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LATTICE RULES: PROJECTION REGULARITY AND UNIQUE REPRESENTATIONS

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ABSTRACT. We introduce a unique characterization for lattice rules which are projection regular. Any such rule, having invariants $n_1, n_2, \ldots, n_s$, may be expressed, uniquely, in the form

$$Qf = \frac{1}{n_1 n_2 \cdots n_s} \sum \sum \cdots \sum \bar{f}\left(\frac{j_1 \mathbf{z}_1}{n_1} + \frac{j_2 \mathbf{z}_2}{n_2} + \cdots + \frac{j_s \mathbf{z}_s}{n_s}\right),$$

where the matrix $Z = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s)^T$ is upper unit triangular and individual elements satisfy $0 \le z_r^{(c)} < (n_r/n_c)$, $r < c$.

1. INTRODUCTION

The notion of a lattice rule for numerical integration over the unit s-dimensional cube was introduced in Sloan [4] and Sloan and Kachoyan [5], and further discussed in Sloan and Lyness [6], where it was shown that any s-dimensional lattice rule Q_s can be expressed in the *canonical form*

$$(1.1) \qquad Q_s f = \frac{1}{n_1 n_2 \cdots n_s} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_s=1}^{n_s} \bar{f}\left(\sum_{i=1}^{s} \frac{j_i \mathbf{z}_i}{n_i}\right).$$

Here the *invariants* $n_1, n_2, \ldots, n_s$ are positive integers satisfying

$$(1.2) \qquad n_{i+1} | n_i, \qquad i = 1, \ldots, s-1;$$

$\mathbf{z}_i \in \mathbb{Z}^s$ for $i = 1, \ldots, s$; and $\bar{f}$ is a 1-periodic extension of f. (For a more precise specification of $\bar{f}$ see Sloan and Lyness [6].) The *abscissa set* of the rule Q_s is the set

$$(1.3) \qquad A(Q_s) = \left\{ \left\{ \sum_{i=1}^{s} \frac{j_i \mathbf{z}_i}{n_i} \right\} : j_i = 1, \ldots, n_i, \; i = 1, \ldots, s \right\},$$

where $\{\mathbf{v}\}$ denotes the vector whose components are the fractional parts of those of $\mathbf{v}$. Clearly, $\{\mathbf{v}\} \in [0, 1)^s$, the half-open unit cube. The *order* of Q_s is

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the number $\nu(Q_s)$ of distinct elements of the abscissa set; since the abscissa set (1.3) of a rule (1.1) in canonical form is by definition nonrepetitive, the order is $\nu(Q_s) = n_1 n_2 \cdots n_s$.

The invariants in (1.1) are uniquely defined, but there is a wide choice available for the vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s$. The purpose of this paper is to show that for a particular class of lattice rules a representation of the form (1.1) can be prescribed in which even the vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s$ are uniquely determined.

The rules we consider have *principal projections* that are well behaved in a certain sense. The d-dimensional principal projection of Q_s is the rule Q_d obtained by retaining only the first d components of each abscissa. In Lyness and Sloan [1] it is shown that, in terms of the invariants of Q_s, its principal projection Q_d has $\nu(Q_d) \le n_1 n_2 \cdots n_d$ distinct abscissas. A rule Q_s for which equality is achieved for every principal projection, that is for which

$$\nu(Q_d) = n_1 n_2 \cdots n_d, \qquad d = 1, 2, \ldots, s, \tag{1.4}$$

is said to be *projection regular*. Thus a projection regular rule is one in which each principal projection has as many distinct points as possible for a rule with the given invariants. Projection regular rules are the focus of this paper.

The *Z-matrix* of the canonical form (1.1) is the $s \times s$ matrix defined by

$$Z = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s)^T. \tag{1.5}$$

We shall establish the following characterization of projection regular rules.

Theorem 1.1. *A lattice rule Q_s is projection regular if and only if there exists a canonical form for which the Z-matrix is upper unit triangular.*

This result will lead us to a *standard form* for projection regular rules, in which not only is the Z-matrix upper unit triangular, but also all elements of the Z-matrix are uniquely determined.

In §2 we collect together transformations of the Z-matrix which leave the rule unaltered. In §3 principal projections are defined, and we establish the easier part of the above theorem, namely that if Z is upper unit triangular, then Q_s is projection regular, while §4 is devoted to the more difficult converse. In §5 we use the results to establish the promised standard form for projection regular rules. The paper concludes with a brief discussion in §6.

2. Transformations of the Z-matrix

We now consider a more general (possibly repetitive) form of the lattice rule Q_s,

$$Q_s f = \frac{1}{n_1 n_2 \cdots n_t} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_t=1}^{n_t} \bar{f}\left(\sum_{i=1}^{t} \frac{j_i \mathbf{z}_i}{n_i} \right), \tag{2.1}$$

where t and $n_1, n_2, \ldots, n_t$ are positive integers (not necessarily satisfying the property (1.2)), and $\mathbf{z}_i \in \mathbb{Z}^s$, $i = 1, \ldots, s$. In this more general situation the

Z-matrix is a $t \times s$ matrix,

$$Z = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_t)^T. \tag{2.2}$$

Our concern in this section is to find transformations which change Z to a new matrix Z',

$$Z' = (\mathbf{z}'_1, \mathbf{z}'_2, \dots, \mathbf{z}'_t)^T, \tag{2.3}$$

while leaving the rule Q_s unchanged. Throughout this section the numbers $t, n_1, \dots, n_t$ are considered fixed.

Given the rule Q_s in the form (2.1), it is convenient to introduce

$$\mathbf{c}_i = \mathbf{z}_i/n_i, \qquad i = 1, 2, \dots, t. \tag{2.4}$$

Then the *lattice* corresponding to this rule (see Sloan and Kachoyan [5]) can be written

$$L = \left\{ \sum_{i=1}^{t} j_i \mathbf{c}_i + \mathbf{k} : j_i \in \mathbb{Z} \text{ for } i = 1, \dots, t, \text{ and } \mathbf{k} \in \mathbb{Z}^s \right\}. \tag{2.5}$$

Note that $L \cap [0, 1)^s = A(Q_s)$.

It is clear that a given transformation of the Z-matrix leaves the rule Q_s unchanged if, and only if, it leaves the corresponding lattice L unchanged.

Transformations which leave the lattice L, defined by (2.5), unchanged are: for given $i \in \{1, 2, \dots, t\}$, replace $\mathbf{c}_i$ by

$$\mathbf{c}'_i = -\mathbf{c}_i; \tag{2.6}$$

or by

$$\mathbf{c}'_i = \mathbf{c}_i + \mathbf{k}, \quad \text{where } \mathbf{k} \in \mathbb{Z}^s; \tag{2.7}$$

or by

$$\mathbf{c}'_i = \mathbf{c}_i + \mathbf{c}_l, \qquad \text{where } l \neq i. \tag{2.8}$$

It can be shown that all the affine transformations that leave L unchanged are combinations of these three.

We are interested only in those transformations which retain the form (2.4), i.e., with n_i being given. We require in addition the conditions

$$\mathbf{z}_i := n_i \mathbf{c}_i \in \mathbb{Z}^s, \quad \mathbf{z}'_i := n_i \mathbf{c}'_i \in \mathbb{Z}^s, \qquad i = 1, 2, \dots, s.$$

The following theorem includes all these.

Theorem 2.1. *The rule Q_s given by equation* (2.1) *is unchanged if for given $i \in \{1, 2, \dots, t\}$ the vector $\mathbf{z}_i$ is replaced by any of*

(a) $\mathbf{z}'_i = r\mathbf{z}_i$ *if $r \in \mathbb{Z}$ and r and n_i are relatively prime*;
(b) $\mathbf{z}'_i = \mathbf{z}_i + n_i\mathbf{k}$, *where* $\mathbf{k} \in \mathbb{Z}^s$;
(c) $\mathbf{z}'_i = \mathbf{z}_i + nn_i\mathbf{z}_l/n_l$, *where* $l \neq i$, $n \in \mathbb{Z}$, *and* $n_l | nn_i$.

Proof. Parts (b) and (c) follow immediately from (2.7) and (2.8), with the second condition in (c) coming from (2.4), or, equivalently, from the requirement that $\mathbf{z}_i$, $\mathbf{z}_i' \in \mathbb{Z}^s$. Part (a) is more interesting. Note that it subsumes (2.6) when $r = -1$. Since r and n_i are relatively prime, it is well known that

$$\{r, 2r, \dots, n_i r\} = \{1, 2, \dots, n_i\} \qquad (\text{mod } n_i).$$

It follows that $\{\{jr\mathbf{z}_i/n_i\}: j = 1, 2, \dots, n_i\}$ is merely a reordering of $\{\{j\mathbf{z}_i/n_i\}: j = 1, 2, \dots, n_i\}$, so that the rule (2.1) is unchanged. □

In the important special case in which Q_s is given in canonical form (1.1) we have, for each pair of invariants n_i, n_l, either $n_l | n_i$ or $n_i | n_l$ (or both, if $n_l = n_i$). There is therefore good reason to be interested in the following special cases of Theorem 2.1(c).

Corollary 2.2. *The rule Q_s given by (2.1) is unchanged if the vector $\mathbf{z}_i$ is replaced by*

(a) $\mathbf{z}_i' = \mathbf{z}_i + n(n_i/n_l)\mathbf{z}_l$ *if* $n_l | n_i$, $l \neq i$, *and* $n \in \mathbb{Z}$;
(b) $\mathbf{z}_i' = \mathbf{z}_i + k\mathbf{z}_l$ *if* $n_i | n_l$, $l \neq i$, *and* $k \in \mathbb{Z}$.

3. Projection regular rules

We begin with a definition from Lyness and Sloan [1].

Definition. The d-dimensional principal projection of an s-dimensional quadrature rule for the unit cube is the d-dimensional rule obtained by omitting the final $s - d$ components of each abscissa.

Specifically, the d-dimensional principal projection of the quadrature rule

$$Q_s f = \sum_{j=1}^{N} w_j f(\zeta_j^{(1)}, \zeta_j^{(2)}, \dots, \zeta_j^{(s)})$$

is

$$Q_d f = \sum_{j=1}^{N} w_j f(\zeta_j^{(1)}, \zeta_j^{(2)}, \dots \zeta_j^{(d)}).$$

The d-dimensional principal projection Q_d of a lattice rule Q_s is itself a lattice rule (see Sloan and Lyness [6, Theorem 5.1]). This leads to the following characterization of Q_d.

Theorem 3.1. *If Q_s is an s-dimensional lattice rule, then its d-dimensional principal projection Q_d is characterized by the three properties:*

(a) Q_d *is a d-dimensional lattice rule*;
(b) $(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(s)}) \in A(Q_s) \Rightarrow (\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(d)}) \in A(Q_d)$;
(c) $(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(d)}) \in A(Q_d) \Rightarrow$ *there exists an element of $A(Q_s)$ of the form*

$$(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(d)}, x^{(d+1)}, \dots, x^{(s)}).$$

Proof. This is almost self-evident: given Q_s, abscissas of Q_d are determined by (b). Additional abscissas are excluded by (c). Since Q_d is a lattice rule, each abscissa has the same weight. Thus Q_d is determined by (a), (b), and (c). □

Theorem 3.2. *If* Q_s *is given by* (1.1), *with* $n_1, \dots, n_s$ *satisfying* (1.2) *and the* Z*-matrix in upper unit triangular form, then*

(a) Q_s *is in canonical form*; *and*

(b) *for* $d \in \{1, 2, \dots, s\}$ *the rule*

$$Q_d f = \frac{1}{n_1 n_2 \cdots n_d} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_d=1}^{n_d} \bar{f}\left(\sum_{i=1}^{d} \frac{j_i(z_i^{(1)}, z_i^{(2)}, \dots, z_i^{(d)})}{n_i}\right) \tag{3.1}$$

is the d*-dimensional principal projection of* $Q_s f$, *expressed in canonical form.*

Proof. (a) That (1.1) is nonrepetitive when the Z-matrix is upper unit triangular follows easily by inspecting the abscissa set $A(Q_s)$: for each value of $k = 1, 2, \dots, s$ it contains the n_k distinct points

$$\left\{\left\{j_k \frac{(0, 0, \dots, 0, 1, z_k^{(k+1)}, \dots, z_k^{(s)})}{n_k}\right\} : j_k = 1, 2, \dots, n_k\right\} \tag{3.2}$$

(where the 1 occurs in the kth component), obtained from (1.1) by setting $j_i = 0$ for $i \neq k$. The points obtained for different values of k are distinct. Thus there are $n_1 n_2 \cdots n_s$ distinct points in $A(Q_s)$, and (1.1) is nonrepetitive.

(b) Almost by inspection, Q_d given by (3.1) satisfies the conditions of Theorem 3.1 with respect to Q_s. Thus it is indeed the principal projection of Q_s. Finally, since Q_d is in the same form as Q_s, but with d replacing s, and since Q_d has an upper unit triangular Z-matrix, it follows from (a) with s replaced by d that (3.1) is the canonical form of Q_d. □

Since (3.1) is in canonical form, we have $\nu(Q_d) = n_1 n_2 \cdots n_d$, from which it follows, by the definition in the Introduction, that Q_s is projection regular. Thus Theorem 3.2 yields:

Theorem 3.3. *If* Q_s *is given by* (1.1), *with* $n_1, n_2, \dots, n_s$ *satisfying* (1.2) *and the* Z*-matrix in upper unit triangular form, then* Q_s *is projection regular.*

4. The converse result

In this section we establish:

Theorem 4.1. *If the lattice rule* Q_s *is projection regular, it may be expressed in canonical form with the* Z*-matrix upper unit triangular.*

This is the converse of Theorem 3.3. Taken together, these give Theorem 1.1 stated in the Introduction.

Throughout this section, $n_1, n_2, \dots, n_s$ denote the invariants of Q_s; thus

$$n_{i+1} | n_i, \qquad i = 1, 2, \dots, s-1,$$

and there exists $Z = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_s)^T$ such that

$$(4.1) \qquad Q_s f = \frac{1}{n_1 n_2 \cdots n_s} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_s=1}^{n_s} \bar{f}\left(\sum_{r=1}^{s} \frac{j_r \mathbf{z}_r}{n_r}\right)$$

is in canonical form. Moreover, because Q_s is projection regular, the d-dimensional principal projection Q_d has invariants $n_1, n_2, \dots, n_d$, and has order

$$(4.2) \qquad \nu(Q_d) = n_1 n_2 \cdots n_d .$$

Our proof of Theorem 4.1 is by induction on the columns of the Z-matrix, the inductive step being:

Lemma 4.2. *Let d be an integer satisfying $1 \le d \le s$. If the projection regular rule Q_s can be expressed in the canonical form (4.1) with a Z-matrix whose first $d-1$ columns are in upper unit triangular form, then it may also be expressed in canonical form with a Z-matrix whose first d columns are in upper unit triangular form.*

Proof of Lemma 4.2. It is convenient to denote the dth column of the Z-matrix, i.e., the column currently being treated, by $\boldsymbol{\zeta}$, so that $\zeta_r = z_r^{(d)}$, $r = 1, \dots, s$. Then the d-dimensional principal projection of (4.1) is given by

$$(4.3) \qquad Q_d f = \frac{1}{n_1 n_2 \cdots n_s} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \cdots \sum_{j_s=1}^{n_s} \bar{f}\left(\sum_{r=1}^{s} \frac{j_r (z_r^{(1)}, \dots, z_r^{(d-1)}, \zeta_r)}{n_r}\right).$$

The hypothesis of the lemma is that the elements of the Z-matrix in (4.1) satisfy

$$(4.4) \qquad z_r^{(c)} = \delta_{rc} \quad \text{for } r = c, c+1, \dots, s, \; c = 1, 2, \dots, d-1 .$$

We plan to provide a set of transformations which transform the Z-matrix, without changing Q_s, in such a way that (4.4) continues to hold, but in addition

$$(4.5) \qquad \zeta_d = z_d^{(d)} = 1 ,$$

and

$$(4.6) \qquad \zeta_r = z_r^{(d)} = 0 \quad \text{for } r = d+1, d+2, \dots, s .$$

We shall specify sets of transformations ((4.11) and (4.16) below) which, while leaving the first $d-1$ columns of the Z-matrix unchanged, produce a value of ζ_d satisfying (4.5), and after this a second set ((4.18) below) which set the ζ_r of (4.6) to zero. However, before establishing these transformations we need the following:

Lemma 4.3. *Under the hypotheses of Lemma 4.2 the quantities in (4.3) satisfy*

$$(4.7) \qquad \left(\zeta_d, \frac{n_d}{n_{d+1}}\zeta_{d+1}, \dots, \frac{n_d}{n_s}\zeta_s, n_d\right) = 1 ,$$

where, as before, ζ_r stands for $z_r^{(d)}$.

The quantity on the left-hand side of (4.7) is the greatest common divisor of the numbers separated by commas.

We now prove Lemma 4.3. Because Q_s is projection regular, the abscissa set $A(Q_d)$ is of order $n_1 n_2 \cdots n_d$. We are interested in the subgroup S_d of $A(Q_d)$ obtained by setting $j_1 = j_2 = \cdots = j_{d-1} = 0$ in (4.3). Because the first $d-1$ columns of the Z-matrix are already in upper unit triangular form, we obtain from (4.3)

$$S_d = \left\{ \left(0, 0, \ldots, 0, \left\{\sum_{r=d}^{s} \frac{j_r \zeta_r}{n_r}\right\}\right) : \begin{array}{l} j_r = 1, 2, \ldots, n_r, \\ r = d, d+1, \ldots, s \end{array} \right\}. \tag{4.8}$$

The dth components of (4.8) form the abscissa set of a one-dimensional lattice rule. Since the only such rule is the trapezoidal rule, the only thing not yet known is the spacing. We assert that S_d contains exactly n_d distinct points, and therefore

$$S_d = \left\{ \left(0, 0, \ldots, 0, \frac{j}{n_d}\right) : j = 0, 1, \ldots, n_d - 1 \right\}. \tag{4.9}$$

To show this, first observe that the summations over $j_1, \ldots, j_{d-1}$ in (4.3), with $j_d, \ldots, j_s$ set equal to $n_d, \ldots, n_s$ respectively, give exactly $n_1 n_2 \cdots n_{d-1}$ distinct abscissas; this is easily verified directly, given that the first $d-1$ columns of the Z-matrix are already in upper unit triangular form. Since $\nu(Q_d) = n_1 n_2 \cdots n_d$, the remaining sums, with $j_1, \ldots, j_{d-1}$ set to zero, must yield exactly n_d abscissas. Thus (4.9) is established.

It follows from the equivalence of (4.8) and (4.9) that there must exist integers $j_d, j_{d+1}, \ldots, j_s$ and l satisfying $1 \le j_r \le n_r$ such that

$$\frac{1}{n_d} = j_d \frac{\zeta_d}{n_d} + j_{d+1} \frac{\zeta_{d+1}}{n_{d+1}} + \cdots + j_s \frac{\zeta_s}{n_s} + l,$$

or equivalently

$$1 = j_d \zeta_d + j_{d+1} \frac{n_d}{n_{d+1}} \zeta_{d+1} + \cdots + j_s \frac{n_d}{n_s} \zeta_s + l n_d. \tag{4.10}$$

Any factor shared by each term on the right must occur in the left. This establishes Lemma 4.3.

Returning to Lemma 4.2, we are now ready to construct a sequence of transformations of the Z-matrix which change only the dth row of the matrix, and leave all other rows unchanged. We shall exploit Lemma 4.3 above to show these give $z_d^{(d)} = 1$. The transformations consist of row operations of the form

$$\mathbf{z}_d(i) = \mathbf{z}_d(i-1) + k_i \frac{n_d}{n_i} \mathbf{z}_i, \qquad i = d+1, \ldots, s, \tag{4.11}$$

where the initial vector $\mathbf{z}_d(d)$ is the dth row of the Z-matrix as given to us in the hypotheses of Lemma 4.2. Note that the leading $d-1$ zeros in

$\mathbf{z}_d$ are not altered by the transformations, and that by Corollary 72.2(a) the transformations (4.11) do not change Q_s.

Consistently with our earlier notation, we denote the dth component of $\mathbf{z}_d(i)$ by $\zeta_d(i)$, so that the leading nonzero component of (4.11) takes the form

$$\zeta_d(i) = \zeta_d(i-1) + k_i \frac{n_d}{n_i}\zeta_i, \qquad i = d+1, \ldots, s. \tag{4.12}$$

We are now ready to specify k_i in (4.11): it is chosen so that

$$\left(\zeta_d(i-1) + k_i \frac{n_d}{n_i}\zeta_i, n_d\right) = \left(\zeta_d(i-1), \frac{n_d}{n_i}\zeta_i, n_d\right), \tag{4.13}$$

the existence of an integer with this property being guaranteed by Lemma A1 in the appendix. Since (4.12) and (4.13) imply

$$(\zeta_d(i), n_d) = \left(\zeta_d(i-1), \frac{n_d}{n_i}\zeta_i, n_d\right), \qquad i = d+1, \ldots, s,$$

it follows recursively that at the end of the sequence of transformations we have

$$(\zeta_d(s), n_d) = \left(\zeta_d(d), \frac{n_d}{n_{d+1}}\zeta_{d+1}, \ldots, \frac{n_d}{n_s}\zeta_s, n_d\right),$$

and hence, from Lemma 4.3,

$$(\zeta_d(s), n_d) = 1. \tag{4.14}$$

Let us now denote by ζ_d instead of $\zeta_d(s)$ the value of $z_d^{(d)}$ achieved at the end of the sequence of transformations (4.11). Since $(\zeta_d, n_d) = 1$, there exists an integer q such that

$$q\zeta_d \equiv 1 \pmod{n_d}. \tag{4.15}$$

Since $(q, n_d) = 1$, the transformation

$$\mathbf{z}_d' = q\mathbf{z}_d \tag{4.16}$$

leaves Q_s unchanged by Theorem 2.1(a), and at the same time ensures

$$\zeta_d' := z_d'^{(d)} \equiv 1 \pmod{n_d}. \tag{4.17}$$

An application of Theorem 2.1(b) then allows us to replace ζ_d' by 1 without changing Q_s. We now have the value of $\zeta_d = z_d^{(d)}$ required in (4.5).

A final set of transformations

$$\mathbf{z}_r' = \mathbf{z}_r - \zeta_r\mathbf{z}_d, \qquad r = d+1, \ldots, s \tag{4.18}$$

(where $\mathbf{z}_d$ denotes the *current* dth row of the Z-matrix, satisfying (4.5)), now replaces the elements $z_r^{(d)} = \zeta_r$ in the lower part of the dth column of the Z-matrix by zeros, while at the same time, by Corollary 2.2(b), leaves Q_s unaltered. The proof of Lemma 4.2 is now complete. □

Theorem 4.1 now follows from Lemma 4.2 by induction on d.

5. A UNIQUE REPRESENTATION FOR PROJECTION REGULAR LATTICE RULES

In this section we establish a unique standard form for all projection regular lattice rules.

Definition. Q_s of (1.1) is in *standard form* when the elements of its Z-matrix satisfy

$$z_r^{(c)} = 0, \qquad 1 \le c < r \le s, \tag{5.1}$$
$$z_c^{(c)} = 1, \qquad c = 1, 2, \dots, s,$$

$$0 \le z_r^{(c)} < n_r/n_c, \qquad 1 \le r < c \le s. \tag{5.2}$$

Thus Z is an upper unit triangular matrix and so, from Theorem 3.3, a rule Q_s in standard form must be a projection regular rule.

In this section we establish

Theorem 5.1. *A projection regular lattice rule with invariants $n_1, n_2, \dots, n_s$ can be expressed, uniquely, in standard form.*

To establish the theorem, we use Theorem 4.1, and also prove in turn Lemmas 5.2 and 5.3 below. The latter deals with uniqueness.

Lemma 5.2. *When Q_s is given in canonical form (1.1) with upper unit triangular Z-matrix, it may be re-expressed with a Z-matrix in standard form.*

Proof. This can be effected by transformations of the form

$$\mathbf{z}_r' = \mathbf{z}_r - \left[\frac{z_r^{(c)} n_c}{n_r}\right] \frac{n_r}{n_c} \mathbf{z}_c \quad \text{for } r < c, \tag{5.3}$$

which according to Corollary 2.2(a) leave Q_s unaffected. The transformation (5.3) affects only the rth row $\mathbf{z}_r$ of the matrix. Since $\mathbf{z}_c$ has $c-1$ leading zero components, it leaves unaltered the first $c-1$ elements of $\mathbf{z}_r$, and generally alters the other elements. In particular, since $z_c^{(c)} = 1$, it replaces $z_r^{(c)}$ by

$$z_r'^{(c)} = z_r^{(c)} - \left[\frac{z_r^{(c)} n_c}{n_r}\right] \frac{n_r}{n_c}, \tag{5.4}$$

which satisfies (5.2). It is necessary to carry out these transformations in an order arranged so that, once $z_r^{(c)}$ has been replaced by $z_r'^{(c)}$, the new element is not affected by any subsequent transformation. It is readily verified that a suitable order is obtained so long as the transformation involving $z_r^{(c)}$ is carried out only after all transformations involving a lower column number $c' < c$ are completed. This establishes Lemma 5.2. □

Lemma 5.3. *The rule Q_s may be represented in standard form in only one way.*

Proof. Suppose that $Z = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_s)^T$ and $W = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s)^T$ are two alternative Z-matrices for the rule Q_s, both in standard form, whose first

$d-1$ columns coincide. Given $r \in \{1, 2, \ldots, d-1\}$, the vectors $\mathbf{z}_r/n_r$ and $\mathbf{w}_r/n_r$ belong to the lattice corresponding to Q_s, and therefore so too does $(\mathbf{z}_r - \mathbf{w}_r)/n_r$. On retaining only the leading d components of each vector, we conclude that the d-vector

$$\left(0, 0, \ldots, 0, \frac{z_r^{(d)} - w_r^{(d)}}{n_r}\right) \tag{5.5}$$

belongs to the lattice corresponding to the d-dimensional principal projection Q_d. From this it follows that

$$\left(0, 0, \ldots, 0, \frac{z_r^{(d)} - w_r^{(d)}}{n_r}\right) = \sum_{i=1}^{d} j_i \frac{(z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(d)})}{n_i} \pmod 1$$

for some integers $j_1, j_2, \ldots, j_d$. Because of the upper unit triangular nature of Z, it is easily seen that this can be satisfied only if

$$j_i \equiv 0 \pmod{n_i}, \qquad i = 1, 2, \ldots, d-1;$$

thus we conclude that

$$\left(0, 0, \ldots, 0, \frac{z_r^{(d)} - w_r^{(d)}}{n_r}\right) = j_d \frac{(0, 0, \ldots, 0, z_d^{(d)})}{n_d} = j_d \frac{(0, \ldots, 0, 1)}{n_d},$$

or

$$z_r^{(d)} - w_r^{(d)} = j_d \frac{n_r}{n_d}$$

for some integer j_d. Thus $z_r^{(d)}$ differs from $w_r^{(d)}$ by some integer multiple of n_r/n_d. Since by (5.2) both $z_r^{(d)}$ and $w_r^{(d)}$ are in the interval $[0, n_r/n_d)$, it follows that they coincide.

We recall that our hypothesis is that the first $d-1$ columns of Z and W coincide. The argument above shows that all elements of the dth column also coincide, and Lemma 5.3 then follows by induction. □

Lemmas 5.2 and 5.3 together with Theorem 4.1 establish Theorem 5.1.

6. Concluding remarks

In Sloan and Lyness [6], we introduced a classification of lattice rules based on the set of invariants $n_1, n_2, \ldots, n_s$. In this paper, for projection regular rules only, we have extended the argument to obtain a unique categorization, based on a standard form of the associated Z-matrix.

It seems likely that the class of projection regular rules is wide enough to contain many interesting rules. However, not all rules are projection regular, even after an interchange of coordinate axes. For example, the 42-point 3-dimensional rank-1 rule

$$Qf = \frac{1}{42} \sum_{j=1}^{42} \bar{f}\left(j \frac{(2, 3, 16)}{42}\right) \tag{6.1}$$

has no one-dimensional projections of order 42, and so cannot be projection regular, even after interchange.

It is interesting to note that the search for good lattice rules of rank 1 by Maisonneuve [2] was in fact restricted to projection regular rules, through the first component of $\mathbf{z} = \mathbf{z}_1$ being forced to have the value 1. Her list (Maisonneuve [2]) includes optimal rank-1 rules, i.e., the most economic rule (or rules) having a particular Zaremba ρ index (Zaremba [7]). A brief check by the present authors showed that in the range $10 < N < 150$ with $s = 3$ she found all of the projection regular rules of this form. If all rank-1 rules are allowed, then the rule (6.1) above should replace the one with $N = 44$ on her list, since both have $\rho = 6$ and (6.1) is more economic. The rest of Maisonneuve's results in this range would be unchanged by the wider search.

APPENDIX

Lemma A1. *Let* l, m, n *be integers, with* $n > 0$. *There exists an integer* k *such that*

$$(l + km, m) = (l, m, n).$$

Proof. Since the result is trivial if l or m is zero, we may assume l, m nonzero. It is sufficient to prove the result for the case $(l, m, n) = 1$, since if $(l, m, n) = a$ we may apply that result to $l' = l/a$, $m' = m/a$, $n' = n/a$.

Given $(l, m, n) = 1$, we are required to choose k so that $(l + km, n) = 1$. Let p_i be the ith prime, and write l, m, n as prime power decompositions,

$$l = \pm \prod p_i^{\lambda_i}, \qquad m = \pm \prod p_i^{\mu_i}, \qquad n = \prod p_i^{\nu_i}.$$

Since $(l, m, n) = 1$, it follows that for each prime p_i at least one of λ_i, μ_i, ν_i is zero. Defining

$$\alpha_i = \begin{cases} 1 & \text{if } \nu_i \neq 0,\ \lambda_i = \mu_i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we may choose

$$k = \prod p_i^{\alpha_i}.$$

Then it is straightforward to verify that any prime p_i which divides n does *not* divide $l + km$, since it divides one term but not the other. □

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