A DIRECT BOUNDARY ELEMENT METHOD
FOR SIGNORINI PROBLEMS

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ABSTRACT. In this paper, a Signorini problem is reduced to a variational inequality on the boundary, and a direct boundary element method is presented for its solution. Furthermore, error estimates for the approximate solutions of Signorini problems are given. In addition, we show that the Signorini problem may be formulated as a saddle-point problem on the boundary.

1. INTRODUCTION

The Signorini problems are a class of very important variational inequalities, which arise in many practical problems such as the elasticity with unilateral conditions [17, 5], the fluid mechanics problems in media with semipermeable boundaries [4, 7], the electropaint process [1], etc. In the literature, there are many authors who have studied the problems, established the existence and uniqueness, and obtained the regularity results for Signorini problems (see Brézis [3] and Friedman [6]). Furthermore, the numerical solution of the Signorini problem by the finite element method has been discussed [7, 8]. We know that the solution of the Signorini problem satisfies a linear partial differential equation in the domain, even when the problem is nonlinear. Hence, it is natural and advantageous to apply the boundary element method to Signorini problems. An indirect boundary element method for solving Signorini problems has been presented in [9].

In this paper, we will present a direct boundary element method for solving the same problem. First, using the Calderón projector for the traces and the normal derivative of the solution of the Signorini problem, this latter is reduced to a variational inequality on the boundary Γ based on a direct boundary element method. The resulting variational inequality on the boundary involves both weakly singular and hypersingular boundary integral operators. The bilinear form arising in this boundary inequality is continuous and coercive on suitable subspaces of the Sobolev space $H^{1/2}(\Gamma) \times H^{-/12}(\Gamma)$. This leads to

existence and uniqueness for the solution of the boundary variational inequality. Furthermore, the Signorini problem can also be formulated as a saddle-point problem involving only boundary integral operators on Γ, which finally we can solve with a uniform boundary element Galerkin method and obtain quasioptimal error estimates for the Galerkin error.

Let Ω be a bounded domain in $\mathbf{R}^2$ with a smooth boundary Γ. Suppose $\Gamma = \Gamma_0 \cup \Gamma_1$ (as shown in Figure 1), with $\Gamma_0 \neq \varnothing$. We define

$$\overset{*}{H}{}^1(\Omega) = \{u \in H^1(\Omega), \, u|_{\Gamma_0} = 0\},$$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

$$L(v) = \int_\Omega fv \, dx + \int_{\Gamma_1} gv \, ds,$$

$$K = \{v \in \overset{*}{H}{}^1(\Omega), \, v \geq 0 \text{ a.e. on } \Gamma_1\},$$

$$\widetilde{K} = \{\text{trace of } v \text{ on } \Gamma, \, v \in K\},$$

$$\overset{*}{H}{}^{1/2}(\Gamma) = \{\text{trace of } v \text{ on } \Gamma, \, v \in \overset{*}{H}{}^1(\Omega)\},$$

$$\overset{*}{H}{}^{-/12}(\Gamma) = \left\{\mu \in H^{-1/2}(\Gamma), \, \int_\Gamma \mu \, ds = 0\right\},$$

$$V = \overset{*}{H}{}^{1/2}(\Gamma) \times \overset{*}{H}{}^{-1/2}(\Gamma), \quad \text{with norm } \|(v, \mu)\|_V^2 = \|v\|_{1/2,\Gamma}^2 + \|\mu\|_{-1/2,\Gamma}^2,$$

where $H^m(\Omega)$ and $H^\alpha(\Gamma)$ denote the usual Sobolev spaces, m, α are two real numbers (see [15]), and $f \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ are given functions. We know that K is a closed convex set in $\overset{*}{H}{}^1(\Omega)$.
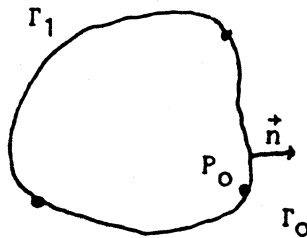

FIGURE 1

We consider the following variational inequality:

(1.1)   find $u \in K$ such that

$$a(u, v - u) \geq L(v - u) \; \forall v \in K.$$

Problem (1.1) is a model for Signorini problems. The following theorems are known [8].

Theorem 1.1. *The variational inequality* (1.1) *has a unique solution.*

Theorem 1.2. *The solution of problem* (1.1) *is characterized by*

$$
\begin{aligned}
-\Delta u &= f && \text{a.e. in } \Omega, \\
u &= 0 && \text{a.e. on } \Gamma_0, \\
u &\geq 0, \ \partial u/\partial n \geq g && \text{a.e. on } \Gamma_1, \\
u(\partial u/\partial n - g) &= 0 && \text{a.e. on } \Gamma_1.
\end{aligned}
$$

(1.2)

Next, let u_0 be the solution of the following boundary value problem:

(1.3)
$$
\begin{aligned}
-\Delta u_0 &= f && \text{in } \Omega, \\
u_0 &= 0 && \text{on } \Gamma.
\end{aligned}
$$

Hence, we have by the first Green's formula with $u \in \overset{*}{H}{}^1(\Omega)$ and $f \in L^2(\Omega)$

(1.4)
$$
\begin{aligned}
-a(u_0, v + u_0 - u) = &- \int_\Omega f(v + u_0 - u)\, dx \\
&- \int_{\Gamma_1} \frac{\partial u_0}{\partial n}(v + u_0 - u)\, ds \quad \forall v \in K.
\end{aligned}
$$

From (1.1) and (1.4) we obtain with $u, u_0 \in K$,

$$
a(u - u_0, v - (u - u_0)) \geq \int_{\Gamma_1} \left(g - \frac{\partial u_0}{\partial n} \right) (v - (u - u_0))\, ds \quad \forall v \in K.
$$

Let $w = u - u_0$ and $g^* = g - \partial u_0/\partial n$; then w solves the following problem:

Find $w \in K$ such that

$$
a(w, v - w) \geq \int_{\Gamma_1} g^*(v - w)\, ds \quad \forall v \in K.
$$

Therefore, without loss of generality, we may assume that $f \equiv 0$. In this case, problem (1.1) is reduced to

Find $u \in K$ such that

(1.1)*
$$
a(u, v - u) \geq \int_{\Gamma_1} g(v - u)\, ds \quad \forall v \in K.
$$

2. An equivalent boundary variational inequality for problem (1.1)*

Suppose that u is the solution of problem (1.1)*; then in the domain Ω, $\Delta u = 0$. Let $\lambda = \partial u/\partial n|_\Gamma \in H^{-1/2}(\Gamma)$. By Green's formula we obtain

(2.1) $\quad u(x) = \int_\Gamma \frac{\partial G(x, y)}{\partial n_y} u(y)\, ds_y - \int_\Gamma G(x, y)\lambda(y)\, ds_y \quad \forall x \in \Omega,$

where $G(x, y) = \frac{1}{2\pi} \log |x - y|$, $x \neq y$; n_y denotes the outward unit normal to $\Gamma = \partial\Omega$ at $y \in \Gamma$. By the properties of the single-layer and double-layer potentials, we obtain the first relationship between λ and $u|_\Gamma$:

(2.2) $\quad \frac{1}{2}u(x) = \int_\Gamma \frac{\partial G(x, y)}{\partial n_y} u(y)\, ds_y - \int_\Gamma G(x, y)\lambda(y)\, ds_y \quad \forall x \in \Gamma.$

Furthermore, using the behavior of the derivative of the single-layer and double-layer potentials [10, 12, 16], we get

$$(2.3) \qquad \frac{1}{2}\lambda(x) = \int_\Gamma \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y)\, ds_y - \int_\Gamma \frac{\partial G(x,y)}{\partial n_x}\lambda(y)\, ds_y \quad \forall x \in \Gamma,$$

where

$$\int_\Gamma \frac{\partial^2 G(x,y)}{\partial n_x \partial n_y} u(y)\, ds_y = \frac{d}{ds_x} \int_\Gamma G(x,y) \frac{du(y)}{ds_y}\, ds_y \quad \forall x \in \Gamma.$$

Multiplying (2.2) by a function $\mu \in \overset{*}{H}{}^{-1/2}(\Gamma)$, we get

$$(2.4) \qquad \begin{aligned} &-\frac{1}{2}\int_\Gamma u(x)\mu(x)\, ds_x + \int_\Gamma \int_\Gamma \frac{\partial G(x,y)}{\partial n_y} u(y)\mu(x)\, ds_y\, ds_x \\ &\quad - \int_\Gamma \int_\Gamma G(x,y)\lambda(y)\mu(x)\, ds_y\, ds_x = 0. \end{aligned}$$

Let

$$a_0(\mu,\lambda) = -\int_\Gamma \int_\Gamma G(x,y)\lambda(y)\mu(x)\, ds_y\, ds_x,$$

$$b(\mu,u) = \frac{1}{2}\int_\Gamma u(x)\mu(x)\, ds_x - \int_\Gamma \int_\Gamma \frac{\partial G(x,y)}{\partial n_y} u(y)\mu(x)\, ds_y\, ds_x;$$

then equation (2.4) can be rewritten as follows:

$$(2.4)^* \qquad\qquad -b(\mu,u) + a_0(\mu,\lambda) = 0 \quad \forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma).$$

For any $v \in \overset{*}{H}{}^1(\Omega)$, we have, for u satisfying $\Delta u = 0$ in Ω,

$$(2.5) \qquad\qquad a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Gamma \lambda v\, ds.$$

Inserting (2.3) into (2.5) and integrating by parts, we obtain

$$\begin{aligned} a(u,v) &= -\int_\Gamma \int_\Gamma G(x,y)\frac{du(y)}{ds_y}\frac{dv(x)}{ds_x}\, ds_y\, ds_x \\ &\quad - \int_\Gamma \int_\Gamma \frac{\partial G(x,y)}{\partial n_x}\lambda(y)v(x)\, ds_y\, ds_x + \frac{1}{2}\int_\Gamma \lambda v\, ds \\ &= a_0\left(\frac{du}{ds},\frac{dv}{ds}\right) + b(\lambda,v). \end{aligned}$$

Hence, the variational inequality $(1.1)^*$ is reduced to the following boundary variational inequality:

Find $(u,\lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma)$ such that

$$(2.6) \qquad a_0\left(\frac{du}{ds},\frac{dv}{ds}-\frac{du}{ds}\right) + b(\lambda,v-u) \geq \int_{\Gamma_1} g(v-u)\, ds \quad \forall v \in \widetilde{K},$$

$$-b(\mu,u) + a_0(\mu,\lambda) = 0 \quad \forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma);$$

or,

(2.7)

find $(u, \lambda) \in \widetilde{K} \times \overset{*}{H}^{-1/2}(\Gamma)$ such that

$$A(u, \lambda; v - u, \mu) \geq \int_{\Gamma_1} g(v - u)\, ds \quad \forall v \in \widetilde{K}, \ \mu \in \overset{*}{H}^{-1/2}(\Gamma),$$

where

$$A(u, \lambda; v, \mu) = a_0\left(\frac{du}{ds}, \frac{dv}{ds}\right) + b(\lambda, v) - b(\mu, u) + a_0(\mu, \lambda)$$

is a bilinear form on $V \times V$.

We now prove the following lemmas:

Lemma 2.1. *There is a constant $\alpha_0 > 0$ such that*

(2.8) $$\|dv/ds\|_{-1/2, \Gamma} \geq \alpha_0 \|v\|_{1/2, \Gamma} \quad \forall v \in \overset{*}{H}^{1/2}(\Gamma).$$

Proof. Let $\theta = 2\pi s / L$, where L denotes the length of the boundary Γ and s denotes the length of the segment from the point $P_0 \in \Gamma$ to a point $p \in \Gamma$ along the boundary Γ. Hence, we only need to prove (2.8) when Γ is the circle with radius 1. On the other hand, we know that the space $\overset{*}{C}^\infty(\Omega) = \{v \in C^\infty(\Omega),$ support of $v \subset \overline{\Omega}\backslash\Gamma_0\}$ is dense in $\overset{*}{H}^1(\Omega)$. If we can prove (2.8) for any $v \in \overset{*}{C}^\infty(\Omega)$, then (2.8) holds for $v \in \overset{*}{H}^{1/2}(\Gamma)$.

Suppose Γ is the circle with radius 1. For any $v \in \overset{*}{C}^\infty(\Omega)$, on Γ we have

$$v = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} v(\theta)\cos n\theta\, d\theta, \qquad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} v(\theta)\sin n\theta\, d\theta, \qquad n = 1, 2, \ldots.$$

We know that [14, p. 21]

(2.9) $$\|v\|_{1/2, \Gamma}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(1 + n^2)^{1/2}(a_n^2 + b_n^2).$$

Furthermore, we have

$$\frac{dv}{d\theta} = \sum_{n=1}^{\infty}(nb_n \cos n\theta - na_n \sin n\theta),$$

(2.10) $$\left\|\frac{dv}{d\theta}\right\|_{-1/2, \Gamma}^2 = \sum_{n=1}^{\infty}(1 + n^2)^{-1/2}(n^2 b_n^2 + n^2 a_n^2)$$

$$\geq \frac{1}{2}\sum_{n=1}^{\infty}(1 + n^2)^{1/2}(a_n^2 + b_n^2).$$

On the other hand, we know that

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} v(\theta) \, d\theta = -\frac{1}{\pi} \int_0^{2\pi} \frac{dv(\theta)}{d\theta} \theta \, d\theta.$$

By $v|_{\Gamma_0} = 0$, that is, $v(\theta) = 0$, $-\delta_1 \le \theta \le \delta_2$ $(\delta_1, \delta_2 > 0)$, there is a function $\phi(\theta) \in C^\infty(\Gamma)$ such that $\phi|_{\Gamma_1} = 1$, $\phi(\theta) = 0$, for $|\theta| \le \frac{1}{2} \min(\delta_1, \delta_2)$. Hence, we get

$$a_0 = -\frac{1}{\pi} \int_0^{2\pi} \frac{dv}{d\theta} \theta \phi(\theta) \, d\theta,$$

where $\theta \phi(\theta) \in C^\infty(\Gamma)$. Furthermore, we obtain

$$|a_0| \le \frac{1}{\pi} \|\theta\phi(\theta)\|_{1/2,\Gamma} \left\| \frac{dv}{d\theta} \right\|_{-1/2,\Gamma}$$

(2.11)
$$\equiv C_0 \left\| \frac{dv}{d\theta} \right\|_{-1/2,\Gamma}, \qquad C_0 = \frac{1}{\pi} \|\theta\phi(\theta)\|_{1/2,\Gamma} > 0,$$

$$\left\| \frac{dv}{d\theta} \right\|_{-1/2,\Gamma}^2 \ge \frac{1}{C_0^2} |a_0|^2.$$

Combining (2.9), (2.10), and (2.11), we obtain the inequality (2.8) with $\alpha_0 = \min(1/\sqrt{2}, 1/\sqrt{2}C_0)$. □

Lemma 2.2. $A(u, \lambda; v, \mu)$ *is a bounded bilinear form on* $V \times V$; *that is, there exists a constant* $M > 0$ *such that*

(2.12) $|A(u, \lambda; v, \mu)| \le M\|(u, \lambda)\|_v \|(v, \mu)\|_v$ $\forall (u, \lambda), (v, \mu) \in V.$

Furthermore, there is a constant $\beta > 0$ *such that*

(2.13) $A(v, \mu; v, \mu) \ge \beta \|(v, \mu)\|_v^2$ $\forall (v, \mu) \in V.$

Proof. It is straightforward to check that $A(u, \lambda; v, \mu)$ is a bounded bilinear form on $V \times V$ (see [11]). We now prove inequality (2.13). We have

(2.14) $A(v, \mu; v, \mu) = a_0 \left(\dfrac{dv}{ds}, \dfrac{dv}{ds} \right) + a_0(\mu, \mu).$

We recall (see [13]) that the bounded bilinear form $a_0(\lambda, \mu)$ is $\overset{*}{H}{}^{-1/2}(\Gamma)$-elliptic, i.e., there exists a positive constant β_0 such that

(2.15) $a_0(\mu, \mu) \ge \beta_0 \|\mu\|_{-1/2,\Gamma}^2$ $\forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma).$

Hence,

(2.16) $a_0 \left(\dfrac{dv}{ds}, \dfrac{dv}{ds} \right) \ge \beta_0 \left\| \dfrac{dv}{ds} \right\|_{-1/2,\Gamma}^2$ $\forall v \in H^{1/2}(\Gamma).$

Combining (2.14), (2.15), (2.16), and (2.8), we get inequality (2.13) with $\beta = \min(\beta_0, \alpha_0\beta_0) > 0$. □

Lemma 2.3. *Suppose* $u \in \overset{*}{H}^{1/2}(\Gamma) \cap H^{1/2+\alpha}(\Gamma)$ *and* $\lambda \in \overset{*}{H}^{-1/2}(\Gamma) \cap H^{-1/2+\alpha}(\Gamma)$
$(0 \leq \alpha \leq 1)$; *then there exists a constant* $M_\alpha > 0$ *such that*

$$(2.17) \qquad |A(u, \lambda; v, \mu)| \leq M_\alpha (\|u\|^2_{1/2+\alpha, \Gamma} + \|\lambda\|^2_{-1/2+\alpha, \Gamma})^{1/2}$$
$$\cdot (\|v\|^2_{1/2-\alpha, \Gamma} + \|\mu\|^2_{1/2-\alpha, \Gamma})^{1/2} \quad \forall (v, \mu) \in V.$$

Proof. We consider the operators

$$K_0 : H^r(\Gamma) \to H^{r+1}(\Gamma), \qquad -\tfrac{3}{2} \leq r,$$

where $K_0 \phi = -\int_\Gamma G(x, y) \phi(y) \, ds_y \;\; \forall \phi(y) \in H^r(\Gamma)$, and

$$K_1 : H^r(\Gamma) \to H^r(\Gamma), \qquad -\tfrac{1}{2} \leq r,$$

where $K_1 \phi = \int_\Gamma (\partial G(x, y)/\partial n_y) \phi(y) \, ds_y \;\; \forall \phi \in H^r(\Gamma)$. We know (see [10]) that
K_0 and K_1 are bounded operators. Let $\langle v, \mu \rangle = \int_\Gamma v \mu \, ds$; then we obtain

$$A(u, \lambda; v, \mu) = \left\langle K_0 \left(\frac{du}{ds} \right), \frac{dv}{ds} \right\rangle + \langle K_0(\lambda), \mu \rangle$$
$$+ \frac{1}{2} \langle v, \lambda \rangle - \frac{1}{2} \langle u, \mu \rangle - \langle K_1 v, \lambda \rangle + \langle K_1 u, \mu \rangle.$$

Furthermore, we have for $0 \leq \alpha \leq 1$

$$\left| \left\langle K_0 \left(\frac{du}{ds} \right), \frac{dv}{ds} \right\rangle \right| \leq \left\| K_0 \left(\frac{du}{ds} \right) \right\|_{1/2+\alpha, \Gamma} \left\| \frac{dv}{ds} \right\|_{-1/2-\alpha, \Gamma}$$
$$\leq \|K_0\| \left\| \frac{du}{ds} \right\|_{-1/2-\alpha, \Gamma} \|v\|_{1/2-\alpha, \Gamma}$$
$$\leq \|K_0\| \|u\|_{1/2+\alpha, \Gamma} \|v\|_{1/2-\alpha, \Gamma},$$
$$|\langle v, \lambda \rangle| \leq \|\lambda\|_{-1/2+\alpha, \Gamma} \|v\|_{1/2-\alpha, \Gamma},$$
$$|\langle K_1 v, \lambda \rangle| \leq \|\lambda\|_{-1/2+\alpha, \Gamma} \|K_1 v\|_{1/2-\alpha, \Gamma}$$
$$\leq C_\alpha \|\lambda\|_{-1/2+\alpha, \Gamma} \|v\|_{1/2-\alpha, \Gamma},$$

where

$$C_\alpha = \sup_{v \in H^{1/2-\alpha}(\Gamma)} \frac{\|K_1 v\|_{1/2-\alpha, \Gamma}}{\|v\|_{1/2-\alpha, \Gamma}}.$$

Similarly, we get

$$|\langle K_0 \lambda, \mu \rangle| \leq \|K_0\| \|\lambda\|_{-1/2+\alpha, \Gamma} \|\mu\|_{-1/2-\alpha, \Gamma},$$
$$|\langle u, \mu \rangle| \leq \|u\|_{1/2+\alpha, \Gamma} \|\mu\|_{-1/2-\alpha, \Gamma},$$
$$|\langle K_1 u, \mu \rangle| \leq \widetilde{C}_\alpha \|u\|_{1/2+\alpha, \Gamma} \|\mu\|_{-1/2-\alpha, \Gamma},$$

where $\widetilde{C}_\alpha = \sup_{u \in H^{1/2+\alpha}(\Gamma)} \|K_1(u)\|_{1/2+\alpha, \Gamma} / \|u\|_{1/2+\alpha, \Gamma}$. Then inequality (2.17)
follows immediately. $\square$

By Lemma 2.2, an application of Theorem 2.1 in [8, Chapter I] yields the
following result.

Theorem 2.1. *Suppose that* $g \in H^{-1/2}(\Gamma_1)$; *then the variational inequality* (2.7) *(or* (2.6)*) has a unique solution* $(u, \lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma)$.

Suppose $(u, \lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma)$ is the solution of (2.7). By formula (2.1), we obtain the function $u(x) \in \overset{*}{H}{}^1(\Omega)$ and $\partial u/\partial n = \lambda$; then, for any $v \in \overset{*}{H}{}^1(\Omega)$ and $\mu \in \overset{*}{H}{}^{-1/2}(\Gamma)$, we have with $\Delta u = 0$ in Ω:

$$a(u, v - u) = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Gamma \lambda(v - u) \, ds = A(u, \lambda; v - u, \mu).$$

Hence, $u(x) \in \overset{*}{H}{}^1(\Omega)$ is the solution of the original problem $(1.1)^*$. This means that the boundary variational inequality (2.7) is equivalent to problem $(1.1)^*$. Furthermore, we have the following

Theorem 2.2. *The variational inequality* (2.6) *is equivalent to the following saddle-point problem*:

(2.18)
$$\text{Find } (u, \lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma) \text{ such that}$$
$$L(u, \mu) \leq L(u, \lambda) \leq L(v, \lambda) \quad \forall v \in \widetilde{K}, \ \mu \in \overset{*}{H}{}^{-1/2}(\Gamma),$$

where

$$L(v, \mu) = \frac{1}{2} a_0 \left(\frac{dv}{ds}, \frac{dv}{ds} \right) + b(\mu, v) - \frac{1}{2} a_0(\mu, \mu) - \int_{\Gamma_1} gv \, ds.$$

Proof. Suppose that $(u, \lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma)$ is the solution of (2.18). Then, for any $\mu \in \overset{*}{H}{}^{-1/2}(\Gamma)$ and any real number ε, $\lambda + \varepsilon\mu \in \overset{*}{H}{}^{-1/2}(\Gamma)$; we have

$$L(u, \lambda + \varepsilon\mu) \leq L(u, \lambda),$$

that is,

$$\varepsilon[b(\mu, u) - a_0(\mu, \lambda)] - \frac{\varepsilon^2}{2} a_0(\mu, \mu) \leq 0.$$

Since ε is an arbitrary constant, we obtain

$$-b(\mu, u) + a_0(\mu, \lambda) = 0 \quad \forall \mu \in \overset{*}{H}{}^{-1/2}(\Gamma).$$

On the other hand, for any $u, v \in \widetilde{K}$, we know that $u + t(v - u) \in \widetilde{K}$ $(0 \leq t \leq 1)$; then $\widetilde{K}$ is convex, and we get

$$L(u, \lambda) \leq L(u + t(v - u), \lambda) \quad \forall v \in \widetilde{K},$$

that is,

$$t \left[a_0 \left(\frac{du}{ds}, \frac{dv}{ds} - \frac{du}{ds} \right) + b(\lambda, v - u) - \int_{\Gamma_1} g(v - u) \, ds \right]$$
$$+ \frac{t^2}{2} a_0 \left(\frac{d(v - u)}{ds}, \frac{d(v - u)}{ds} \right) \geq 0.$$

Since $1 \geq t \geq 0$ is an arbitrary constant, we get

$$a_0 \left(\frac{du}{ds}, \frac{d(v-u)}{ds} \right) + b(\lambda, v-u) \geq \int_{\Gamma_1} g(v-u)\, ds \quad \forall v \in \widetilde{K}.$$

This means that $(u, \lambda) \in \widetilde{K} \times \overset{*}{H}{}^{-1/2}(\Gamma)$ is the solution of (2.6). Each of the above steps is reversible, and we conclude that if (u, λ) is a solution of (2.6), then it is a solution of (2.18). □

3. THE DISCRETE APPROXIMATION OF
THE BOUNDARY VARIATIONAL INEQUALITY (2.7)

Suppose that S_{h_1} and S_{h_2} are two finite-dimensional subspaces of $\overset{*}{H}{}^{1/2}(\Gamma)$ and $\overset{*}{H}{}^{-1/2}(\Gamma)$, respectively. Let $\widetilde{K}_{h_1} = \{v_h, \ v_h \in S_{h_1} \cup \widetilde{K}\}$. Moreover, we assume that $\widetilde{K}_{h_1}$ is a closed convex nonempty subset of S_{h_1}.

Now we consider the discrete problem

(3.1)
Find $(u_h, \lambda_h) \in \widetilde{K}_{h_1} \times S_{h_2}$ such that

$$A(u_h, \lambda_h; v_h - u_h, \mu_h) \geq \int_{\Gamma_1} g(v_h - u_h)\, ds \quad \forall v_h \in \widetilde{K}_{h_1}, \ \mu_h \in S_{h_2}.$$

It is straightforward to prove:

Theorem 3.1. *The problem* (3.1) *has a unique solution* $(u_h, \lambda_h) \in \widetilde{K}_{h_1} \times S_{h_2}$. *Furthermore, we obtain the error estimate stated in the following theorem.*

Theorem 3.2. *Suppose that the solution of* (2.7), (u, λ), *satisfies* $u \in \widetilde{K} \cup H^{1/2+\alpha}(\Gamma)$, $\lambda \in \overset{*}{H}{}^{-1/2}(\Gamma) \cup H^{-1/2+\alpha}(\Gamma)$, *and* $g \in H^{-1/2+\alpha}(\Gamma)$, *where* $0 \leq \alpha \leq 1$; *then the following error estimate holds:*

(3.2)
$$\|(u - u_h, \lambda - \lambda_h)\|_v^2 \leq C_\alpha \left\{ \inf_{v_h \in \widetilde{K}_{h_1}} [\|u - v_h\|_{1/2,\Gamma}^2 + \|u - v_h\|_{1/2-\alpha,\Gamma}] \right.$$
$$\left. + \inf_{\mu_h \in S_{h_2}} [\|\lambda - \mu_h\|_{-1/2,\Gamma}^2 + \|\lambda - \mu_h\|_{-1/2-\alpha,\Gamma}] \right\},$$

where C_α *is a constant independent of* h_1 *and* h_2.
Proof. Taking $v = u_h$ and $\mu = \lambda_h - \lambda$ in (2.7), we have

(3.3)
$$-A(u, \lambda; u_h - u, \lambda_h - \lambda) \leq -\int_{\Gamma_1} g(u_h - u)\, ds.$$

Similarly, we get

(3.4)
$$-A(u_h, \lambda_h; v_h - u_h, \mu_h - \lambda_h) \leq -\int_{\Gamma_1} g(v_h - u_h)\, ds \quad \forall v_h \in \widetilde{K}_{h_1}, \ \mu_h \in S_{h_2}.$$

On the other hand,

$$\|(u - u_h, \lambda - \lambda_h)\|_V^2 \leq \tfrac{1}{\beta} A(u - u_h, \lambda - \lambda_h; u - u_h, \lambda - \lambda_h)$$

$$= \tfrac{1}{\beta}\{A(u - u_h, \lambda - \lambda_h; u - v_h, \lambda - \mu_h)$$

$$+ A(u, \lambda; v_h - u, \mu_h - \lambda)$$

$$- A(u_h, \lambda_h; v_h - u_h, \mu_h - \lambda_h)$$

$$- A(u, \lambda; u_h - u, \lambda_h - \lambda)\}$$

(3.5)

$$\leq \tfrac{1}{\beta}\left\{ A(u - u_h, \lambda - \lambda_h; u - v_h, \lambda - \mu_h) \right.$$

$$+ A(u, \lambda; v_h - u, \mu_h - \lambda)$$

$$\left. - \int_{\Gamma_1} g(v_h - u)\, ds \right\} \quad \forall v_h \in \widetilde{K}_{h_1}, \ \mu_h \in S_{h_2}.$$

Combining (3.5), (2.12), and (2.17), we obtain

$$\|(u - u_h, \lambda - \lambda_h)\|_V^2 \leq \tfrac{1}{\beta}\{M\|(u - u_h, \lambda - \lambda_h)\|_V\|(u - v_h, \lambda - \mu_h\|_V$$

$$+ M_\alpha(\|u\|_{1/2+\alpha,\Gamma}^2 + \|\lambda\|_{-1/2+\alpha,\Gamma}^2)^{1/2}$$

$$\cdot (\|u - v_h\|_{1/2-\alpha,\Gamma}^2 + \|\lambda - \mu_h\|_{1/2-\alpha,\Gamma}^2)^{1/2}$$

$$+ \|g\|_{-1/2+\alpha,\Gamma_1}\|u - v_h\|_{1/2-\alpha,\alpha,\Gamma}\}.$$

Then the error estimate (3.2) follows immediately. □

Assume that the boundary Γ of Ω is represented as

(3.6) $\chi_1 = \chi_1(S), \quad \chi_2 = \chi_2(S), \qquad 0 \leq S \leq L,$

and $\chi_j(0) = \chi_j(L)$, $j = 1, 2$. Furthermore, Γ is divided into segments $\{T\}$ by the points $\chi^i = (\chi_1(S_i), \chi_2(S_i))$, $i = 1, 2, \ldots, N_1$, that include the two endpoints of Γ_0, with $S_1 = 0$, $S_{N_1+1} = L$; we define

$$h_1 = \max_{1 \leq i \leq N_1} |S_{i+1} - S_i|.$$

This partition of Γ is denoted by $\mathscr{T}_{h_1}$. Let

$$S_{h_1} = \{v_h \in C^0(\Gamma), \ v_h|_T \text{ is a linear function}$$

(3.7) $\forall T \in \mathscr{T}_{h_1} \text{ and } v_h|_{\Gamma_0} = 0\},$

$$\widetilde{K}_{h_1} = \{v_h \in S_{h_1}, \ v_h \geq 0 \text{ on } \Gamma_1\}.$$

Similarly, we have another partition $\mathscr{T}_{h_2}$. Let

(3.8) $S_{h_2} = \{\mu_h|_T \text{ is a constant } \forall T \in \mathscr{T}_{h_2} \text{ and } \int_\Gamma \mu_h\, ds = 0\}.$

Obviously, $\widetilde{K}_{h_1}$ is a closed convex subset of S_{h_1}, and it is nonempty. The subspaces S_{h_1} and S_{h_2} are two regular finite element spaces in the sense of

Babuška and Aziz [2] that satisfy the following approximation property:

$$
(3.9) \quad \begin{aligned}
\inf_{v_h \in \widetilde{K}_{h_1}} \{\|u - v_h\|^2_{1/2,\Gamma} + \|u - v_h\|_{1/2-\alpha,\Gamma}\} \\
\leq C^1_\alpha h_1^{2\alpha}[\|u\|^2_{1/2+\alpha,\Gamma} + \|u\|_{1/2+\alpha,\Gamma}],
\end{aligned}
$$

$$
(3.10) \quad \begin{aligned}
\inf_{\mu_h \in S_{h_2}} \{\|\lambda - \mu_h\|^2_{-1/2,\Gamma} + \|\lambda - \mu_h\|_{-1/2-\alpha,\Gamma}\} \\
\leq C^2_\alpha h_2^{2\alpha}[\|\lambda\|^2_{-1/2+\alpha,\Gamma} + \|\lambda\|_{-1/2+\alpha,\Gamma}].
\end{aligned}
$$

Combining (3.9), (3.10), and (3.2), we obtain

Corollary. *Suppose that the subspaces S_{h_1} and S_{h_2} are given by (3.7) and (3.8); moreover, let the solution (u, λ) satisfy the assumptions of Theorem 3.2. Then the following error estimate holds:*

$$
\begin{aligned}
\|(u - u_h, \lambda - \lambda_h)\|^2_v \leq \widetilde{C}_\alpha \{h_1^{2\alpha}[\|u\|^2_{1/2+\alpha,\Gamma} + \|u\|_{1/2+\alpha,\Gamma}] \\
+ h_2^{2\alpha}[\|\lambda\|^2_{-1/2+\alpha,\Gamma} + \|\lambda\|_{-1/2+\alpha}]\},
\end{aligned}
$$

where $\widetilde{C}_\alpha$ is a constant independent of h_1 and h_2.

4. The solution of the boundary variational inequality (3.1)

In this section, we will present an approach for solving problem (3.1). Let $a_1(u, v) = a_0(du/ds, dv/ds)$; then problem (3.1) can be rewritten as follows:

Find $(u_h, \lambda_h) \in \widetilde{K}_{h_1} \times S_{h_2}$ such that

$$
(3.1)^* \quad \begin{aligned}
a_1(u_h, v_h - u_h) + b(\lambda_h, v_h - u_h) \geq \int_{\Gamma_1} g(v_h - u_h)\, ds, \quad v_h \in \widetilde{K}_{h_1}, \\
- b(\mu_h, u_h) + a_0(\mu_h, \lambda_h) = 0, \quad \forall \mu_h \in S_{h_2}.
\end{aligned}
$$

As a first step, for any given $u \in H^{1/2}(\Gamma)$, we solve the following problem:

$$
(4.1) \quad \begin{aligned}
\text{Find } \lambda_h(u) \in S_{h_2} \text{ such that} \\
a_0(\mu_h, \lambda_h) = b(\mu_h, u) \quad \forall \mu_h \in S_{h_2}.
\end{aligned}
$$

Problem (4.1) has a unique solution $\lambda_h(u)$, and

$$
\|\lambda_h(u)\|_{-/12,\Gamma} \leq \|b\|\|u\|_{1/2,\Gamma}/\beta_0.
$$

Let $\mu_i(x)$, $i = 1, 2, \ldots, N_2$, denote a basis for the space S_{h_2}, and

$$
\lambda_h(u) = \sum_{i=1}^{N_2} \lambda_i(u)\mu_i(x) = \boldsymbol{\lambda}^T \cdot \boldsymbol{\mu}(x),
$$

where $\boldsymbol{\lambda}^T = (\lambda_1(u), \lambda_2(u), \ldots, \lambda_{N_2}(u))$ and $\boldsymbol{\mu}(x)^T = (\mu_1(x), \mu_2(x), \ldots, \mu_{N_2}(x))$. From (4.1), we obtain

$$
(4.2) \quad \mathbf{G}\boldsymbol{\lambda} = \mathbf{F}(u),
$$

where $\mathbf{G} = (a_0(\mu_i, \mu_j))_{N_2 \times N_2}$ and

$$\mathbf{F}(u)^T = (b(\mu_1, u), b(\mu_2, u), \ldots, b(\mu_{N_2}, u)).$$

$\mathbf{G}$ is a symmetric and positive definite matrix. Hence we get

$$(4.3) \qquad\qquad\qquad \lambda = \mathbf{G}^{-1}\mathbf{F}(u)$$

and $\lambda_h(u) = \lambda^T \cdot \boldsymbol{\mu}(x)$. Let $A_{h_2}(u, v) = a_1(u, v) + b(\lambda_h(u), v)$; then $A_{h_2}(u, v)$ is a bilinear form on $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Furthermore, we have:

Lemma 4.1. (i) $A_{h_2}(u, v)$ *is a bounded bilinear form on* $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$; *that is, there exists a constant* $M > 0$ *independent of* h_2 *such that*

$$(4.4) \qquad\qquad |A_{h_2}(u, v)| \le M\|u\|_{1/2,\Gamma}\|v\|_{1/2,\Gamma} \quad \forall u, v \in H^{1/2}(\Gamma).$$

(ii) $A_{h_2}(u, v)$ *is symmetric.*

(iii) $A_{h_2}(u, v)$ *is* $\overset{*}{H}{}^{1/2}(\Gamma)$-*elliptic uniformly for* h_2; *that is, there exists a constant* $\gamma > 0$ *such that*

$$(4.5) \qquad\qquad A_{h_2}(v, v) \ge \gamma\|v\|_{1/2,\Gamma}^2 \quad \forall v \in \overset{*}{H}{}^{1/2}(\Gamma).$$

Proof. (i) We have

$$
\begin{aligned}
|A_{h_2}(u, v)| &= |a_1(u, v) + b(\lambda_h(u), v)| \\
&\le \|a_1\|\,\|u\|_{1/2,\Gamma}\|v\|_{1/2,\Gamma} + \|b\|\,\|\lambda_h(u)\|_{-1/2,\Gamma}\|v\|_{1/2,\Gamma} \\
&\le \{\|a_1\| + \|b\|^2/\beta_0\}\|u\|_{1/2,\Gamma}\|v\|_{1/2,\Gamma}.
\end{aligned}
$$

The inequality (4.4) holds with $M = \|a_1\| + \|b\|^2/\beta_0$.

(ii) From (4.1), we get

$$
\begin{aligned}
a_0(\lambda_h(v), \lambda_h(u)) &= b(\lambda_h(v), u), \\
a_0(\lambda_h(u), \lambda_h(v)) &= b(\lambda_h(u), v).
\end{aligned}
$$

By symmetry of $a_0(\lambda, \mu)$, we obtain $b(\lambda_h(v), u) = b(\lambda_h(u), v)$. Hence,

$$A_{h_2}(u, v) = a_1(u, v) + b(\lambda_h(u), v) = a_1(v, u) + b(\lambda_h(v), u) = A_{h_2}(v, u).$$

(iii) We have

$$
\begin{aligned}
A_{h_2}(v, v) &= a_1(v, v) + b(\lambda_h(v), v) \\
&= a_1(v, v) + a_0(\lambda_h(v), \lambda_h(v)) \\
&= A(v, \lambda_h(v); v, \lambda_h(v)) \ge \beta\|v\|_{1/2,\Gamma}^2 \quad \forall v \in \overset{*}{H}{}^{1/2}(\Gamma). \qquad \square
\end{aligned}
$$

In the second step we reduce the boundary variational inequality $(3.1)^*$ to:

Find $u_h \in \widetilde{K}_{h_1}$ such that

$$(4.6) \qquad A_{h_2}(u_h, v_h - u_h) \ge \int_{\Gamma_1} g(v_h - u_h)\,ds \quad \forall v_h \in \widetilde{K}_{h_1}.$$

Let

$$J(u) = \tfrac{1}{2}A_{h_2}(u, u) - \int_{\Gamma_1} gu\,ds.$$

By Lemma 4.1 we obtain [8, p. 10]:

Theorem 4.1. *The boundary variational problem* (4.6) *is equivalent to the following minimization problem on* $\widetilde{K}_{h_1}$:

$$(4.7) \qquad\qquad J(u_h) = \min_{v_h \in \widetilde{K}_{h_1}} J(v_h).$$

A method for solving problem (4.7) can be found in [7]. After solving (4.7), we can get $\lambda(u_h)$ from problem (4.1); then formula (2.1) gives the solution of the original problem.

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