

MULTIVARIATE INTERPOLATION AND CONDITIONALLY POSITIVE DEFINITE FUNCTIONS. II

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ABSTRACT. We continue an earlier study of certain spaces that provide a variational framework for multivariate interpolation. Using the Fourier transform to analyze these spaces, we obtain error estimates of arbitrarily high order for a class of interpolation methods that includes multiquadrics.

1. INTRODUCTION

This paper continues a study, [11], of certain subspaces C_h of $C(\mathbf{R}^n)$, the continuous complex-valued functions on n -space \mathbf{R}^n . The spaces C_h provide a variational framework for the following interpolation problem: given numerical values at a scattered set of points in \mathbf{R}^n , make a good choice of a function f in $C(\mathbf{R}^n)$ that takes on those values.

For the reader's convenience we review some basic features of the development in [11]. The starting point is the selection of an integer $m \geq 0$ and a continuous function h on \mathbf{R}^n that is conditionally positive definite of order m . For example: $m = 1$, $h(x) = -\sqrt{1 + |x|^2}$. Using h , a space C_h with a semi-inner product $(\cdot, \cdot)_h$ is constructed. C_h is a subspace of $C(\mathbf{R}^n)$, and the null space of $(\cdot, \cdot)_h$ is P_{m-1} , the polynomials on \mathbf{R}^n of degree $m - 1$ or less. A key property of C_h is this: if x_1, \dots, x_N are distinct points in \mathbf{R}^n and v_1, \dots, v_N are complex numbers, then among all functions f in C_h that satisfy the interpolation conditions $f(x_i) = v_i$, the quadratic $\|f\|_h^2 = (f, f)_h$ is minimized by a function of the form $f = s + p$, where p is in P_{m-1} and

$$(1.1) \quad s(x) = \sum_{i=1}^N c_i h(x - x_i)$$

with $\sum_{i=1}^N c_i x_i^\alpha = 0$ for all $|\alpha| < m$. For the example mentioned, (1.1) is a multiquadric interpolant.

Because the spaces C_h are translation-invariant, the Fourier transform is a natural tool for analyzing them; it plays a central role here. To clarify basic ideas and make an orderly division of our results, we avoided Fourier techniques in

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[11]. We did, however, rely on them in our earlier investigation [10], which was in fact prompted by the Fourier methods in Duchon [5]. Use of Fourier transforms allows us to give improved descriptions of the spaces C_h (see §3) and allows us to single out certain cases where error estimates of order $l \geq m$ are possible (see §4). These estimates apply to the multiquadric case as well as to related examples given in §5; for each example given there, the integer l can be arbitrarily large.

2. PRELIMINARIES

In this section we recall some notation and results involving Fourier transforms and conditionally positive definite functions.

Let $\mathcal{D}(\mathbf{R}^n)$ denote the space of complex-valued functions on \mathbf{R}^n that are compactly supported and infinitely differentiable. The Fourier transform of a function φ in \mathcal{D} is

$$(2.1) \quad \widehat{\varphi}(\xi) = \int e^{-i\langle x, \xi \rangle} \varphi(x) dx.$$

In order to make use of theorems from Gelfand and Vilenkin [7], we adopt their definition of m th-order conditional positive definiteness. (Equivalence with the definition used in [11] can be seen from Proposition 2.4 and Theorem 6.1 below.) Thus, for a continuous function h we assume

$$(2.2) \quad \int h(x) \varphi * \bar{\varphi}(x) dx \geq 0$$

holds whenever $\varphi = p(D)\psi$ with ψ in \mathcal{D} and $p(D)$ a linear homogeneous constant coefficient differential operator of order m . Here $\bar{\varphi}(x) = \overline{\varphi(-x)}$ and $*$ denotes the convolution product

$$\varphi_1 * \varphi_2(t) = \int \varphi_1(x) \varphi_2(t - x) dx.$$

Note that (2.2) can be rewritten as

$$(2.3) \quad \iint h(x - y) \varphi(x) \overline{\varphi(y)} dx dy \geq 0.$$

The following result can be found in Chapter II, Section 4.4 of [7]; we incorporate a remark at the end of that section concerning the case where h is continuous.

Theorem 2.1. *Let h be continuous and conditionally positive definite of order m . Then it is possible to choose a positive Borel measure μ on $\mathbf{R}^n \sim \{0\}$, constants a_γ , $|\gamma| \leq 2m$ and a function χ in \mathcal{D} such that: $1 - \widehat{\chi}(\xi)$ has a zero of order $2m + 1$ at $\xi = 0$; both of the integrals $\int_{0 < |\xi| < 1} |\xi|^{2m} d\mu(\xi)$, $\int_{|\xi| \geq 1} d\mu(\xi)$ are finite; for all $\psi \in \mathcal{D}$,*

$$(2.4) \quad \int h(x) \psi(x) dx = \int \left[\widehat{\psi}(\xi) - \widehat{\chi}(\xi) \sum_{|\gamma| < 2m} D^\gamma \widehat{\psi}(0) \frac{\xi^\gamma}{\gamma!} \right] d\mu(\xi) + \sum_{|\gamma| \leq 2m} D^\gamma \widehat{\psi}(0) \frac{a_\gamma}{\gamma!}.$$

This uniquely determines the measure μ and the constants a_γ , for $|\gamma| = 2m$. In addition, for every choice of complex numbers c_α , $|\alpha| = m$,

$$(2.5) \quad \sum_{|\alpha|=m} \sum_{|\beta|=m} a_{\alpha+\beta} c_\alpha \bar{c}_\beta \geq 0.$$

The choice of χ affects the value of the coefficients a_γ , for $|\gamma| < 2m$. Note that the value of the right side of (2.4) does not change if, for suitable φ , $\widehat{\chi}$ is replaced by $\widehat{\chi} + \varphi$ and the a_γ , for $|\gamma| < 2m$, are replaced by $a_\gamma + \int \varphi(\xi) \xi^\gamma d\mu(\xi)$.

As can be seen from

$$(2.6) \quad (-i)^{|\gamma|} \int x^\gamma \varphi(x) dx = D^\gamma \widehat{\varphi}(0),$$

changing a coefficient a_γ on the right-hand side of (2.4) corresponds to changing $h(x)$ on the left side by adding a constant multiple of x^γ .

For $m = 0$, (2.4) reduces to $\int h \psi = \int \widehat{\psi} d\lambda$, where λ is the Borel measure on \mathbf{R}^n given by

$$\lambda(E) = \mu(E \sim \{0\}) + a_0 \delta(E).$$

Here δ is the measure corresponding to a unit mass at the origin; $\delta(E) = 1$ if $0 \in E$ and $\delta(E) = 0$ otherwise. Recall that Borel measures that are finite on compact sets are called Radon measures. We make the usual identification of a Radon measure on an open set $\Omega \subset \mathbf{R}^n$ with the corresponding distribution in $\mathcal{D}'(\Omega)$ and write $\langle \lambda, \psi \rangle = \int \psi d\lambda$. Also, if $f \in L^1_{\text{loc}}(\mathbf{R}^n)$, we identify it with the distribution in \mathcal{D}' given by $\langle f, \psi \rangle = \int \psi(x) f(x) dx$. Thus, for $m = 0$, (2.4) says $\langle h, \varphi \rangle = \langle \lambda, \widehat{\varphi} \rangle$.

For an illustration of the theorem when $m \neq 0$, take $n = 2$, $m = 1$, $h(x) = -\sqrt{1 + |x|^2}$. Then $d\mu(\xi) = w(\xi) d\xi$ with

$$w(\xi) = \frac{(1 + |\xi|) e^{-|\xi|}}{(2\pi)^2 |\xi|^3}$$

and $a_\gamma = 0$ for $|\gamma| = 2$. If χ is even, then the coefficients a_γ for $|\gamma| = 1$ are also 0. The remaining coefficient is $a_0 = -(1 + \int [1 - \widehat{\chi}(\xi)] w(\xi) d\xi)$. Details for this and related examples are given in §5.

We use $T^k \varphi$ to denote the k th-order Taylor polynomial for φ about 0:

$$(2.7) \quad T^k \varphi(\xi) = \sum_{|\alpha| \leq k} D^\alpha \varphi(0) \frac{\xi^\alpha}{\alpha!}.$$

The integral on the right side of (2.4) can then be written as $\int \widehat{\psi} - \widehat{\chi} T^{2m-1} \widehat{\psi} d\mu$.

The Schwartz space of rapidly decreasing C^∞ functions and its dual, the space of tempered distributions, are denoted by the usual letters \mathcal{S} and \mathcal{S}' .

Proposition 2.2. *Let k be a positive integer and let σ be a Radon measure on $\mathbf{R}^n \sim \{0\}$ such that $\int |\xi|^k (1 + |\xi|^k)^{-1} d|\sigma|(\xi) < \infty$. Let s be a continuous*

function such that $|\xi|^k s(\xi)$ is bounded on \mathbf{R}^n and $1 - s(\xi) = O(|\xi|^k)$ at $\xi = 0$. Let

$$(2.8) \quad u(x) = \int \left[e^{-i\langle x, \xi \rangle} - s(\xi) \sum_{r=0}^{k-1} \frac{(-i\langle x, \xi \rangle)^r}{r!} \right] d\sigma(\xi).$$

Then $u \in C(\mathbf{R}^n)$, $u(x) = o(|x|^k)$ as $|x| \rightarrow \infty$ and for all φ in \mathcal{S}

$$(2.9) \quad \int u(x)\varphi(x) dx = \int (\widehat{\varphi} - sT^{k-1}\widehat{\varphi}) d\sigma.$$

Proof. Let $E(t) = e^{-it} - \sum_{r=0}^{k-1} (-it)^r / r!$ and note that $u = u_0$, where

$$u_a(x) = \int_{|\xi|>a} (1 - s(\xi)) e^{-i\langle x, \xi \rangle} + s(\xi)E(\langle x, \xi \rangle) d\sigma(\xi).$$

From $|E(t)| \leq |t|^k$ we have $|s(\xi)E(\langle x, \xi \rangle)| \leq |x|^k |\xi|^k |s(\xi)|$. Our assumptions on σ and s ensure that $1 - s(\xi)$ and $|\xi|^k |s(\xi)|$ belong to $L^1(\sigma)$. Continuity of u can be established using dominated convergence.

To prove $u(x) = o(|x|^k)$, note that $|u_0(x) - u_a(x)| \leq (c_1(a) + c_2(a)|x|^k)$, where $c_1(a)$ and $c_2(a)$ are the results of integrating $|1 - s(\xi)|$ and $|\xi|^k |s(\xi)|$ over $0 < |\xi| \leq a$ with respect to $|\sigma|$. Given $\varepsilon > 0$, choose $a > 0$ so that $c_1(a) < \varepsilon$ and $c_2(a) < \varepsilon$. From $|E(t)| \leq 2|t|^{k-1}$ and $a > 0$ we have $u_a(x) = O(|x|^{k-1})$ as $|x| \rightarrow \infty$. Thus, we may choose $R \geq 1$ such that $|u_a(x)| \leq \varepsilon|x|^k$ for all $|x| > R$. Then, for $|x| > R$,

$$|u(x)| \leq |u_a(x)| + |u_0(x) - u_a(x)| \leq \varepsilon|x|^k + \varepsilon + \varepsilon|x|^k.$$

It follows that $u(x) = o(|x|^k)$.

To establish (2.9), apply Fubini's theorem and use

$$\int \frac{(-i\langle x, \xi \rangle)^r}{r!} \varphi(x) dx = \sum_{|\alpha|=r} D^\alpha \widehat{\varphi}(0) \frac{\xi^\alpha}{\alpha!}.$$

This can be verified by using $(y_1 + \dots + y_n)^r / r! = \sum_{|\alpha|=r} y^\alpha / \alpha!$ and (2.6). \square

If u is defined by (2.8) with $\sigma = \mu$, $k = 2m$ and $s = \widehat{\chi}$, then from (2.4), (2.9) and (2.6) we have $\langle h - u, \psi \rangle = \langle q, \psi \rangle$ for all ψ in \mathcal{D} . Here, $q(x) = \sum_{|\gamma| \leq 2m} a_\gamma (-ix)^\gamma / \gamma!$.

Corollary 2.3. *Suppose h is continuous and positive definite of order m . If $m > 0$, then there are unique constants a_γ , $|\gamma| = 2m$, such that*

$$h(x) - \sum_{|\gamma|=2m} a_\gamma (-ix)^\gamma / \gamma! = o(|x|^{2m}), \quad \text{as } |x| \rightarrow \infty.$$

These constants are the same as those appearing in (2.4).

For ease in dealing with (2.5), we develop some related notation. Let V_m be the space of vectors $v = (v_\alpha)_{|\alpha|=m}$ and let A be the operator on V_m defined

by $Av = w$ where $w_\alpha = \sum_{|\beta|=m} A_{\alpha,\beta} v_\beta$ and $A_{\alpha,\beta} = a_{\alpha+\beta}/(\alpha!\beta!)$. Because of (2.5), A must be real-symmetric. Thus $Av = 0$ if and only if $v^T \overline{Av} = 0$. Equivalently, the null space, N_A , of A is the null space of the semi-inner product $(v, w)_A = v^T \overline{Aw}$. Let $H_A = V_m/N_A$ be the Hilbert space obtained by identifying v and w whenever $\|v - w\|_A = 0$. The elements of H_A are the cosets $v + N_A$, and as w varies over such a coset, Aw remains fixed.

By applying Theorem 2.1 we can recover (2.2) for a more convenient set of functions φ . Let

$$(2.10) \quad \mathcal{D}_m = \left\{ \varphi \in \mathcal{D} : \int x^\alpha \varphi(x) dx = 0 \text{ for all } |\alpha| < m \right\}.$$

Clearly, $\mathcal{D}_m = \{ \varphi \in \mathcal{D} : \widehat{\varphi}(\xi) = O(|\xi|^m) \text{ at } \xi = 0 \}$. If $\psi = \varphi * \tilde{\varphi}$, then $\widehat{\psi} = |\widehat{\varphi}|^2$, so

$$D^\gamma \widehat{\psi} = \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} D^\alpha \widehat{\varphi} D^\beta \overline{\widehat{\varphi}}.$$

Hence, for $\psi = \varphi * \tilde{\varphi}$ with $\varphi \in \mathcal{D}_m$,

$$(2.11) \quad \sum_{|\gamma| \leq 2m} D^\gamma \widehat{\psi}(0) \frac{a_\gamma}{\gamma!} = \sum_{|\alpha|=m} \sum_{|\beta|=m} a_{\alpha+\beta} \frac{D^\alpha \widehat{\varphi}(0)}{\alpha!} \frac{D^\beta \overline{\widehat{\varphi}(0)}}{\beta!} = \|\widehat{\varphi}^{(m)}(0)\|_A^2,$$

where $\widehat{\varphi}^{(m)}(0)$ is the vector v in V_m given by $v_\alpha = D^\alpha \widehat{\varphi}(0)$. From (2.4) we see that if $\varphi \in \mathcal{D}_m$, then

$$(2.12) \quad \int h(x) \varphi * \tilde{\varphi}(x) dx = \int |\widehat{\varphi}|^2 d\mu + \|\widehat{\varphi}^{(m)}(0)\|_A^2,$$

and (2.2) holds. Since \mathcal{D}_m includes the functions φ for which (2.2) was assumed, we conclude that requiring (2.2) for all $\varphi \in \mathcal{D}_m$ is an equivalent definition of h being conditionally positive definite of order m .

Since $\mathcal{D}_{m+1} \subset \mathcal{D}_m$, the latter definition makes it clear that h will be conditionally positive definite of order $m + 1$ if it is conditionally positive definite of order m . If m is replaced by $m + 1$ in Theorem 2.1, with h held fixed, the measure μ will remain the same, the coefficients a_γ , $|\gamma| = 2(m + 1)$, will be 0, and the lower-order coefficients will change to reflect changes in $\widehat{\chi}$ and additional terms in the Taylor polynomial.

In order to apply results from [11], we verify that h is in the space $Q_m(\mathbf{R}^n)$ defined there.

Proposition 2.4. *Let h be continuous and assume (2.2) holds for all $\varphi \in \mathcal{D}_m$. If x_1, \dots, x_N are distinct points in \mathbf{R}^n and c_1, \dots, c_N are constants that satisfy $\sum_{i=1}^N c_i x_i^\alpha = 0$ for all $|\alpha| < m$, then*

$$(2.13) \quad \sum_{i,j=1}^N c_i \bar{c}_j h(x_i - x_j) \geq 0.$$

Proof. Choose g in \mathcal{D} with $\int g(x) dx = 1$ and $g(x) = 0$ for all $|x| \geq 1$. For $\varepsilon > 0$, let $g_\varepsilon = \varepsilon^{-n} g(x/\varepsilon)$ and take $\varphi_\varepsilon(x) = \sum_{k=1}^N c_k g_\varepsilon(x - x_k)$. Then

$\widehat{\varphi}_\varepsilon(\xi) = \tau(\xi)\widehat{g}(\varepsilon\xi)$ with $\tau(\xi) = \sum_{k=1}^N c_k e^{-i\langle x_k, \xi \rangle}$. From

$$D^\alpha \tau(\xi) = \sum_{k=1}^N c_k (-ix_k)^\alpha e^{-i\langle x_k, \xi \rangle}$$

we find $\tau(\xi) = O(|\xi|^m)$ at $\xi = 0$. Thus $\varphi_\varepsilon \in \mathcal{D}_m$ and

$$0 \leq \int h(x)\varphi_\varepsilon * \overline{\varphi}_\varepsilon(x) dx = \iint h(t-y)\varphi_\varepsilon(t)\overline{\varphi}_\varepsilon(y) dt dy.$$

Letting $\varepsilon \rightarrow 0$, we obtain (2.13). \square

The following observations will be used in the next section. Let $\widehat{\mathcal{D}}_m = \{\widehat{\varphi} : \varphi \in \mathcal{D}_m\}$.

Proposition 2.5. *Let $m \geq 0$ and let μ be a positive Borel measure on $\mathbf{R}^n \sim \{0\}$ that satisfies $\int (|\xi|^m/(1 + |\xi|^m))^2 d\mu(\xi) < \infty$. If $2k \geq m$, then $\widehat{\mathcal{D}}_{2k}$ is a dense subset of $L^2(\mu)$.*

Proof. Let $g \in L^2(\mu)$ and $\varepsilon > 0$. Choose $g_1 \in \mathcal{D}(\mathbf{R}^n \sim \{0\})$ so that $\|g - g_1\|_{L^2(\mu)} < \varepsilon$. Then $f(\xi) = |\xi|^{-2k} g_1(\xi)$ is in \mathcal{D} . Since $\widehat{\mathcal{D}}$ is dense in \mathcal{S} , we can find $\psi \in \mathcal{D}$ so that for all ξ in \mathbf{R}^n , $|f(\xi) - \widehat{\psi}(\xi)| \leq \varepsilon/(1 + |\xi|^{2k})$. Multiplying by $|\xi|^{2k}$ gives

$$|g_1(\xi) - |\xi|^{2k} \widehat{\psi}| \leq \frac{\varepsilon |\xi|^{2k}}{1 + |\xi|^{2k}}.$$

Let $\varphi = (-\Delta)^k \psi$. Then $\varphi \in \mathcal{D}$, $\widehat{\varphi}(\xi) = |\xi|^{2k} \widehat{\psi}(\xi)$ and

$$\int |g_1 - \widehat{\varphi}|^2 d\mu \leq \varepsilon^2 \int \left(\frac{|\xi|^{2k}}{1 + |\xi|^{2k}} \right)^2 d\mu(\xi).$$

Thus $\|g - \widehat{\varphi}\|_{L^2(\mu)}$ can be made as small as desired with $\varphi \in \mathcal{D}_{2k}$. \square

Proposition 2.6. *If $T \in \mathcal{D}'$ satisfies $T(\varphi) = 0$ for all φ in \mathcal{D}_m , then T belongs to P_{m-1} .*

Proof. Define $T_\alpha \in \mathcal{D}'$ by $T_\alpha(\varphi) = \int x^\alpha \varphi(x) dx$ and note that $\bigcap \{T_\alpha^{-1}(0) : |\alpha| < m\} = \mathcal{D}_m$. By assumption, \mathcal{D}_m is contained in $T^{-1}(0)$, the null space of T . It follows (see Theorem 1.3 of [9]) that there are constants c_α such that $T = \sum_{|\alpha| < m} c_\alpha T_\alpha$. \square

3. FOURIER DESCRIPTION OF C_h

After analyzing the space $\mathcal{E}_{h,m}$ defined below, we will see that it coincides with the space C_h studied in [11]. Among the results emerging from this analysis is a Fourier transform description of $\mathcal{E}_{h,m}$.

Definition. Let h be a continuous function on \mathbf{R}^n that is conditionally positive definite of order m . We write $f \in \mathcal{E}_{h,m}(\mathbf{R}^n)$ if $f \in C(\mathbf{R}^n)$ and there is a constant $c(f)$ such that for all φ in \mathcal{D}_m

$$(3.1) \quad \left| \int f(x)\varphi(x) dx \right| \leq c(f) \left\{ \iint h(x-y)\varphi(x)\overline{\varphi(y)} dx dy \right\}^{1/2}.$$

If $f \in \mathcal{E}_{h,m}(\mathbf{R}^n)$ we let $c_*(f)$ denote the smallest constant for which (3.1) is true.

It is easily checked that if f_1 and f_2 are in $\mathcal{E}_{h,m}$, then f_1+f_2 and af_1 , $a \in \mathbf{C}$, are also in $\mathcal{E}_{h,m}$ with $c_*(f_1+f_2) \leq c_*(f_1) + c_*(f_2)$ and $c_*(af_1) = |a|c_*(f_1)$. If $f \in P_{m-1}$ and $\varphi \in \mathcal{D}_m$, then $\langle f, \varphi \rangle = 0$, so $f \in \mathcal{E}_{h,m}$ and $c_*(f) = 0$. Conversely, if $c_*(f) = 0$, then $f \in P_{m-1}$ by Proposition 2.6. Thus $c_*(f)$ is a seminorm with null space P_{m-1} ; for $m = 0$, take $P_{-1} = \{0\}$.

Using (2.12), we note that (3.1) is equivalent to

$$(3.2) \quad |\langle f, \varphi \rangle| \leq c(f) \left\{ \|\widehat{\varphi}\|_{L^2(\mu)}^2 + \|\widehat{\varphi}^{(m)}(0)\|_A^2 \right\}^{1/2}$$

for all φ in \mathcal{D}_m . If $v \in V_m$ and

$$(3.3) \quad q(x) = \sum_{|\alpha|=m} (Av)_\alpha (-ix)^\alpha,$$

then $\langle q, \varphi \rangle = \sum_{|\alpha|=m} (Av)_\alpha D^\alpha \widehat{\varphi}(0) = (\widehat{\varphi}^{(m)}(0), \bar{v})_A$, so $q \in \mathcal{E}_{h,m}$ with $c_*(q) = \|\bar{v}\|_A$. If $g \in L^2(\mu)$ and u is defined by (2.8) with $\sigma = g\mu$, $k = m$ and an appropriate choice of s (take $s = 0$ for $m = 0$), then, for $\varphi \in \mathcal{D}_m$, (2.9) gives $\langle u, \varphi \rangle = \int \widehat{\varphi} g d\mu$. It follows that $u \in \mathcal{E}_{h,m}$ with $c_*(u) = \|g\|_{L^2(\mu)}$.

Clearly, $\mathcal{E}_{h,m}$ includes all functions of the form $f = u + q + p$ with u, q as above and $p \in P_{m-1}$. The next result, when combined with Proposition 2.6, shows that all functions in $\mathcal{E}_{h,m}$ can be obtained in this way.

From the behavior of $u(x)$ as $|x| \rightarrow \infty$, described by Proposition 2.2, we see that if $m > 0$ and $f = u + q + p$, then $f(x) = o(|x|^m)$ is equivalent to $q = 0$ (or $Av = 0$). In any case,

$$(3.4) \quad \mathcal{E}_{h,m}(\mathbf{R}^n) \subset \{f \in C(\mathbf{R}^n) : f(x) = O(|x|^m) \text{ as } |x| \rightarrow \infty\}.$$

Proposition 3.1. *Let m, h, μ and a_γ be as in Theorem 2.1. If $f \in \mathcal{E}_{h,m}$, then there is a function $g \in L^2(\mu)$ and a vector $v \in V_m$ such that for all φ in \mathcal{D}_m*

$$(3.5) \quad \langle f, \varphi \rangle = \int \widehat{\varphi} g d\mu + \sum_{|\alpha|=m} (Av)_\alpha D^\alpha \widehat{\varphi}(0).$$

This uniquely determines g and the coset $v + N_A$.

Proof. Define $J: \mathcal{D}_m \rightarrow H = L^2(\mu) \oplus H_A$ by $J\varphi = \widehat{\varphi} \oplus (\widehat{\varphi}^{(m)}(0) + N_A)$. From (3.2) we see that $|\langle f, \varphi \rangle| \leq c_*(f) \|J\varphi\|_H$. From this we deduce that, if $J\varphi_1 = J\varphi_2$, then $\langle f, \varphi_1 \rangle = \langle f, \varphi_2 \rangle$. It follows that there is a bounded linear functional L on the image $J\mathcal{D}_m$ such that $L(J\varphi) = \langle f, \varphi \rangle$ for all φ

in \mathcal{D}_m . Since H is a Hilbert space, we can choose $\bar{g} \oplus (\bar{v} + N_A)$ so that for all φ in \mathcal{D}_m , $\langle f, \varphi \rangle = (J\varphi, \bar{g} \oplus (\bar{v} + N_A))_H$. This gives (3.5).

For uniqueness, we show that $J\mathcal{D}_m$ is dense in H . Let $g_1 \in L^2(\mu)$, $w \in V_m$ and $\eta > 0$ be given. Take $2k > m$ and use Proposition 2.5 to choose $\varphi_1 \in \mathcal{D}_{2k}$ with $\|g_1 - \widehat{\varphi}_1\|_{L^2(\mu)} < \eta$. Note that $J\varphi_1 = \widehat{\varphi}_1 \oplus 0$ since $2k > m$. Put $p(\xi) = \sum_{|\alpha|=m} w_\alpha \xi^\alpha / \alpha!$ and take $\chi \in \mathcal{D}$ so that $1 - \widehat{\chi}(\xi) = O(|\xi|^{m+1})$ at $\xi = 0$. Define $\psi_\varepsilon \in \mathcal{D}$ by $\widehat{\psi}_\varepsilon(\xi) = p(\xi)\widehat{\chi}(\varepsilon^{-1}\xi)$. Then $J\psi_\varepsilon = \widehat{\psi}_\varepsilon \oplus (w + N_A)$. Choosing ε close enough to 0, we have $\|\widehat{\psi}_\varepsilon\|_{L^2(\mu)} < \eta$. Then $\|g_1 + (w + N_A) - J(\varphi_1 + \psi_\varepsilon)\|_H < 2\eta$. \square

If $f \in \mathcal{E}_{h,m}$, let $\Lambda f = g \oplus (v + N_A)$ be the point in $H = L^2(\mu) \oplus H_A$ determined by (3.5). Clearly, the resulting map $\Lambda: \mathcal{E}_{h,m} \rightarrow H$ is linear. That Λ maps onto H is evident from the remarks leading up to Proposition 3.1. From (3.2) and (3.5) we see that $c_*(f) = \|\Lambda f\|_H$. Note $\|\Lambda f\|_H = \{(f, f)_h\}^{1/2} = \|f\|_h$, where $(f_1, f_2)_h = (\Lambda f_1, \Lambda f_2)_H$ is a semi-inner product for $\mathcal{E}_{h,m}$. There is a corresponding inner product on $\mathcal{E}_{h,m}/P_{m-1}$, which is then a Hilbert space isomorphic to H under the quotient map associated with Λ .

The following provides a converse to Proposition 3.1 and clarifies how the Fourier transform relates f to g, v in (3.5).

Proposition 3.2. *Let m, h, μ and a_γ be as in Theorem 2.1. Fix $g \in L^2(\mu)$, $v \in V_m$ and $f \in \mathcal{D}'$. The following are equivalent:*

- (a) (3.5) holds for all φ in \mathcal{D}_m ;
- (b) $f \in \mathcal{S}'$ and for every $|\alpha| = m$, $\xi^\alpha F = \lambda_\alpha$, where F is the inverse Fourier transform of f and λ_α is the Radon measure on \mathbf{R}^n given by

$$(3.6) \quad \lambda_\alpha(E) = \int_{E \sim \{0\}} \xi^\alpha g(\xi) d\mu(\xi) + \alpha!(Av)_\alpha \delta(E).$$

When this is the case, $f \in \mathcal{E}_{h,m}$, $\Lambda f = g \oplus (v + N_A)$ and $(f, f)_h = \int |g|^2 d\mu + v^T \overline{Av}$.

Proof. Let q be as in (3.3) and let u be defined by (2.8) with $\sigma = g\mu$, $k = m$ and a choice of s that satisfies the hypotheses of Proposition 2.2. If (a) holds, then $\langle f, \varphi \rangle = \langle u + q, \varphi \rangle$ for all $\varphi \in \mathcal{D}_m$. By Proposition 2.6, $f - (u + q) = p \in P_{m-1}$. If $\widehat{F} = f$ and $\widehat{\psi}(\xi) = \xi^\alpha \varphi(\xi)$, then

$$\begin{aligned} \langle \xi^\alpha F, \varphi \rangle &= \langle F, \widehat{\psi} \rangle = \langle f, \psi \rangle = \langle u, \psi \rangle + \langle q + p, \psi \rangle \\ &= \int (\widehat{\psi} - sT^{m-1}\widehat{\psi})g d\mu + \sum_{|\alpha| \leq m} b_\alpha D^\alpha \widehat{\psi}(0), \end{aligned}$$

where the constants b_α are determined by $q + p(x) = \sum_{|\alpha| \leq m} b_\alpha (ix)^\alpha$. Thus,

$$(3.7) \quad \langle \xi^\alpha F, \varphi \rangle = \int (\xi^\alpha \varphi(\xi) - 0) g(\xi) d\mu(\xi) + \alpha!(Av)_\alpha \varphi(0),$$

which establishes (b). To see that (b) implies (a), let $f_1 = u + q$ with u and q as above. Then (3.7) holds for F_1 , where $\widehat{F}_1 = f_1$. Hence, $\xi^\alpha F_1 = \lambda_\alpha$. If (b) holds, then $\xi^\alpha F_1 = \xi^\alpha F$ for all $|\alpha| = m$. This implies $F_1 - F = \sum_{|\alpha| < m} b_\alpha D^\alpha \delta$, which says $f_1 - f \in P_{m-1}$. Therefore, (a) and the other assertions about f follow from the corresponding facts about f_1 . \square

For typical choices of h (e.g. those considered in §5) the measure μ is absolutely continuous with respect to Lebesgue measure, $d\mu(\xi) = w(\xi)d\xi$, and $a_\gamma = 0$ for all $|\gamma| = 2m$. In such cases the measures λ_α in (3.6) are given by functions F_α in $L^1_{\text{loc}}(\mathbf{R}^n)$; $d\lambda_\alpha(\xi) = F_\alpha(\xi)d\xi$, where $F_\alpha(\xi) = \xi^\alpha g(\xi)w(\xi)$. From $D^\alpha f = ((-i\xi)^\alpha F)^\wedge = (-i)^m \widehat{\lambda_\alpha}$, we see that $(D^\alpha f)^\wedge = (-i)^m (2\pi)^n \check{F}_\alpha \in L^1_{\text{loc}}(\mathbf{R}^n)$, where $\check{F}_\alpha(\xi) = F_\alpha(-\xi)$. Let

$$(3.8) \quad r(\xi) = \frac{1}{(2\pi)^{2n} |\xi|^{2m} w(-\xi)},$$

with $r(\xi) = \infty$ when $w(-\xi) = 0$. If $d\rho(\xi) = r(\xi)d\xi$, then $(D^\alpha f)^\wedge \in L^2(\rho)$ and

$$\| (D^\alpha f)^\wedge \|_{L^2(\rho)}^2 = \int \frac{\xi^{2\alpha} |g(\xi)|^2}{|\xi|^{2m}} d\mu(\xi).$$

Using (4.2) below with $l = m$,

$$(3.9) \quad \sum_{|\alpha|=m} \frac{m!}{\alpha!} \| (D^\alpha f)^\wedge \|_{L^2(\rho)}^2 = \int |g|^2 d\mu = (f, f)_h.$$

Corollary 3.3. *Let m, h, μ , and a_γ be as in Theorem 2.1. Assume $d\mu(\xi) = w(\xi)d\xi$ and $a_\gamma = 0$ for all $|\gamma| = 2m$. Let ρ be the Borel measure on \mathbf{R}^n defined by $d\rho(\xi) = r(\xi)d\xi$, with r as in (3.8). Then $f \in \mathcal{E}_{h,m}$ if and only if $f \in \mathcal{S}'$ and $(D^\alpha f)^\wedge \in L^2(\rho)$ for every $|\alpha| = m$. In that case, $(f, f)_h$ is given by (3.9).*

The translation invariant nature of $\mathcal{E}_{h,m}$ is evident in the following

Proposition 3.4. *Let τ be a compactly supported Radon measure on \mathbf{R}^n . If f is in $\mathcal{E}_{h,m}$, then so is $\tau * f$. Furthermore, if $\Lambda: \mathcal{E}_{h,m} \rightarrow L^2(\mu) \oplus H_A$ is as defined above and $\Lambda f = g \oplus (v + N_A)$, then $\Lambda(\tau * f) = tg \oplus (t(0)v + N_A)$, where $t(\xi) = \int e^{i\langle x, \xi \rangle} d\tau(x)$.*

Proof. If $\psi(x) = \int \varphi(x+y)d\tau(y)$, then $\langle \tau * f, \varphi \rangle = \langle f, \psi \rangle$ and

$$(3.10) \quad \begin{aligned} \widehat{\psi}(\xi) &= \iint e^{-i\langle x, \xi \rangle} \varphi(x+y) dx d\tau(y) \\ &= \iint e^{-i\langle z-y, \xi \rangle} \varphi(z) dz d\tau(y) = \widehat{\varphi}(\xi)t(\xi). \end{aligned}$$

If $\Lambda f = g \oplus (v + N_A)$, so that (3.5) holds, then for all $\varphi \in \mathcal{D}_m$

$$\begin{aligned} \langle \tau * f, \varphi \rangle &= \int \widehat{\psi} g \, d\mu + \sum_{|\alpha|=m} D^\alpha \widehat{\psi}(0)(Av)_\alpha \\ &= \int \widehat{\varphi} t g \, d\mu + \sum_{|\alpha|=m} t(0) D^\alpha \widehat{\varphi}(0)(Av)_\alpha. \end{aligned}$$

This gives (3.5), with f, g, v replaced by $\tau * f, t g, t(0)v$; the assertions made are now apparent. \square

In the next result, (3.11) is equivalent to $\Lambda(\nu * h) = n \oplus (w + N_A)$ and (3.12) says $\nu(\bar{f}) = (\nu * h, f)_h$. From this it is clear that $\mathcal{E}_{h,m}$ satisfies condition (c) in Theorem 1.1 of [11]. That conditions (a) and (b) are also satisfied can be seen from the discussion above in which the map Λ was introduced. Applying Theorem 1.1 of [11], we conclude that $\mathcal{E}_{h,m} = C_h$.

Proposition 3.5. *Let m, h, μ and a_γ be as in Theorem 2.1. Let ν be a compactly supported Radon measure on \mathbf{R}^n and assume that $\int x^\alpha d\nu(x) = 0$ for all $|\alpha| < m$. Then $\nu * h \in \mathcal{E}_{h,m}$ and for all φ in \mathcal{D}_m*

$$(3.11) \quad \langle \nu * h, \varphi \rangle = \int \widehat{\varphi} n \, d\mu + \sum_{|\alpha|=m} (Av)_\alpha D^\alpha \widehat{\varphi}(0),$$

where $n(\xi) = \int e^{i(x,\xi)} d\nu(x)$ and $w_\beta = D^\beta n(0) = \int (ix)^\beta d\nu(x)$. Furthermore, if $f \in \mathcal{E}_{h,m}$ and $\Lambda f = g \oplus (v + N_A)$, then

$$(3.12) \quad \int \overline{f(x)} d\nu(x) = \int n \bar{g} \, d\mu + w^T \overline{Av}.$$

Proof. If $\psi(z) = \int \varphi(z+y) d\nu(y)$, then from (2.4),

$$(3.13) \quad \langle \nu * h, \varphi \rangle = \langle h, \psi \rangle = \int \widehat{\psi} - \widehat{\chi} T^{2m-1} \widehat{\psi} \, d\mu + \sum_{|\gamma| \leq 2m} D^\gamma \widehat{\psi}(0) \frac{a_\gamma}{\gamma!}$$

and, as in (3.10), $\widehat{\psi} = \widehat{\varphi} n$. Clearly, $D^\alpha n(0) = 0$ for all $|\alpha| < m$. If $\varphi \in \mathcal{D}_m$, then $D^\gamma \widehat{\psi}(0) = 0$ for $|\gamma| < 2m$, and for $|\gamma| = 2m$

$$D^\gamma \widehat{\psi}(0) = \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha! \beta!} D^\alpha \widehat{\varphi}(0) w_\beta.$$

Thus, (3.11) follows from (3.13). To establish (3.12), choose a real-valued function r in \mathcal{D} with $\widehat{r}(0) = 1$, and for $\varepsilon > 0$ let $\overline{\varphi_\varepsilon(x)} = \int \varepsilon^{-n} r\left(\frac{x-y}{\varepsilon}\right) d\nu(y)$. Then $\varphi_\varepsilon \in \mathcal{D}_m$ and

$$\langle f, \varphi_\varepsilon \rangle = \int \widehat{\varphi_\varepsilon} g \, d\mu + \sum_{|\alpha|=m} (Av)_\alpha D^\alpha \widehat{\varphi_\varepsilon}(0).$$

This yields (3.12) because

$$\int \overline{f(x)} d\nu(x) = \lim_{\varepsilon \rightarrow 0} \langle f, \varphi_\varepsilon \rangle \quad \text{and} \quad \widehat{\varphi_\varepsilon}(\xi) = \widehat{r}(\varepsilon\xi) \overline{n(\xi)}. \quad \square$$

For s as in (1.1) we have $s = \nu * h$ with $\int \varphi d\nu = \sum_{i=1}^N c_i \varphi(x_i)$. Thus, such functions s belong to $\mathcal{E}_{h,m}$.

The distribution $D^\kappa h$, $|\kappa| \geq m$, can be obtained as a limit of $\nu * h$'s by choosing ν 's that correspond to appropriate difference operators. Such ν 's satisfy the orthogonality condition $\int x^\alpha d\nu(x) = 0$, $|\alpha| < m$. Hence, the following may be regarded as a limiting case of the situation considered above.

Proposition 3.6. *Let m, h, μ and a_γ be as in Theorem 2.1. Fix κ with $|\kappa| \geq m$ and let $p(\xi) = (i\xi)^\kappa$. Then, $p \in L^2(\mu)$ if and only if the distribution $D^\kappa h$ belongs to $\mathcal{E}_{h,m}$. In that case, $\Lambda((-D)^\kappa h) = p \oplus (w + N_A)$ with $w_\alpha = D^\alpha p(0)$, $|\alpha| = m$.*

Proof. Let $\psi = D^\kappa \varphi$, so $\widehat{\psi} = p\widehat{\varphi}$. If $\varphi \in \mathcal{D}_m$, then, by a calculation like that for (2.11),

$$\sum_{|\gamma| \leq 2m} D^\gamma (p\widehat{\varphi})(0) \frac{a_\gamma}{\gamma!} = \sum_{|\alpha|=m} \sum_{|\beta|=m} a_{\alpha+\beta} \frac{D^\alpha p(0)}{\alpha!} \frac{D^\beta \widehat{\varphi}(0)}{\beta!}.$$

Using (2.4), we have

$$(3.14) \quad \langle (-D)^\kappa h, \varphi \rangle = \langle h, \psi \rangle = \int p\widehat{\varphi} d\mu + \sum_{|\beta|=m} (Aw)_\beta D^\beta \widehat{\varphi}(0)$$

for all $\varphi \in \mathcal{D}_m$. This is (3.5) with $f = (-D)^\kappa h$, $g = p$ and $v = w$. If $p \in L^2(\mu)$ we apply Proposition 3.2 to see that $f \in \mathcal{E}_{h,m}$ and $\Lambda f = p \oplus (w + N_A)$. If $p \notin L^2(\mu)$ we apply Proposition 2.5 to obtain a sequence $\varphi_i \in \mathcal{D}_{2k}$ such that $\int |\widehat{\varphi}_i|^2 d\mu = 1$ and $\int p\widehat{\varphi}_i d\mu \rightarrow \infty$. We take $2k > m$ so that $D^\beta \widehat{\varphi}_i(0) = 0$ when $|\beta| = m$. Then (3.14) gives

$$\langle (-D)^\kappa h, \varphi_i \rangle = \int p\widehat{\varphi}_i d\mu \rightarrow \infty.$$

Since $\|\widehat{\varphi}_i\|_{L^2(\mu)}^2 + \|\widehat{\varphi}_i^{(m)}(0)\|_A^2 = 1$, we see that $f = (-D)^\kappa h$ cannot satisfy (3.2) and hence cannot be in $\mathcal{E}_{h,m}$. \square

4. ERROR ESTIMATES

In this section we derive bounds on the difference between a function g in $\mathcal{E}_{h,m}$ and a function g^X of minimal $\mathcal{E}_{h,m}$ norm that agrees with g on a set $X \subset \mathbf{R}^n$ of 'interpolation points'. These error estimates involve a parameter that measures the spacing of the points in X and are of order l in that parameter; our derivation assumes $l \geq m$ and

$$(4.1) \quad \int |\xi|^{2l} d\mu(\xi) < \infty.$$

For the examples given in §5, this assumption is satisfied for arbitrarily large values of l ; see (5.2) below. In particular, the estimates apply to multiquadric interpolation, since the example there with $a = -1$ gives $h(x) = -2\sqrt{\pi(1 + |x|^2)}$.

Before starting on the error estimates, we look at a related implication of (4.1). Let $p_\alpha(\xi) = (i\xi)^\alpha$. From

$$(4.2) \quad (\xi_1^2 + \dots + \xi_n^2)^l = \sum_{|\alpha|=l} \frac{l!}{\alpha!} \xi^{2\alpha}$$

we observe that (4.1) holds if and only if $p_\alpha \in L^2(\mu)$ for all $|\alpha| = l$. If a distribution has all of its l th order derivatives given by continuous functions, then it will belong to $C^l(\mathbf{R}^n)$. Thus, the following result shows that (4.1) holds if and only if $\mathcal{E}_{h,m} \subset C^l(\mathbf{R}^n)$.

Proposition 4.1. *Let m, h, μ and a_γ be as in Theorem 2.1. Fix α with $|\alpha| \geq m$. Then the following are equivalent:*

- (a) $p_\alpha \in L^2(\mu)$, where $p_\alpha(\xi) = (i\xi)^\alpha$;
- (b) for every f in $\mathcal{E}_{h,m}$, the distribution $D^\alpha f$ belongs to $C(\mathbf{R}^n)$ and there is a constant c_α such that for all f in $\mathcal{E}_{h,m}$, $\|D^\alpha f\|_\infty \leq c_\alpha \|f\|_h$;
- (c) there is a point x_0 in \mathbf{R}^n and a constant c_α such that for all f in $\mathcal{E}_{h,m} \cap C^\infty$, $|D^\alpha f(x_0)| \leq c_\alpha \|f\|_h$.

If these are true, then for all $f \in \mathcal{E}_{h,m}$ and all $y \in \mathbf{R}^n$,

$$D^\alpha f(y) = (f, \delta_y * (-D)^\alpha h)_h.$$

Proof. Let $f \in \mathcal{E}_{h,m}$ and let F be its inverse Fourier transform, so that $\widehat{F} = f$. If $|\alpha| = m$, then, by Proposition 3.2, $\xi^\alpha F = \lambda_\alpha$ with λ_α given by (3.6). If $|\alpha| > m$, then $\alpha = \alpha' + \beta$ with $|\alpha'| = m$. Hence, $\xi^\alpha F = \lambda_\alpha$ with $\lambda_\alpha = \xi^\beta \lambda_{\alpha'}$ where $\lambda_{\alpha'}$ is given by (3.6). If (a) holds, then λ_α is finite; for $|\alpha| = m$, $\int d|\lambda_\alpha| = \int |\xi^\alpha g(\xi)| d\mu(\xi) + |(Av)_\alpha|$ and for $|\alpha| > m$, $\int d|\lambda_\alpha| = \int |\xi^\alpha g(\xi)| d\mu(\xi)$. Thus, $\widehat{\lambda_\alpha}$ is continuous and bounded by $\int d|\lambda_\alpha|$. Since $(iD)^\alpha f = (\xi^\alpha F)^\widehat{=} \widehat{\lambda_\alpha}$, we see that (b) holds with $c_\alpha = \|p_\alpha \oplus (p_\alpha^{(m)}(0) + N_A)\|_H$. Thus, (a) implies (b).

That (b) implies (c) is obvious. To see that (c) implies (a), let ψ be an arbitrary function in $\mathcal{D}(\mathbf{R}^n \sim \{0\})$ and define u by (2.8) with $\sigma = \psi\mu$ and $k = m$. Then, $u \in \mathcal{E}_{h,m}$, $\Lambda u = \psi \oplus 0$ and $\|u\|_h^2 = \int |\psi|^2 d\mu$. In addition, $u \in C^\infty$ and

$$D^\alpha u(x_0) = \int e^{-i\langle x_0, \xi \rangle} (-i\xi)^\alpha \psi(\xi) d\mu(\xi).$$

Thus, (c) gives $|\int e^{-i\langle x_0, \xi \rangle} (-i\xi)^\alpha \psi(\xi) d\mu(\xi)| \leq c_\alpha \|\psi\|_{L^2(\mu)}$. Since this holds for all ψ in $\mathcal{D}(\mathbf{R}^n \sim \{0\})$, a dense subset of $L^2(\mu)$, (a) must be true.

To verify the last assertion, suppose $f \in \mathcal{E}_{h,m}$ with $\Lambda f = g \oplus (v + N_A)$. By Proposition 3.6, $\Lambda((-D)^\alpha h) = p_\alpha \oplus (p_\alpha^{(m)}(0) + N_A)$. Using Proposition 3.4 with $\tau = \delta_y$, we have $t(\xi) = e^{i\langle y, \xi \rangle}$ and

$$(4.3) \quad \Lambda(\delta_y * (-D)^\alpha h) = t p_\alpha \oplus (p_\alpha^{(m)}(0) + N_A).$$

Thus, $(f, \delta_y * (-D)^\alpha h)_h = \int g \overline{ip}_\alpha d\mu + v^T \overline{Ap}_\alpha^{(m)}(0) = (-i)^m \widehat{\lambda}_\alpha(y)$. Here, λ_α is as above so, as already noted, $\widehat{\lambda}_\alpha = (iD)^\alpha \widehat{f}$; this gives the desired equality. \square

Our error estimates will be based on the following

Theorem 4.2. *Let m, h, μ and a_γ be as in Theorem 2.1. Assume that μ satisfies (4.1) with $l \geq \max\{1, m\}$. For a point x_0 in \mathbf{R}^n suppose that σ is a real-valued, compactly supported Radon measure on \mathbf{R}^n such that*

$$(4.4) \quad p(x_0) = \int p(x) d\sigma(x)$$

for all p in P_{l-1} . Then for all f in $\mathcal{E}_{h,m}$,

$$(4.5) \quad \left| f(x_0) - \int f(x) d\sigma(x) \right| \leq c \|f\|_h \int |x - x_0|^l d|\sigma|(x),$$

where $c = \{s + \int |\xi|^{2l} / (l!)^2 d\mu(\xi)\}^{1/2}$ with $s = \sum_{|\alpha|=m} \sum_{|\beta|=m} |A_{\alpha,\beta}|$ for $l = m$ and $s = 0$ for $l > m$.

Proof. Let $\nu = \delta_{x_0} - \sigma$. By (4.4), $\int p(x) d\nu(x) = 0$ for all $p \in P_{l-1}$. Since $l \geq m$, Proposition 3.5 applies to ν , and from (3.12),

$$(4.6) \quad \left| \int \overline{f(x)} d\nu(x) \right| \leq \|n \oplus (w + N_A)\|_H \|f\|_h.$$

Here, $w_\beta = \int (ix)^\beta d\nu(x)$, $|\beta| = m$. If $l > m$, then $w = 0$; if $l = m$, then

$$w_\beta = i^m \int (x - x_0)^\beta d\nu(x) = 0 - i^m \int (x - x_0)^\beta d\sigma(x).$$

Defining $R(\theta)$ by $e^{i\theta} = \sum_{k=0}^{l-1} (i\theta)^k / k! + R(\theta)$, we have $|R(\theta)| \leq |\theta|^l / l!$ and

$$\begin{aligned} e^{-i\langle x_0, \xi \rangle} n(\xi) &= \int e^{i\langle x - x_0, \xi \rangle} d\nu(x) = \int R(\langle x - x_0, \xi \rangle) d\nu(x) \\ &= - \int R(\langle x - x_0, \xi \rangle) d\sigma(x). \end{aligned}$$

If $b = \int |x - x_0|^l d|\sigma|(x)$, then $|n(\xi)| \leq b|\xi|^l / l!$ and, for $l = m$, $|w_\beta| \leq b$. From this we obtain $\|n \oplus (w + N_A)\|_H \leq cb$ and (4.5) follows. \square

To obtain the error estimates mentioned at the beginning of this section, we apply Theorem 4.2 to $f = g - g^X$. Because of the minimum norm property of g^X , $\|f\|_h \leq \|g\|_h$. Since other fixed bounds on $\|f\|_h$ result in acceptable error estimates, the minimum norm requirement on g^X could be relaxed to simply a requirement that $\|g^X\|_h$ not exceed some set bound. If we choose σ so that $\text{supp } \sigma \subset X$, then $\int g - g^X d\sigma = 0$, and (4.5) gives

$$(4.7) \quad \left| g(x_0) - g^X(x_0) \right| \leq c \|f\|_h \int |x - x_0|^l d|\sigma|(x).$$

To make such a choice of σ possible, it may be necessary to restrict x_0 . From (4.4) we see that if $p \equiv 0$ on $\text{supp } \sigma$ then $p(x_0) = 0$. Let

$$N_{l-1}(X) = \{p \in P_{l-1} : p(x) = 0 \text{ for all } x \in X\},$$

$$\langle X \rangle_{l-1} = \{x \in \mathbf{R}^n : p(x) = 0 \text{ for all } p \in N_{l-1}(X)\}.$$

Proposition 4.3. *Let $E_{l-1}(x_0, X)$ be the set of all real-valued, compactly supported Radon measures on \mathbf{R}^n that satisfy both (4.4) and $\text{supp } \sigma \subset X$. Then $E_{l-1}(x_0, X)$ is nonempty if and only if $x_0 \in \langle X \rangle_{l-1}$.*

Proof. Necessity of $x_0 \in \langle X \rangle_{l-1}$ is evident from the preceding discussion. To see that this is also sufficient, consider the linear functionals on P_{l-1} defined by $L_x(p) = p(x)$. Choose a (finite) subset Y of X such that $\{L_y : y \in Y\}$ is linearly independent and $L_x \in \text{span}\{L_y : y \in Y\}$ for all x in X . Then, $N_{l-1}(Y) = N_{l-1}(X)$ and $\langle Y \rangle_{l-1} = \langle X \rangle_{l-1}$. Also, $\{L_y : y \in Y\}$ is a basis for $(P_{l-1}/N_{l-1}(Y))'$; let $\{p_y + N_{l-1}(Y) : y \in Y\}$ be the dual basis. If the polynomials p_y are replaced by their real parts, the result is still dual to $\{L_y : y \in Y\}$. We may therefore assume that each p_y is real-valued. For x_0 in $\langle Y \rangle_{l-1}$, L_{x_0} gives a linear functional on $P_{l-1}/N_{l-1}(Y)$. Thus, $L_{x_0} = \sum_{y \in Y} c_y L_y$ with $c_y = L_{x_0}(p_y)$, and it follows that $\sigma = \sum_{y \in Y} c_y \delta_y$ is in $E_{l-1}(x_0, X)$. \square

Of course, (4.7) will give a better error estimate if σ is chosen from $E_{l-1}(x_0, X)$ so as to minimize $\int |x - x_0|^l d|\sigma|(x)$; we made no attempt to do this with our choice of σ in the preceding proof.

We turn now to an analysis of the rate at which the error estimate goes to zero as the coverage by X improves. For this we fix a region Ω and a function $g \in \mathcal{C}_{h,m}^r$ and, for various X , look at bounds on $|g - g^X|_\Omega$ given by (4.7). Here we use the notation $|f|_\Omega = \sup_{x \in \Omega} |f(x)|$.

The number $d = d(\Omega, X)$ defined by

$$(4.8) \quad d(\Omega, X) = \sup_{y \in \Omega} \inf_{x \in X} |y - x|$$

is a standard measurement of how closely X covers Ω . Using (4.7) and some mild assumptions about Ω , we will show that

$$(4.9) \quad |g - g^X|_\Omega = O(d^l).$$

In order to use (4.7), we assume (4.1). In that case, Proposition 4.1 assures us of a uniform bound for the l th order derivatives of $g - g^X$. From this and (4.9), we can deduce that the derivatives $D^\alpha(g - g^X)$ of intermediate order $0 < |\alpha| < l$ satisfy $O(d^{l-|\alpha|})$ estimates.

To establish (4.9), we proceed along lines used by Duchon [6]. We start by assuming that there are positive constants M, ε_0 such that for every $0 < \varepsilon < \varepsilon_0$,

$$(4.10) \quad \Omega \subset \bigcup \{B(t, \varepsilon M) : t \in T_\varepsilon\},$$

where $T_\varepsilon = \{t \in \mathbf{R}^n : B(t, \varepsilon) \subset \Omega\}$, $B(t, r) = \{x \in \mathbf{R}^n : |x - t| \leq r\}$. Arguments in §1 of [6] show that such constants M, ε_0 will exist if Ω satisfies a cone condition.

Next we select a P_{l-1} -unisolvent set of points $\mathbf{a}(\alpha) \in \mathbf{R}^n$, $|\alpha| < l$. A corresponding set of Lagrange polynomials, $p_\gamma^{\mathbf{a}} \in P_{l-1}$, $|\gamma| < l$, is determined by the requirements: $p_\gamma^{\mathbf{a}}(\mathbf{a}(\alpha)) = 1$, for $\alpha = \gamma$; $p_\gamma^{\mathbf{a}}(\mathbf{a}(\alpha)) = 0$, for $\alpha \neq \gamma$. The matrix $A_{\alpha, \beta} = (\mathbf{a}(\alpha))^\beta$, $|\alpha| < l$, $|\beta| < l$ is nonsingular. If $p_\gamma(x) = \sum_{|\beta| < l} (A^{-1})_{\beta, \gamma} x^\beta$, then $p_\gamma(\mathbf{a}(\alpha)) = (AA^{-1})_{\alpha, \gamma}$, so $p_\gamma = p_\gamma^{\mathbf{a}}$. The function $\alpha \rightarrow \mathbf{a}(\alpha)$ can be identified with a point in $\mathbf{B} = \prod_{|\alpha| < l} B(\mathbf{a}(\alpha), \delta)$. Clearly, $\mathbf{b} \in \mathbf{B}$ if and only if $|\mathbf{b}(\alpha) - \mathbf{a}(\alpha)| < \delta$ for all $|\alpha| < l$. Now choose $\delta > 0$ so that $B_{\alpha, \beta} = (\mathbf{b}(\alpha))^\beta$ is invertible for all $\mathbf{b} \in \mathbf{B}$. As justified by replacing the points $\mathbf{a}(\alpha)$ with the points $\delta^{-1}\mathbf{a}(\alpha)$, we assume $\delta = 1$.

Choose R so that $B(0, R)$ contains all the unit balls $B(\mathbf{a}(\alpha), 1)$, $|\alpha| < l$. The Lagrange polynomials $p_\alpha^{\mathbf{b}}$ depend continuously on \mathbf{b} . Let

$$\lambda(r) = \sup \left\{ \sum_{|\alpha| < l} |p_\alpha^{\mathbf{b}}(x)| : |x| \leq r, \mathbf{b} \in \mathbf{B} \right\}.$$

For $d = d(\Omega, X) < \varepsilon_0/R$, set $\varepsilon = Rd$ and fix a point t in T_ε . The balls $B(t + d\mathbf{a}(\alpha), d)$ are contained in $B(t, Rd) = B(t, \varepsilon) \subset \Omega$. By (4.8), for every $|\alpha| < l$, there is at least one point x_α in $X \cap B(t + d\mathbf{a}(\alpha), d)$. If \mathbf{b} is the point in \mathbf{B} defined by $x_\alpha = t + d\mathbf{b}(\alpha)$, and

$$\sigma = \sum_{|\alpha| < l} p_\alpha^{\mathbf{b}} \left(\frac{x_0 - t}{d} \right) \delta_{x_\alpha}$$

with x_0 arbitrary, then $\text{supp } \sigma \subset X \cap B(t, \varepsilon)$, and (4.4) holds for all $p \in P_{l-1}$; to verify (4.4), take q so that $p(x) = q((x-t)/d)$ and use $\sum_{|\alpha| < l} p_\alpha^{\mathbf{b}}(y)q(\mathbf{b}(\alpha)) = q(y)$ with $y = (x_0 - t)/d$.

Suppose $x_0 \in B(t, \varepsilon M + d)$. Then, $|x_0 - t|/d \leq (RM + 1)$, so $\int d|\sigma| \leq \lambda(RM + 1)$. Also, for $x \in \text{supp } \sigma$,

$$|x - x_0| \leq |x - t| + |t - x_0| \leq (R + RM + 1)d.$$

Thus, $\int |x - x_0|^l d|\sigma| \leq C^0 d^l$ with $C^0 = (R + RM + 1)^l \lambda(RM + 1)$. Since x_0 is any point in $B(t, \varepsilon M + d)$, (4.7) gives $|g - g^X|_{B(t, \varepsilon M + d)} \leq c \|f\|_h C^0 d^l$. By (4.10), if $y \in \Omega$, we can choose $t \in T_\varepsilon$ so that $y \in B(t, \varepsilon M)$. Then $B(y, d) \subset B(t, \varepsilon M + d)$, so for every $y \in \Omega$,

$$(4.11) \quad |g - g^X|_{B(y, d)} \leq c C^0 \|f\|_h d^l.$$

This is more than required for (4.9), but will be useful for derivative estimates.

By Proposition 4.1, $f = g - g^X$ is in $C^l(\mathbf{R}^n)$. For $y \in \Omega$, $\theta \in \mathbf{R}$ and $u \in \mathbf{R}^n$ with $|u| = 1$, let $\varphi(\theta) = f(y + \theta u)$. Then

$$(4.12) \quad \varphi^{(k)}(\theta) = k! \sum_{|\alpha|=k} \frac{u^\alpha}{\alpha!} D^\alpha f(y + \theta u).$$

By (b) in Proposition (4.1), $|\varphi^{(l)}|_{\mathbf{R}} \leq C' \|f\|_h$ with $C' = l! \sum_{|\alpha|=l} c_\alpha / \alpha!$. From (4.11) we also have a bound on $|\varphi|_I$ where I is the interval $[-d, d]$. For $0 < k < l$, the results of Gorny [8] summarized in [12] then give

$$(4.13) \quad |\varphi^{(k)}(0)| \leq C_k \|f\|_h d^{l-k},$$

where $C_k = 16(2e)^k (c C^0)^{1-k/l} [\max(C', l! 2^{-l} c C^0)]^{k/l}$. Note that C_k can be calculated from n, l, m, h and M ; the choice of R depends only on l and n , so C^0 requires only l, n, M , while c and C' require only m, h, l, n . Combining (4.12) and (4.13) gives

$$(4.14) \quad \sup_{|u|=1} \left| \sum_{|\alpha|=k} \frac{u^\alpha}{\alpha!} D^\alpha f(y) \right| \leq \frac{C_k}{k!} \|f\|_h d^{l-k}$$

for every $y \in \Omega$. Since

$$|v|_k = \sup_{|u|=1} \left| \sum_{|\alpha|=k} \frac{u^\alpha}{\alpha!} v_\alpha \right|$$

is a norm for V_k , we conclude that $|D^\alpha f|_\Omega = O(d^{l-|\alpha|})$ for every $|\alpha| \leq l$. To summarize, we state

Theorem 4.4. *Let m, h, μ and a_γ be as in Theorem 2.1. Assume (4.1) holds with $l \geq \max\{1, m\}$, and suppose Ω is a subset of \mathbf{R}^n that satisfies (4.10) for some $M, \varepsilon_0 > 0$. Then there are positive constants C, d_0 such that if $f \in \mathcal{E}_{h,m}$ vanishes on a set X and the number $d = d(\Omega, X)$ defined by (4.8) is less than d_0 , then for all $|\alpha| \leq l$,*

$$(4.15) \quad |D^\alpha f|_\Omega \leq C \|f\|_h d^{l-|\alpha|}.$$

5. EXAMPLES

In this section we look at some examples of conditionally positive definite functions h . For these examples we determine the measure μ_\cdot and coefficients $a_\gamma, |\gamma| = 2m$, that appear in (2.4). As can be seen from (5.2) below, these examples all satisfy (4.1) and do so for arbitrarily large choices of l . Thus the error estimates in §4 apply, showing that for interpolation based on any of the h 's given here, approximation of arbitrarily high order can be achieved.

For $a \in \mathbf{R}$, let w_a be the function on \mathbf{R}^n defined by

$$(5.1) \quad w_a(\xi) = \frac{2 K_{(n-a)/2}(|\xi|)}{(2\pi)^{n/2} 2^{a/2} |\xi|^{(n-a)/2}},$$

where K_ν is a modified Bessel function of the second kind. From the behavior of $K_\nu(r)$ at $r = 0$ and $r = \infty$ we note that

$$(5.2) \quad \int |\xi|^{2l} w_a(\xi) d\xi < \infty$$

if and only if $a + 2l > 0$. For $a \in \mathbf{R}$, $a \neq 0, -2, -4, \dots$, let

$$(5.3) \quad h_a(x) = \frac{\Gamma(a/2)}{(1 + |x|^2)^{a/2}},$$

and for $a = -2k$, $k = 0, 1, 2, \dots$, define h_a by

$$(5.4) \quad \begin{aligned} h_{-2k}(x) &= \lim_{a \rightarrow -2k} [h_a(x) - \Gamma(a/2)(1 + |x|^2)^k] \\ &= \frac{(-1)^{k+1}}{k!} (1 + |x|^2)^k \log(1 + |x|^2). \end{aligned}$$

The last equality can be verified by using $\Gamma(\frac{a}{2} + k + 1) = (\frac{a}{2} + k) \cdots (\frac{a}{2}) \Gamma(\frac{a}{2})$ together with

$$\frac{d}{dt} \Big|_{t=k} (1 + |x|^2)^t = \lim_{a \rightarrow -2k} \frac{(1 + |x|^2)^{-a/2} - (1 + |x|^2)^k}{(-a/2) - k}.$$

Lemma 5.1. *If $\widehat{\varphi} \in \mathcal{D}(\mathbf{R}^n \sim \{0\})$, then for all a in \mathbf{R}*

$$(5.5) \quad \int h_a(x)\varphi(x)dx = \int \widehat{\varphi}(\xi)w_a(\xi) d\xi.$$

Proof. A basic fact used in the theory of Bessel potentials is that (5.5) holds for all $\varphi \in \mathcal{S}$ if $a > 0$; see [2], [3] or [4]. For $\widehat{\varphi} \in \mathcal{D}(\mathbf{R}^n \sim \{0\})$ an analytic continuation argument gives (5.5) for $a \neq 0, -2, -4, \dots$. To obtain (5.5) for the remaining values of $a = -2k$, we take limits. If $f(t) = (1 + |x|^2)^t$ and $a \neq 0, -2, -4, \dots$, then

$$\left[h_a(x) - \Gamma\left(\frac{a}{2}\right) (1 + |x|^2)^k \right] = \left(\frac{a}{2} + k\right) \Gamma\left(\frac{a}{2}\right) \int_0^1 f'(k - \left(\frac{a}{2} + k\right)s) ds.$$

Estimates from this can be used to justify an application of Lebesgue's dominated convergence theorem that shows

$$\int h_{-2k}(x)\varphi(x)dx = \lim_{a \rightarrow -2k} \int \left[h_a(x) - \Gamma\left(\frac{a}{2}\right) (1 + |x|^2)^k \right] \varphi(x) dx.$$

Now $\widehat{\varphi} \in \mathcal{D}(\mathbf{R}^n \sim \{0\})$, so $\int (1 + |x|^2)^k \varphi(x)dx = 0$. We therefore have $\int h_{-2k}(x)\varphi(x)dx = \lim_{a \rightarrow -2k} \int \widehat{\varphi}(\xi)w_a(\xi)d\xi$, which gives (5.5) for $a = -2k$. \square

Theorem 5.2. *If m is a nonnegative integer and $a + 2m > 0$, then (2.4) holds with $h = h_a$, $d\mu(\xi) = w_a(\xi)d\xi$, and $a_\gamma = 0$ for $|\gamma| = 2m$.*

Proof. If $m = 0$, then $a > 0$. As already mentioned, (5.5) holds for all φ in \mathcal{S} if $a > 0$; thus, we have (2.4) with $m = 0$ and $a > 0$. For the rest of the proof we assume $m \geq 1$. Let

$$u_a(x) = \int \left[e^{-i\langle x, \xi \rangle} - \widehat{\chi}(\xi) \sum_{k=0}^{2m-1} \frac{(-i\langle x, \xi \rangle)^k}{k!} \right] w_a(\xi) d\xi.$$

By Proposition 2.2 we have $u_a \in C(\mathbf{R}^n)$, $u_a(x) = o(|x|^{2m})$, and for all φ in \mathcal{S}

$$\int u_a(x)\varphi(x) dx = \langle S_a, \widehat{\varphi} \rangle,$$

where $\langle S_a, \psi \rangle = \int [\psi - \widehat{\chi} T^{2m-1} \psi](\xi) w_a(\xi) d\xi$. Let T_a be the tempered distribution defined by $\int h_a(x) \varphi(x) dx = \langle T_a, \widehat{\varphi} \rangle$. By (5.5), $\langle T_a, \psi \rangle = \langle S_a, \psi \rangle$ for all $\psi \in \mathcal{D}(\mathbf{R}^n \sim \{0\})$. Thus, $(T_a - S_a)^\wedge = h_a - u_a$ is a polynomial q . Both h_a and u_a are $o(|x|^{2m})$ at $|x| = \infty$, so $\deg q < 2m$. The desired instance of (2.4) now follows from $\langle h_a - q, \varphi \rangle = \langle S_a, \widehat{\varphi} \rangle$. \square

6. EQUIVALENCE OF DEFINITIONS

Theorem 6.1 below, when combined with Proposition 2.4, shows the equivalence of the definition of conditional positive definiteness adopted here with that used in [11]. As in [11], we define P_{m-1}^\perp to be the space of all finite measures ν on \mathbf{R}^n that have support consisting of a finite set of points and satisfy $\nu(p) = 0$ for all $p \in P_{m-1}$. The space obtained by relaxing the support requirement to allow compact sets, rather than only finite sets, will be denoted by $\langle P_{m-1}^\perp \rangle$. If $\nu = \sum_{i=1}^N c_i \delta_{x_i}$, then

$$\nu (\overline{\nu * \bar{h}}) = \sum_{i=1}^N \sum_{j=1}^N c_i \bar{c}_j h(x_i - x_j),$$

and $\nu \in P_{m-1}^\perp$ if and only if $\sum_{i=1}^N c_i x_i^\alpha = 0$ for all $|\alpha| < m$. If $d\nu(x) = \varphi(x) dx$ then

$$\nu (\overline{\nu * \bar{h}}) = \iint \varphi(x) \overline{\varphi(y)} h(x - y) dx dy,$$

and ν is in $\langle P_{m-1}^\perp \rangle$ if $\varphi \in \mathcal{D}_m$.

Theorem 6.1. *Let h be an arbitrary function in $C(\mathbf{R}^n)$. If $\nu (\overline{\nu * \bar{h}}) \geq 0$ holds for all $\nu \in P_{m-1}^\perp$, then it holds for all $\nu \in \langle P_{m-1}^\perp \rangle$.*

Proof. Fix ν in $\langle P_{m-1}^\perp \rangle$ and let K be its support. Recall that the finite Borel measures on K form the dual $C(K)'$ of $C(K)$, the continuous functions on K with the sup norm topology. The norms involved in this duality will be written as follows: for $f \in C(K)$, $\|f\|_K = \sup_{x \in K} |f(x)|$; for $\sigma \in C(K)'$, $\|\sigma\| = \int d|\sigma|$. Let $h_y(x) = h(y - x)$. K is compact, so for every $\varepsilon > 0$ there is a finite set $F_\varepsilon \subset K$ such that, if $y \in K$, then $|h_y - h_{y_0}|_K < \varepsilon$ for at least one $y_0 \in F_\varepsilon$. If σ is in the weak* neighborhood

$$U(\nu, F_\varepsilon, \varepsilon) = \left\{ \sigma \in C(K)' : |(\sigma - \nu)(\bar{h}_{y_0})| < \varepsilon \text{ for all } y_0 \in F_\varepsilon \right\}$$

and $y \in K$, then, for a suitable choice of $y_0 \in F_\varepsilon$,

$$|(\sigma - \nu)(\bar{h}_y)| = |(\sigma - \nu)(\bar{h}_y - \bar{h}_{y_0}) + (\sigma - \nu)(\bar{h}_{y_0})| \leq (\|\sigma - \nu\| + 1)\varepsilon.$$

Since $(\sigma - \nu) * \bar{h}(y) = (\sigma - \nu)(\bar{h}_y)$, we get $|(\sigma - \nu) * \bar{h}|_K \leq (\|\sigma - \nu\| + 1)\varepsilon$ for all $\sigma \in U(\nu, F_\varepsilon, \varepsilon)$. For such σ let w be the number defined by

$$w = \sigma (\overline{\sigma * \bar{h}}) - \nu (\overline{\nu * \bar{h}}) = \sigma ((\sigma - \nu) * \bar{h}) + (\sigma - \nu) (\overline{\nu * \bar{h}})$$

and observe $|w| \leq \|\sigma\| |(\sigma - \nu) * \bar{h}|_K + |(\sigma - \nu)(\overline{\nu * \bar{h}})|$.

Let $B = \{\sigma \in C(K)': \|\sigma\| \leq \|\nu\|\}$ and take $C = B \cap \langle P_{m-1}^\perp \rangle$, $S = B \cap P_{m-1}^\perp$. By arguments given below, S is weak* dense in C . This allows us to choose

$$\sigma \in S \cap \{\sigma \in U(\nu, F_\varepsilon, \varepsilon) : |(\sigma - \nu)(\overline{\nu * \bar{h}})| < \varepsilon\}.$$

For that choice we have $\sigma(\overline{\sigma * \bar{h}}) \geq 0$ and

$$|w| \leq \|\sigma\| (\|\sigma - \nu\| + 1) \varepsilon + \varepsilon \leq \|\nu\| (2\|\nu\| + 1) \varepsilon + \varepsilon.$$

Since w is arbitrarily small, we see that $\nu(\overline{\nu * \bar{h}})$ must be arbitrarily close to points on the positive real axis and hence must be greater than or equal to zero.

C is convex and weak* compact so, by the Krein-Milman theorem, C is the closed convex hull of its extreme points. Since S is convex, it will be weak* dense if it contains all of the extreme points of C . Suppose σ_0 is an extreme point of C that is not in S . Then $\text{supp } \sigma_0$ cannot be a finite set, so we can subdivide it into $J = 2(1 + \dim P_{m-1})$ disjoint subsets E_1, \dots, E_J with $|\sigma_0|(E_j) \neq 0$. Let $\sigma_j(E) = \sigma_0(E_j \cap E)$ and take $c_{\alpha,j} = \int x^\alpha d\sigma_j(x)$. By a dimension argument, there is a point $a \in \mathbf{R}^J \sim \{0\}$ that satisfies the equations

$$\sum_{j=1}^J a_j \|\sigma_j\| = 0; \quad \sum_{j=1}^J a_j c_{\alpha,j} = 0, \quad |\alpha| < m.$$

For $t \in \mathbf{R}$, let $\sigma^t = \sum_{j=1}^J (1 + t a_j) \sigma_j$. Then, $\sigma^t \in \langle P_{m-1}^\perp \rangle$, and if $(1 + t a_j) \geq 0$,

$$\|\sigma^t\| = \sum_{j=1}^J (1 + t a_j) \|\sigma_j\| = \sum_{j=1}^J \|\sigma_j\| = \|\sigma_0\| \leq \|\nu\|.$$

Thus, $\sigma^t \in C$ for all t in an interval about 0. This contradicts the assumption that σ_0 was an extreme point of C because $\sigma^t = \sigma_0$ only if $t = 0$, as seen from the fact that $a \neq 0$ and $\|\sigma_j\| \neq 0$ for all $j = 1, \dots, J$. \square

BIBLIOGRAPHY

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, *The theory of splines and their applications*, Academic Press, New York, 1967.
2. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials*. I, Ann Inst. Fôurier **11** (1961), 385-475.
3. A. P. Calderon, *Lebesgue spaces of differentiable functions and distributions*, Proc. Sympos. Pure Math., vol. 5, Amer. Math. Soc., Providence, R.I., 1961, pp. 33-49.
4. W. F. Donoghue, *Distributions and Fourier transforms*, Academic Press, New York, 1969.
5. J. Duchon, *Splines minimizing rotation-invariant semi-norms in Sobolev spaces*, in Constructive Theory of Functions of Several Variables (W. Schempp and K. Zeller, eds.), Springer-Verlag, Berlin-Heidelberg, 1977, pp. 85-100.
6. ———, *Sur l'erreur d'interpolation des fonctions de plusieurs variables par les D^m -splines*, RAIRO Anal. Numér. **12** (1978), 325-334.
7. I. M. Gelfand and N. Ya. Vilenkin, *Generalized functions*, Vol. 4, Academic Press, New York, 1964.
8. A. Gorny, *Contribution à l'étude des fonctions dérivables d'une variable réelle*, Acta Math. **71** (1939), 317-358.

9. J. L. Kelly and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, NJ, 1963.
10. W. R. Madych and S. A. Nelson, *Multivariate interpolation: a variational theory*, manuscript, 1983.
11. ———, *Multivariate interpolation and conditionally positive definite functions*, *Approx. Theory and its Applications* **4** (1988), no. 4, 77-89.
12. D. S. Mitrinović, *Analytic inequalities*, Springer-Verlag, Berlin-Heidelberg, 1970.

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