}^n)} < \delta \quad \text{implies} \quad E(x) < E(y) \quad \text{for } y \in X_\Gamma^{\text{tp}}.$$

Theorem 8. *Suppose that* Γ *is rectifiable. Let* $\Delta^{\text{tp}} \supset \{\Omega_n\}_{n=1}^{\infty}$ *be such that* $\lim_{n\to\infty} |\Omega_n| = 0$, *and let* $x \in X_\Gamma^{\text{tp}}$ *be an isolated stable minimal surface. Then there exists a sequence* $\{x_n \in X_{\Gamma,\Omega_n}^{\text{tp}}\}_{n=1}^{\infty}$ *of stable d-minimal surfaces which converges to* x *in the sense of* (4.1) *and* (4.2).

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