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SOME NEW POLYHEDRA WITH VERTEX DEGREE 4 AND/OR 5 ONLY

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ABSTRACT. A table of 4- and 5-hedra of orders up to and including 22 is given.

In 1981 we reported on the number of polyhedral graphs [5]. That work was a byproduct of the search for the lowest-order squared square, which was found in March 1978 [3]. The squaring problem is closely related to the theory of 3-connected planar graphs, as was first shown by Brooks, Smith, Stone, and Tutte [1] in 1940. In 1962 we developed the necessary techniques for computer manipulation of 3-connected planar graphs. These techniques were reported in [2]. The set of 4- and 5-hedra is a subset of the set of 3-connected planar graphs. In that paper, a code for 3-connected graphs was introduced in which the essential properties of planarity are preserved.

It is assumed that the graph is drawn on the sphere. The vertices are numbered arbitrarily from 1 to K, where K is the number of vertices of the graph. The sides or meshes are numbered arbitrarily from 1 to M, where M is the number of sides (or meshes). The boundary contains a set of vertices. A code of a side is obtained as follows: while walking in the positive sense along the boundary of the side, starting with V_i, we encounter V_j, V_k, V_l, ..., until we return to V_i. The sequence V_i, V_j, V_k, V_l, ..., V_i is a code of the side.

Example. A possible code of side 1 of the reference graph is 1 2 6 5 1, as can be seen from Figure 1; but we can also take 2 6 5 1 2, 6 5 1 2 6, or 5 1 2 6 5.

A code of the graph is the sequence of codes of all its sides, separated by zeros. At the end, two more zeros are added.

Example. A code of the reference graph is as follows:

1 2 6 5 1 0 2 3 6 2 0 3 5 6 3 0 3 4 5 3 0 1 5 4 1 0 1 4 3 2 1 0 0

In case we deal with more than nine vertices, it is more convenient to code the vertices with (capital letters, where $A = 1$, $B = 2$, $C = 3$, etc.

Example. The above code of the reference graph reads:

$A\,B\,F\,E\,A\,0\,B\,C\,F\,B\,0\,C\,E\,F\,C\,0\,C\,D\,E\,C\,0\,A\,E\,D\,A\,0\,A\,D\,C\,B\,A\,0\,0$

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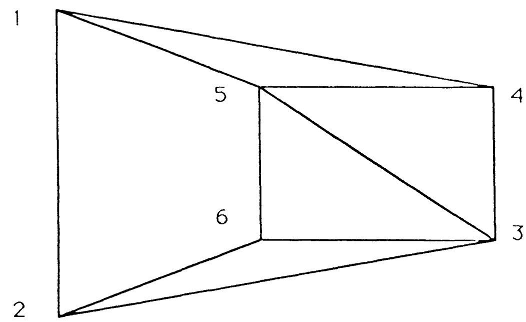

FIGURE 1. Reference graph

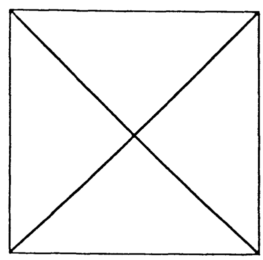

FIGURE 2. Ancestor graph

All 3-connected graphs can be generated starting with the ancestor graph consisting of eight edges, using a theorem of Tutte [6]. See Figure 2.

Tutte considered the set S_B of 3-connected planar graphs having B edges. Let $s \in S_B$ and let s' be its dual. Then if s is not a wheel, at least one of the graphs s or s' can be constructed from an element σ of S_{B-1} by addition of an edge joining two vertices of σ. A wheel is a planar graph with an even number of vertices (B), with one vertex of degree $1/2B$, and $B-1$ vertices of degree 3. See also Figure 3.

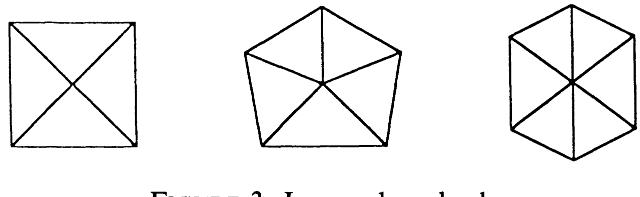

FIGURE 3. Low-order wheels

The process of adding wires is referred to as the generation process. The generation process produces many duplicates. It is therefore necessary to develop an identification algorithm by which graphs can uniquely be identified. The basic ideas for identifying graphs were given in [2]. The identification problem, including the calculation of the order of the automorphism group, was completely solved in 1978 [4].

As reported in [5], we generated and identified all 3-connected planar graphs of orders 9 up to and including 22. Those of orders 23 and 24 were only generated for graphs with 10 vertices. The results have been stored on magnetic tape. Very recently, Dr. King from the chemistry department of New Georgia University asked us to search our tapes for graphs with the property that either the original or its dual consists of vertices with degree 4 and/or 5 only. We found 40 such graphs which are listed in Table 1.

First, the code is given, then the order of the automorphism group, next an identification of selfduality (0=not selfdual, 1=selfdual) and finally the identification number. For explanation of the identification number we refer to [4].

Example. The reference graph

*ABFEA*0*BCFB*0*CEFC*0*CDEC*0*AEDA*0*ADCBA*00 0002 1 00000000075523

TABLE 1. 4- and 5-hedra

EFAE0ADEA0EDCE0FECF0CBFC0FBAF0DABD0BCDB00

0048000000000000000075537

FGBF0EFBE0BAEB0FEDF0GFDG0GDCG0GCBG0DEAD0CDAC0ABCA00

0020000000000000003777423

GHBG0FGBAF0GFEG0HGEH0HEDCH0HCBH0FADF0DEFD0CDAC0BCAB00

0016000000000001706424537

GHCG0FGCF0FCBF0BAFB0GFEG0HGEH0HEDH0HDBCH0EFAE0DEAD0BDAB00

0004000000000001746324563

GHCG0FGCF0FCBF0GFEG0HGEH0HEDH0HDCH0FBAF0AEFA0DEAD0DABD0BCDB00

0008000000000001533672741

HICH0GHCBG0GBAG0HGFH0IHFEI0IEDI0CDBC0IDCI0FGAF0AEFA0DEABD00

0012000000007415030145 37

HICH0GHCG0GCBG0HGFH0IHFEI0IEDI0IDCI0FGBF0FBAF0AEFA0DEAD0CDABC00

0002000000003656560204 37

HICH0GHCBG0GBAG0HGFH0IHFEI0IEDI0IDCI0FGAF0EFAE0ABEA0DEBD0CDBC00

0004000000007515230151 17

HIDCH0GHCG0GCBG0HGFH0IHFI0IFEI0IEDI0GBAG0FGAF0EFAE0DEABD0BCDB00

0004000000007614624251 17

HIDCH0GHCG0GCBG0HGFH0IHFI0IFEI0IEDI0GBAG0FGAF0EFAE0BEAB0DEBD0BCDB00

0008000000006707265460 63

HICH0GHCG0GCBG0HGFH0IHFEI0IEDI0IDCI0FGBF0FBAF0AEFA0DEAD0BDAB0CDBC00

0002000000007612615662 16

IJDI0HIDCH0HCBH0IHGI0JIGFJ0JFEJ0JEDJ0GHBG0FGBAF0EFAE0DEACD0BCAB00

0004000007406404430245 37

JIIJ0JHCJ0GJCBAG0IJGI0IGFI0HIFEDH0DBCD0CHDC0EABE0BDEB0FGAF0AEFA00

0020000007406405024245 37

IJDI0HIDCH0HCBH0IHGFI0JIFJ0JFEJ0JEDJ0GHBG0GBAG0FGAEF0DEACD0BCAB00

0016000007406405044055 37

HICH0GHCG0GCBG0HGFH0HFEH0EIHE0IEDI0IDCI0FGBF0FBAF0AEFA0DEAD0BDAB0CDBC00

0012000000006545665333 30

IJCI0HGICH0HCBH0HBAH0IGFEI0JIEJ0JEDJ0JDCJ0GHAG0AFGA0BFAB0DFBCD0EFDE00

0002000003526526014442 53

TABLE 1 (*continued*)

IHCI0JICJ0GJCBG0GBAG0JGFJ0EDHE0IJFI0FEHIF0DBCD0CHDC0DEABD0FGAF0AEFA00
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IJDI0HIDH0HDCBH0IHGI0IGFI0FJIF0JFEJ0JEDJ0GHBG0FGBAF0EFAE0CEAC0ABCA0DECD00
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0016000000000000000662325261662552

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