Only steady state problems are solved in Part II, and this is done without using time-dependent methods. In supersonic regions there is a hyperbolic system with a time-like direction, and some of the methods from Chapter 2 are used. Chapter 3 seems to be intended as (or should be) a theoretical foundation for all other applications which are presented. However, this chapter is only 14 pages long and does not contain much theory. The methods are based on shooting in one space direction, and the authors point out that this leads to approximation of initial value problems for elliptic equations, which is an ill-posed problem. There are some remarks about that, but they end up with the conclusion that "the number of lines must be properly selected", where "lines" refer to the number of points in one space direction. There is no analysis to guide the reader in that selection.

Part II is a detailed description of all the equations which are necessary to apply the numerical methods. This is in a sense the strength of the book. For example, anyone who wants to know an approximation formula for the specific enthalpy expressed in terms of pressure and density will find it with all the coefficients given in tables. The internal boundary conditions used at discontinuities and other singularities are worked out in detail. Numerical results are presented for blunt body computations, and for bodies composed of spheres, cones and cylinders. These computations are impressive, not the least considering that the computers used were no real supercomputers. The problems probably belong to the most difficult ones that have been solved by shock-fitting methods.

Unfortunately, as indicated above, the book is not as informative as it could have been. The theory and the methods are presented in a very technical way, which makes it difficult to understand the basic structure and ideas. The book could possibly be useful for engineers who want to solve flow problems with a similar structure using the same methods, but for graduate students it is of less value. Nevertheless, for someone, who really wants to work his way through the book, there is certainly a large amount of material to be found and to learn from.

BERTIL GUSTAFSSON

Department of Computer Sciences
Uppsala University and
National Defense Research Institute (FOA)
Stockholm, Sweden

7[41A15, 41A63, 65D07].—CHARLES K. CHUI, *Multivariate Splines*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 54, SIAM, Philadelphia, 1988, vi+189 pp., 25 cm. Price Softcover $19.00.

Since the late seventies the subject of what may be called "multivariate splines" has become a rapidly growing field of mathematical research. However, it soon became clear that there is no hope to ever arrive at anything nearly as unified as the univariate theory. In fact, trying to extend particular features of univariate splines into a multivariate setting typically leads to totally different multivariate concepts. In view of this diversity it would be unfair to interprete the general title "Multivariate Splines" as a promise to address all the relevant aspects of the subject that have evolved during the past ten years. For instance, the classical 'physical' meaning of 'spline' suggests replacing a thin wooden rod by a thin plate. In analogy to the univariate case one could then generate multivariate interpolating splines by selecting among all interpolants to given data one that minimizes a certain functional approximating the bending energy of the plate. However, unlike the univariate case, the resulting splines are not even piecewise polynomial anymore and lack the perhaps most important feature of (univariate) splines, namely their local structure. In this book 'spline' is simply a synonym for a piecewise polynomial that exhibits a certain degree of global differentiability. The lectures compiled in this book reflect essentially one major 'philosophy' of how to approach multivariate splines. The following remarks are to bring that out and to relate it, if possible, to various other viewpoints.

The material in Chapter 1 is to motivate the development of subsequent multivariate counterparts. As the author points out himself, it is not meant to give a balanced overview of the univariate spline theory. Chapter 2 deals with multivariate versions of truncated powers and so-called box splines. The box spline is an important instance of a multivariate B-spline-like density that can be obtained by generalizing a fundamental geometric interpretation of the univariate B-spline due to Curry and Schoenberg. This geometric construction principle yields compactly supported piecewise polynomials which automatically enjoy a certain global smoothness. The main issue then becomes to build suitable spline spaces by properly selecting spanning sets of such B-splines. This has initiated a systematic development of what one may call a 'B-spline approach'. In the particular case of box splines, the analogy with the univariate cardinal B-spline suggests using its multi-integer translates as spanning sets. But already the question of (local) linear independence of such translates turns out to be rather delicate and has interesting algebraic and combinatorial aspects. Here the discussion of box splines is confined to some basic facts about recurrence relations, difference representations, connections with multivariate truncated powers and polynomial reproduction. These results pertain to a somewhat different philosophy. In fact, instead of trying to build splines by taking linear combinations of properly selected 'B-splines', thereby automatically inducing a partition for the resulting space of piecewise polynomials, an alternate (in some sense opposite) approach is to specify first a partition, a class of polynomials and a certain global smoothness in order to then characterize the space of piecewise polynomials—splines satisfying those constraints. In the univariate case, both points of view lead to the same concept, whereas in higher dimensions they diverge. Choosing the latter approach, one typically has to face the following tasks: determine the dimension of the spline space; identify a basis; among all possible bases find one with functions of possibly small or even minimal support; determine the approximation power of the space.

The book essentially follows this pattern for various types of partitions. As outlined in Chapter 3, for spaces based on certain uniform triangulations of the plane, it is possible to progress relatively far with the above program. This is also where box splines, as one of the few links between the two approaches, come into play. It turns out that the translates of those box splines that belong to a given spline space span a subspace which already has the full approximation power of the whole space. Nevertheless, the problem of determining the approximation power of spaces spanned by the translates of several distinct box splines is still far from being solved. Here the discussion of this issue is essentially confined to spaces spanned by the translates of a single box spline or, more generally, of a single compactly supported function. In this case an elegant and fairly complete theory based on Poisson's summation formula is available. In my opinion, the development of this topic, outlined in Chapter 8, is a highlight of the book. When dealing with nonuniform partitions, already the first step, namely the 'dimension problem', turns out to cause serious difficulties. It may not be surprising that the problem of determining the dimension of spline spaces over nonuniform triangulations in the plane was addressed first in the context of finite element analysis. An early example of C^1-piecewise quadratics demonstrated that the dimension will in general not only depend on the degree of the polynomials, the degree of smoothness and the combinatorial structure of the triangulation, expressed, e.g., in terms of the number of edges or vertices, but also on the actual location of the vertices. This is the main issue of Chapter 4. Finding sharp lower and upper bounds in terms of the degrees and the combinatorial structure of the triangulation constitutes, therefore, an important progress on the dimension problem. Meanwhile, using methods from homological algebra, the existence of a 'generic' dimension has been established which typically agrees with the lower bound. In view of these major difficulties one often has to be content with identifying suitable subspaces spanned by appropriate functions of possibly small support in order to deal with interpolation and approximation problems (conceptually getting back closer to the 'B-spline approach'). Such subspaces can be spanned by so-called 'vertex splines' (splines whose support contains at most one vertex in its interior), whose construction is outlined in Chapter 6. The main tool in this context is the Bernstein representation of polynomials as described in Chapter 5. Again, as in the case of uniform partitions, the role of quasi-interpolants as a fundamental tool for determining the approximation power of spline spaces is stressed in Chapter 6. This part establishes certainly the closest contact to finite element analysis, as many classical shape functions can be recovered through vertex splines. However, in contrast to the classical finite element philosophy, the basic point of view here is still trying to assemble individual elements to appropriate B-spline-like functions of small support.

Chapter 7 is quite different in spirit. It proposes ways of computing piecewise polynomial surfaces. Specifically, subdivision techniques are described that facilitate an efficient graphical display of a given surface represented in terms of

Bernstein polynomials or box splines. The basic idea, which has evolved in the context of Computer Aided Geometric Design (CAGD), is to represent such a spline or polynomial in terms of bases for finer and finer grids or partitions. The coefficients in the resulting refined representations turn out to approach the surface and are typically computed more efficiently than the exact values of the spline. For Bernstein polynomials this is usually done with the aid of de Casteljau's algorithm, while the so-called line average algorithm applies to box spline surfaces. In both cases, the number of operations per refined coefficient (i.e., per approximate value on the surface) is only proportional to the degree of the surface. Therefore, I am not sure about any advantage in using the explicit representation (7.1) for the refined Bernstein coefficients. Its evaluation seems to be more costly and less stable than the averaging techniques of subdivision algorithms.

Although neither explicit implementations of any computational scheme nor corresponding numerical effects are discussed in this book, the constructive aspects of the theory are clearly emphasized, and corresponding direct or potential applications are outlined. Numerous explicit formulae and concrete spline representations (see, for instance, Chapter 6 on vertex splines) make parts of the book almost appear as a reference work, which may well be suitable also for those who are primarily interested in computational applications without having to work through the whole theory first. Aside from the issue of graphical display, discussed in Chapter 7, various other applications such as shape-preserving approximation and interpolation are outlined in Chapter 10. (I wonder why the more theoretically oriented material in Chapters 8 and 9 pertaining mainly to splines on uniform partitions was not placed immediately after Chapter 3.) In this sense the book means to address not only the experts in the field but also those working in other areas where the application of splines may be relevant.

The book reads very comfortably. In fact, with a few exceptions, the author dispenses with giving any proofs. Instead, the given facts are illustrated by many examples and figures. This allows covering an enormous amount of material and touching on many ideas in a volume of such relatively small size. On the other hand, I find the general style of presentation rather 'leveled' in that facts on very different levels of mathematical sophistication and generality are often put together with 'equal weights'. In this sense, I find the presentation somewhat short on 'vertical' orientation regarding the highlights of the field, on more information about the type of qualitatively new methods, and along with that, on a clearer idea of principal constraints as well as potentials for future research activities. To some extent, this seems to be a price one has to pay for the rich 'horizontal' orientation provided by the range of clearly structured topics. In fact, the main strength of the book lies in the combination of the contents and the extensive list of references which, I think, is one of the author's central objectives. To really penetrate into any of the subjects discussed in the book, one has to consult the original literature. In view of my introductory remarks, the importance of such a first, well organized and very informative 'guide' to a major part of a highly diverse and active field of research can hardly be overestimated.

WOLFGANG DAHMEN

Fachbereich Mathematik
Freie Universität Berlin
1000 Berlin 33
West Germany

8[65-06, 65Kxx, 90Cxx].—JEAN-PAUL PENOT (Editor), *New Methods in Optimization and Their Industrial Uses*, International Series of Numerical Mathematics, Vol. 87, Birkhäuser, Basel, 1989, ix+227 pp., 24 cm. Price $55.00.

These are the proceedings of two symposia, one held in Pau, October 19–29, 1987, the other in Paris, November 19, 1987. Included are 14 contributions, all in English except for three, which are in French. Among the recent advances discussed are Karmarkar's algorithm in linear programming, methods for global optimization, simulated annealing methods and trust region algorithms. Also represented are topics related to computer science, such as expert systems developments and the automatic analysis of the effects of round-off and data errors. Industrial applications include those to biomedicine, structural problems in the automotive industry, and stock management problems.

W. G.

9[33-06, 33A65].—JAIME VINUESA (Editor), *Orthogonal Polynomials and Their Applications*, Lecture Notes in Pure and Appl. Math., Vol. 117, Marcel Dekker, New York and Basel, 1989, x+207 pp., $25\frac{1}{2}$ cm. Price $99.75.

This volume is the proceedings of the International Congress on Orthogonal Polynomials held in Laredo, Spain, September 7–12, 1987. It contains the text of six invited lectures and eleven contributed papers.

W. G.