

## OPTIMAL-ORDER NONNESTED MULTIGRID METHODS FOR SOLVING FINITE ELEMENT EQUATIONS II: ON NON-QUASI-UNIFORM MESHES

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**ABSTRACT.** Nonnested multigrid methods are proved to be optimal-order solvers for finite element equations arising from elliptic problems in the presence of singularities caused by re-entrant corners and abrupt changes in the boundary conditions, where the multilevel grids are appropriately refined near singularities and are not necessarily nested. Therefore, optimal and realistic finer grids (compared with nested local refinements) could be used because of the freedom in generating nonnested multilevel grids.

### 1. INTRODUCTION

Multigrid methods provide optimal-order solvers for linear systems of finite element equations arising from elliptic boundary value problems (cf. references in [7] and [5]). The convergence of multigrid methods was proved by many authors (cf., e.g., [8], [2] and [6]) under the assumptions of quasi-uniformity of grids and  $H^{1+\alpha}$ -regularity. However, quasi-uniform multigrid methods are not well suited for problems with singularities because finite elements do not capture singularities well on such meshes, and the convergence rate would deteriorate owing to singularities ( $Cm^{-\alpha/2}$  for  $H^{1+\alpha}$ -regularity, cf. [2]).

Yserentant proved the convergence of multigrid methods on non-quasi-uniform grids (cf. [10]). However, it is not quite clear how to implement the methods proposed by Yserentant in practical situations. It is difficult to construct, say, four or five nested locally-refined grids which reasonably satisfy the mesh refinement condition proposed by Babuška, Kellogg and Pitkäranta in [1] (see (1.3a-b) below). Couplings of triangles, the minimal-angle condition (see (1.2) below) and the control of the growth of total grid points, give rise to many difficulties in local, nested grid refinements. To avoid these difficulties, it is natural to use nonnested multigrid methods (cf. [11] or [12]). The nonnesting of multilevel grids is well illustrated by the example shown in Figure 1, where the triangulations on the triangle  $OAB$  are generated by simply refining each triangle on the coarser level into four subtriangles and then moving all grid points appropriately to the corner  $O$ . This paper is devoted to proving the

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convergence of the nonnested multigrid methods on such non-quasi-uniform meshes (satisfying (1.3a–b)).

Although a mesh could have a quite arbitrary relation with its lower level in nonnested multigrid methods, each mesh must be related to the coarser one in such a way that the intergrid transfer operators ( $I_k$ , depending only on the locations of the fine-grid points on the coarse-level triangles) can be easily generated by computer codes. Also, we should construct meshes such that more information can be passed by intergrid transfer operators. A subdividing-moving technique (described above) was used by Scott and the author for generating nonnested, tetrahedral meshes in their general 3-d nonnested multigrid code NMGTM (cf. [9]). Because of many variations in generating nonnested meshes, we leave discussions on grid generations and numerical experiments to [13]. For example, a mapping technique (maps nested, quasi-uniform meshes to nonnested, non-quasi-uniform ones) is also discussed in [13]. Nonnested multigrid methods on quasi-uniform meshes are also analyzed by Bramble, Pasciak and Xu within their framework in [3].

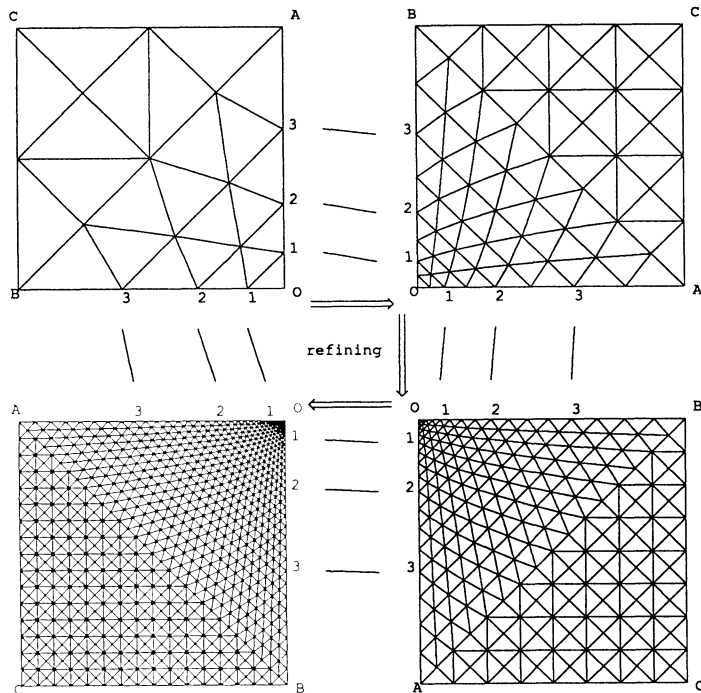


FIGURE 1

*Nonnested refinements on  $\triangle OAB$ . The square  $OACB$  (rotated above) is a part of the domain.  $\triangle ABC$  has standard multi-grid refinements, but a subdividing-moving method is applied in  $\triangle OAB$ . Here, the grid points on  $OA$  satisfy  $|x_i - x_{i-1}| = 2^{-k} |x_i|^{1/2}$  (see (1.3a–3b)).*

For simplicity, we consider the model problem

$$\begin{aligned} -\Delta u + u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_D, \\ \partial u / \partial n &= 0 & \text{on } \Gamma_N, \end{aligned}$$

where  $\Omega$  is a bounded polygonal domain in  $\mathbf{R}^2$  with the boundary subdivided into two parts  $\Gamma_D$  and  $\Gamma_N$ . We assume  $\Gamma_D$  has a positive measure. Finite element approximation problems are defined as follows: Find  $u_k \in V_k$  such that

$$(1.1) \quad a(u_k, v) = F(v) \quad \forall v \in V_k, \quad k = 1, 2, \dots,$$

where  $a(u, v) = \int_{\Omega} (\partial_1 u \partial_1 v + \partial_2 u \partial_2 v + uv) dx$ ,  $F(v) = \int_{\Omega} f v dx$ , and  $V_k$  is the space of continuous, piecewise linear (on a grid  $\mathcal{T}_k$ ) functions which vanish on  $\Gamma_D$ . Here,  $\{\mathcal{T}_k, k = 1, 2, \dots\}$  is a family of nondegenerate triangulations on  $\Omega$ : there exists an  $\alpha_0 > 0$  such that

$$(1.2) \quad \min_{K \in \mathcal{T}_k} \min\{\text{the three angles of } K\} > \alpha_0 \quad \forall k.$$

But we can have  $\mathcal{T}_k \not\subset \mathcal{T}_{k+1}$ . Let  $H_D^1(\Omega)$  be the space of  $H^1$  functions which vanish on  $\Gamma_D$  in the sense of traces; then  $V_k \subset H_D^1(\Omega)$ .

Now we state the essential assumption on the triangulations (cf. [1]): Let the points  $x_1, \dots, x_n$  be the vertices of the domain  $\Omega$ , including the points of abrupt changes from Dirichlet to natural boundary conditions. For each vertex  $x_i$  we define  $\kappa_i = 1$  if the two sides having the endpoint  $x_i$  belong either both to  $\Gamma_D$  or both to  $\Gamma_N$ , and  $\kappa_i = 1/2$  otherwise. Let  $0 < \theta_i \leq 2\pi$  be the interior angle of  $\Omega$  at the point  $x_i$ ; because of possible changes in the boundary conditions, the case  $\theta_i = \pi$  is permitted. Let  $\alpha_i = \min\{1, \kappa_i \pi / \theta_i\}$ . Then  $1/4 \leq \alpha_i \leq 1$  holds. We choose  $\beta_i$  with  $1 - \alpha_i < \beta_i < 1$  and  $\beta_i = 0$  if  $\alpha_i = 1$ . Let  $\phi_{\beta}(x) = \prod_{i=1}^n |x - x_i|^{\beta_i}$  for  $\beta = (\beta_1, \dots, \beta_n)$ , where  $|\cdot|$  denotes the Euclidean distance. We assume that the triangulations  $\{\mathcal{T}_k\}$  satisfy the condition: Let  $K \in \mathcal{T}_k$  be a (closed) triangle; then

$$(1.3a) \quad \underline{c} h_k \max_{x \in K} \phi_{\beta}(x) \leq h_K \leq \bar{c} h_k \min_{x \in K} \phi_{\beta}(x) \quad \text{if } x_i \notin K \text{ for all } i,$$

$$(1.3b) \quad \underline{c} h_k \max_{x \in K} \phi_{\beta}(x) \leq h_K \leq \bar{c} h_k \max_{x \in K} \phi_{\beta}(x) \quad \text{if } x_i \in K \text{ for some } i.$$

Here,  $\underline{c}$  and  $\bar{c}$  are some positive constants independent of  $K$  and  $k$ , and  $h_K$  is the diameter of  $K$ . The  $h_k$  in (1.3a–b) is a global “mesh size” of  $\mathcal{T}_k$ . For convenience, we simply let  $h_k = 2^{-k}$  in this paper. It was shown by Babuška et al. in [1] that  $\dim V_k \leq C h_k^{-2}$ . We assume that  $\dim V_k \simeq C h_k^{-2} \simeq (\dim V_{k+1})/4 \forall k$ . The meshes generated by the subdividing-moving method mentioned above will satisfy this condition naturally. Further, we assume that the meshes are

locally comparable:

$$(1.4) \quad \begin{aligned} & \sup_{K \in \mathcal{T}_k} \{ \text{cardinality}(\{K' \in \mathcal{T}_{k \pm 1} \mid K' \cap K \neq \emptyset\}) \} \leq \beta_0, \\ & \sup_{K \in \mathcal{T}_k} \{ h_K / \min\{h_{K'} \mid K' \in \mathcal{T}_{k \pm 1}, K' \cap K \neq \emptyset\} \} \leq \beta_0, \end{aligned} \quad \forall k$$

for some  $\beta_0 > 0$ . In fact, (1.4) should be consequences of (1.2) and (1.3a–b).

We define some weighted Sobolev norms associated with the function  $\phi_\beta(x)$  defined above. Let

$$(1.5) \quad (v, w)_\beta = \int_\Omega \phi_\beta^2(x) v(x) w(x) dx$$

be a weighted (weight  $\phi_\beta^2$ )  $L_2$ -inner product and  $\|\cdot\|_{0,\beta}$  the norm induced by  $(\cdot, \cdot)_\beta$ . Let  $H^{0,\beta}(\Omega)$  be the space of all Lebesgue measurable functions having finite  $\|\cdot\|_{0,\beta}$ -norm. Replacing the  $\phi_\beta^2$  by  $\phi_\beta^{-2}$  in (1.5), we define  $(v, w)_{-\beta}$  and  $H^{0,-\beta}(\Omega)$  similarly. Let  $\|\cdot\|_{2,\beta}^2 = \|\cdot\|_{H^1(\Omega)}^2 + \sum_{|\alpha|=2} \|D^\alpha \cdot\|_{0,\beta}^2$  and  $H^{2,\beta}(\Omega)$  be the corresponding space. Here,  $H^1(\Omega)$  is the usual Sobolev space. It was shown in [1] that  $H^{2,\beta}(\Omega) \subset C(\Omega)$  and that  $H^1(\Omega) \subset H^{0,\beta}(\Omega)$ . Therefore, nodal-value interpolation operators are well defined for functions in  $H^{2,\beta}(\Omega)$ , and  $V_k \subset H^{0,\beta}(\Omega) \forall k$ .

By the Riesz representation theorem, the bilinear form  $a(\cdot, \cdot)$  defines a linear, symmetric, positive definite operator  $A_k$  in  $(V_k, (\cdot, \cdot)_{-\beta})$ :  $a(v, w) = (A_k v, w)_{-\beta} \forall v, w \in V_k$ . We define a family of discrete norms on  $V_k$ :

$$(1.6) \quad |||v|||_{s,k}^2 = (v, A_k^s v)_{-\beta} \quad \forall v \in V_k, \quad 0 \leq s \leq 2.$$

We note that  $|||\cdot|||_{0,k} = \|\cdot\|_{0,-\beta}$  and that  $|||\cdot|||_{1,k} = \|\cdot\|_{H^1(\Omega)}$ . For simplicity, we let  $|||\cdot||| = |||\cdot|||_{1,k}$ . Let  $I_k: C(\Omega) \cap H_D^1(\Omega) \rightarrow V_k$  be the usual nodal-value interpolation operator associated with the triangulation  $\mathcal{T}_k$ .

We now define a nonsymmetric (cf. Definition 4.1 in §4), nonnested multigrid scheme (cf. [2]). The full multigrid method has two iterative processes. The overall process involves solving problems (1.1) sequentially for  $k = 1, 2, \dots$  to get  $\tilde{u}_k$ . To solve (1.1) on the  $k$ th level, we interpolate the  $\tilde{u}_{k-1}$  as an initial guess and then repeat several times the second, recursive process:

**Definition 1.1.** (The  $k$ th level nonnested multigrid scheme.)

- (1) For  $k = 1$ , (1.1) or the following (1.8) is solved by any method.
- (2) For  $k > 1$ , a final guess  $w_{m+1}$  will be generated from an initial guess  $w_0$  as follows. First,  $m$  presmoothings will be performed:

$$(1.7) \quad \begin{aligned} (w_l - w_{l-1}, v)_{0,-\beta} &= \lambda_k^{-1} (F(v) - a(w_{l-1}, v)) \\ &\quad \forall v \in V_k, \quad l = 1, 2, \dots, m, \end{aligned}$$

where  $\lambda_k$  is the maximal eigenvalue of  $A_k$  and  $F(\cdot)$  is either the  $F(\cdot)$  in (1.1) or the  $\tilde{F}(\cdot)$  in (1.8) below. To define  $w_{m+1}$ , we need to construct a  $(k - 1)$ st

level residual problem: Find  $\bar{q} \in V_{k-1}$  such that

$$(1.8) \quad a(\bar{q}, v) = F(I_k v) - a(w_m, I_k v) \stackrel{\text{def}}{=} \tilde{F}(v) \quad \forall v \in V_{k-1}.$$

Let  $q \in V_{k-1}$  be the approximation of  $\bar{q}$  obtained by applying  $p$  iterations of the  $(k-1)$ st level scheme to (1.8) starting with initial guess zero; then we let

$$(1.9) \quad w_{m+1} = w_m + I_k q.$$

If meshes are nested, the  $I_k$  in (1.8) and (1.9) is an identity. However, an intergrid transfer operator, “ $I_k$ ”, is necessary in computation, no matter whether the meshes are nested or not. If we remove the  $I_k$  in (1.8),  $\bar{q}$  would be the  $a(\cdot, \cdot)$ -projection of an iterative error. Correspondingly, we define two operators  $Q_{k-1}, P_{k-1} : H_D^1(\Omega) \rightarrow V_{k-1}$  by

$$(1.10) \quad \begin{aligned} a(Q_{k-1}v, w) &= a(v, I_k w), \\ a(P_{k-1}v, w) &= a(v, w), \end{aligned} \quad \forall w \in V_{k-1}.$$

## 2. PRELIMINARY RESULTS

In this section, we will prove some lemmas concerning  $I_k$  and introduce some results of Babuška, Kellogg and Pitkäranta in [1]. In the rest of this paper, “ $\lesssim$ ” stands for “ $\leq C$ ” with the constant  $C$  independent of the functions being estimated and the level number  $k$ .

**Lemma 2.1.** *Let (1.2) and (1.4) hold; then*

$$\|w - I_k v\| \lesssim \|w - v\| \quad \forall v \in V_{k-1}, \quad \forall w \in V_k.$$

*Proof.* See Proposition 2.1 of [12]. The equivalence of  $|\cdot|_{H^1(\Omega)}$  and  $\|\cdot\|_{H^1(\Omega)} = \|\cdot\|$  in  $H_D^1(\Omega)$  follows from a general Poincaré inequality in [4, Theorem 1.2.1].  $\square$

**Lemma 2.2.** *Let (1.2)–(1.4) hold; then*

$$\|v - I_k v\|_{0, -\beta} \lesssim h_k \|v\| \quad \forall v \in V_{k-1}.$$

To prove the lemma, we use a lemma in [1].

**Lemma 2.3** (Lemma 4.3 in [1]). *Let  $K$  be a triangle with one vertex at 0,  $\alpha \neq 0$ , and let  $u$  be defined on  $K$  with weak first derivatives which satisfy  $\int_K |x|^\alpha |\nabla u|^2 dx < \infty$ . Then there is a constant  $A$  depending on  $u$  and a constant  $C(\alpha, \alpha_K)$ , independent of  $u$  but depending on  $\alpha$  and the minimal angle  $\alpha_K$  of  $K$ , such that*

$$\int_K |x|^{\alpha-2} |u - A|^2 \leq C \int_K |x|^\alpha |\nabla u|^2 dx,$$

and

$$|A| \leq C \int_T |x|^\alpha |\nabla u| dx. \quad \square$$

*Proof of Lemma 2.2.* For any  $K \in \mathcal{T}_k$ , let  $S_K = \cup\{K' \in \mathcal{T}_{k-1} \mid K' \cap K \neq \emptyset\}$ ; then  $K \subset S_K$ . Let  $v \in V_{k-1}$ ; then  $v - I_k v$  is piecewise linear on  $K$  and vanishes at the three vertices of  $K$ . We consider two cases:  $x_i \notin K \ \forall i$  or some vertex  $x_i$  of  $\Omega$  is a vertex  $K$ . In the first case, we use (1.3a) and (1.4) to derive

$$(2.1) \quad \begin{aligned} \int_K \phi_\beta^{-2} (v - I_k v)^2 dx &\leq \bar{c}^2 h_k^2 h_K^{-2} \int_K \left( h_K \max_{x \in K} \{2|\nabla v|\} \right)^2 dx \\ &= 4\bar{c}^2 h_k^2 \int_K \max_{x \in S_K} |\nabla v|^2 dx \lesssim h_k^2 \int_{S_K} |\nabla v|^2 dx. \end{aligned}$$

If  $x_i \in K$ , without loss of generality, we may assume  $x_i = 0$ . Applying Lemma 2.3 and the Schwarz inequality, we obtain

$$\begin{aligned} \int_K \phi_\beta^{-2} |v - I_k v|^2 dx &\lesssim \int_K |x|^{-2\beta_i} |v - I_k v|^2 dx \\ &\lesssim 2 \int_K |x|^{-2\beta_i} |(v - I_k v) - A|^2 dx + 2A^2 \int_K |x|^{-2\beta_i} dx \\ &\lesssim \int_K |x|^{2-2\beta_i} |\nabla (v - I_k v)|^2 dx \\ &\quad + \left( \int_K |x|^{2-2\beta_i} |\nabla (v - I_k v)| dx \right)^2 \int_K \phi_\beta^{-2} dx \\ &\lesssim \int_K |x|^{2-2\beta_i} |\nabla (v - I_k v)|^2 dx \left( 1 + \int_K |x|^{2-2\beta_i} dx \int_\Omega \phi_\beta^{-2} dx \right). \end{aligned}$$

Here, we used the fact that  $|x|^{-2\beta} \sim \phi_{\beta_i}^{-2}$  on  $K$ . By (1.3b), it follows that  $h_K^2 \leq \bar{c}^2 h_k^2 \max\{\phi_\beta^2\} \lesssim h_k^2 |x|^{2\beta_i} \lesssim h_k^2 h_K^{2\beta_i}$ . Therefore,

$$(2.2) \quad \begin{aligned} \int_K \phi_\beta^{-2} |v - I_k v|^2 dx &\lesssim h_K^{2-2\beta_i} \int_K |\nabla (v - I_k v)|^2 dx \\ &\lesssim h_k^2 \int_K \max_{x \in K} \{4|\nabla v|^2\} dx \lesssim h_k^2 \int_{S_K} |\nabla v|^2 dx. \end{aligned}$$

Combining (2.1) and (2.2), we conclude, by (1.4), that

$$\begin{aligned} \|v - I_k v\|_{0,-\beta}^2 &= \sum_{K \in \mathcal{T}_k} \int_K \phi_\beta^{-2} |v - I_k v|^2 dx \lesssim h_k^2 \sum_{K \in \mathcal{T}_k} \int_{S_K} |\nabla v|^2 dx \\ &\lesssim h_k^2 \sum_{K' \in \mathcal{T}_{k-1}} \int_{K'} |\nabla v|^2 dx = Ch_k^2 |v|_{H^1(\Omega)}^2 \lesssim h_k^2 \|v\|^2. \quad \square \end{aligned}$$

**Theorem 2.4** (Theorem 3.2 in [1]). *Let  $f \in H^{0,\beta}(\Omega)$  and  $u \in H_D^1(\Omega)$  satisfy  $a(u, v) = F(v) \ \forall v \in H_D^1(\Omega)$ . Then  $u \in H^{2,\beta}(\Omega)$  and*

$$\|u\|_{2,\beta} \leq C \|f\|_{0,\beta}. \quad \square$$

**Lemma 2.5** (Lemma 4.5 in [1]). *Let (1.2) and (1.3a–b) hold; then*

$$\|u - I_k u\|_{H^1(\Omega)} \leq Ch_k \|u\|_{2,\beta} \quad \forall u \in H^{2,\beta}(\Omega) \cap H_D^1(\Omega). \quad \square$$

**Theorem 2.6** (Theorem 5.2 in [1]). *Let (1.2) and (1.3a–b) hold; then*

$$\|u - P_k u\|_{0,-\beta} \leq Ch_k \|u - P_k u\|_{H^1(\Omega)} \quad \forall u \in H_D^1(\Omega). \quad \square$$

### 3. OPTIMAL COMPUTATIONAL ORDER OF $W$ -CYCLE METHODS

In this section, we will show some lemmas concerning coarse-level corrections and coarse-level projections. We then prove an inverse inequality. Finally, the two-level and multilevel  $W$ -cycle nonnested multigrid methods will be shown to have a constant rate of convergence. The proof given here follows a route of Bank and Dupont in [2] and is almost identical to a proof given by the author in [12] for nonnested multigrid methods on quasi-uniform meshes.

**Lemma 3.1.** *Let (1.2)–(1.4) hold; then*

$$|||v - P_{k-1} v||| \lesssim h_k |||v|||_{2,k} \quad \forall v \in V_k.$$

*Proof.* For any  $v \in V_k$ , let  $\tilde{v} = P_{k-1} v$ . By the Schwarz inequality, we have

$$(3.1) \quad |||v - \tilde{v}|||^2 = a(v, v - \tilde{v}) = a(v, v - P_k \tilde{v}) \leq |||v|||_{2,k} |||v - P_k \tilde{v}|||_{0,k}.$$

We then use a duality argument to estimate the second term: Let  $u \in H_D^1(\Omega)$  solve

$$a(u, w) = (\phi_\beta^{-2}(v - P_k \tilde{v}), w) \quad \forall w \in H_D^1(\Omega).$$

By Theorem 2.4,  $u \in H^{2,\beta}(\Omega)$  and  $\|u\|_{2,\beta} \lesssim \|\phi_\beta^{-2}(v - P_k \tilde{v})\|_{0,\beta} = \|v - P_k \tilde{v}\|_{0,-\beta}$ . Therefore, by Lemma 2.5, we can derive

$$(3.2) \quad \begin{aligned} |||v - P_k \tilde{v}|||_{0,k}^2 &= (v - P_k \tilde{v}, v - P_k \tilde{v})_{-\beta} = a(u, v - P_k \tilde{v}) \\ &= a(u, v - \tilde{v}) + a(u, \tilde{v} - P_k \tilde{v}) \\ &\leq |||u - I_{k-1} u||| |||v - \tilde{v}||| + |||u - I_k u||| |||\tilde{v} - P_k \tilde{v}||| \\ &\lesssim \|u\|_{2,\beta} (h_{k-1} |||v - \tilde{v}||| + h_k |||\tilde{v} - P_k \tilde{v}|||) \\ &\lesssim h_k \|u\|_{2,\beta} (3 |||v - \tilde{v}||| + |||P_k(v - \tilde{v})|||) \\ &\lesssim h_k |||v - P_k \tilde{v}|||_{0,k} |||v - \tilde{v}|||. \end{aligned}$$

Combining (3.1) and (3.2), the lemma is proved.  $\square$

**Lemma 3.2.** *Let (1.2)–(1.4) hold; then*

$$|||P_{k-1} v - Q_{k-1} v||| \lesssim h_k |||v|||_{2,k} \quad \forall v \in V_k.$$

*Proof.* For any  $v \in V_k$ , by (1.10), it follows that

$$a(P_{k-1}v - Q_{k-1}v, w) = a(v, w) - a(v, I_k w) \quad \forall w \in V_{k-1}.$$

By Lemma 2.2 and Theorem 2.6, we can complete the proof as follows:

$$\begin{aligned} \|\|P_{k-1}v - Q_{k-1}v\|\| &= \sup_{w \in V_{k-1}, \|\|w\|\|=1} a(v, w - I_k w) \\ (3.3) \quad &= \sup_w a(v, P_k w - I_k w) \leq \sup_w \|\|v\|\|_{2,k} \|\|P_k w - I_k w\|\|_{0,k} \\ &\leq \|\|v\|\|_{2,k} \sup_w (\|\|w - I_k w\|\|_{0,\beta} + \|\|w - P_k w\|\|_{0,\beta}) \lesssim h_k \|\|v\|\|_{2,k}. \quad \square \end{aligned}$$

We remark that the  $P_k$  in (3.1) and (3.3) are introduced only for the purpose of applying the Schwarz inequality.

**Lemma 3.3.** *Let (1.2)–(1.3b) hold; then*

$$(3.4) \quad \|\|v\|\| \lesssim h_k^{-1} \|\|v\|\|_{0,k} \quad \forall v \in V_k.$$

*Proof.* Let  $v \in V_k$  and  $K \in \mathcal{T}_k$ ; then  $v$  is linear on  $K$ . By (1.2)–(1.3b), we get

$$\int_K |\nabla v|^2 dx \lesssim \max_{x \in K} v^2(x) = C \int_K h_K^{-2} v^2 dx \lesssim h_k^{-2} \int_K \phi_\beta^{-2}(x) v^2 dx.$$

Summing over all triangles in  $\mathcal{T}_k$ , (3.4) is proved.  $\square$

**Theorem 3.4** (Two-level methods). *Let (1.2)–(1.4) hold and  $q = \bar{q}$  in (1.9). For any  $0 < \gamma < 1$ , there exists an integer  $m$  independent of the level number  $k$  such that*

$$\|\|u_k - w_{m+1}\|\| \leq \gamma \|\|u_k - w_0\|\|,$$

where  $u_k$ ,  $w_i$ ,  $q$  and  $\bar{q}$  are defined in (1.1) and (1.7)–(1.9).

*Proof.* Let the iterative errors be denoted by  $e_l = u_k - w_l$  for  $0 \leq l \leq m+1$ . Our goal is to estimate  $e_{m+1} = e_m - I_k \bar{q}$ . Noting that  $\bar{q} = Q_{k-1} e_m$ , we can apply Lemmas 2.1, 3.1 and 3.2 sequentially to get

$$(3.5) \quad \|\|e_{m+1}\|\| \lesssim \|\|e_m - \bar{q}\|\| \lesssim \|\|e_m - P_{k-1} e_m\|\| + \|\|P_{k-1} e_m - \bar{q}\|\| \lesssim h^k \|\|e_m\|\|_{2,k}.$$

By (3.4), it follows that the maximal eigenvalue of  $A_k$ ,  $\lambda_k$ , is bounded by  $Ch_k^{-2}$ . Therefore, the following estimates for the smoothing (1.7) hold (see, e.g., [2]),

$$(3.6) \quad \|\|e_m\|\| \leq \|\|e_0\|\| \quad \text{and} \quad \|\|e_m\|\|_{2,k} \lesssim h_k^{-1} m^{-1/2} \|\|e_0\|\|.$$

We conclude that  $\|\|e_{m+1}\|\| \leq Cm^{-1/2} \|\|e_0\|\|$ . Let  $m \geq C^2/\gamma^2$ ; then the theorem is proved.  $\square$

It is standard to derive the constant rate of convergence of  $W$ -cycle methods from the constant rate of convergence of two-level methods.



**Theorem 3.5** (*W-cycle methods*). *Let (1.2)–(1.4) hold. For any  $0 < \gamma < 1$ , there exists a  $\hat{\gamma} \leq \gamma$  and an integer  $m$ , all independent of the level number  $k$ , such that, if*

$$(3.7) \quad |||q - \bar{q}||| \leq \hat{\gamma}^p |||\bar{q}|||$$

for some  $p > 1$ , then  $|||u_k - w_{m+1}||| \leq \hat{\gamma} |||u_k - w_0|||$ .

*Proof.* By the induction hypothesis (3.7) and by (3.5), it follows that

$$\begin{aligned} |||e_{m+1}||| &\lesssim |||e_m - \bar{q}||| + |||\bar{q} - q||| \lesssim (1 + \hat{\gamma}^p) |||e_m - \bar{q}||| + \hat{\gamma}^p |||e_m||| \\ &\lesssim h^k |||e_m|||_{2,k} + \hat{\gamma}^p |||e_m|||. \end{aligned}$$

By (3.6), we get  $|||e_{m+1}||| \leq (Cm^{-1/2} + C\hat{\gamma}^p) |||e_0|||$ . To complete the proof, we first choose  $\hat{\gamma}$  small enough such that  $C\hat{\gamma}^p \leq \hat{\gamma}/2$ , and then choose  $m$  large enough such that  $Cm^{-1/2} \leq \hat{\gamma}/2$ .  $\square$

In practice, we replace the  $(\cdot, \cdot)_{-\beta}$  in the presmoothing (1.7) by some simpler inner product  $b_k(\cdot, \cdot)$ , which induces a norm equivalent to the  $H^{0, -\beta}$ -norm (cf. [2]). In [10], Yserentant gave a good  $b_k(\cdot, \cdot)$ :

$$b_k(u, v) = h_k^2 \sum_{K \in \mathcal{T}_k} \sum_{n_i \in \{\text{three vertices of } K\}} u(n_i)v(n_i) \quad \forall u, v \in V_k.$$

After such a modification of (1.7), Theorem 3.4 and 3.5 remain valid. It is then standard to show the linear order of computation of the *W-cycle* methods (cf., e.g., [2]).

#### 4. CONVERGENCE OF *V-CYCLE* METHODS

In this section, we will define symmetric nonnested multigrid methods first and then study the rate of convergence of the (variable) *V-cycle*, symmetric, nonnested multigrid methods.

**Definition 4.1** (The  $k$ th level symmetric nonnested multigrid scheme).

(1) For  $k = 1$ , (1.1) or (1.8) is solved exactly or by doing  $m_1$  smoothings of (1.7).

(2) For  $k > 1$ ,  $w_{2m_k+1}$  will be generated from  $w_0$  as follows:

(2-a)  $m_k$  presmoothings of (1.7) are performed to generate  $w_{m_k}$ ;

(2-b)  $w_{m_k}$  is corrected by  $I_k q$  (see (1.8)–(1.9)) to generate  $w_{m_k+1}$ ;

(2-c)  $m_k$  postsmoothings of (1.7) are performed to generate  $w_{2m_k+1}$ .

We note that the number of presmoothings and postsmoothings,  $m_k$ , may depend on the level number  $k$ . By (1.10),  $I_k Q_{k-1}$  is a selfadjoint operator in  $(V_k, a(\cdot, \cdot))$ :

$$(4.1) \quad a(I_k Q_{k-1} v, v) = a(Q_{k-1} v, Q_{k-1} v) = a(v, I_k Q_{k-1} v) \quad \forall v \in V_k.$$

We can combine Lemmas 2.1 and 3.1–3.3 to get

$$\begin{aligned}
 (4.2) \quad & |||v - I_k Q_{k-1} v||| \lesssim |||v - Q_{k-1} v||| \\
 & \lesssim |||v - P_{k-1} v||| + |||P_{k-1} v - Q_{k-1} v||| \\
 & \lesssim h_k |||v|||_{2,k} = c_0 \lambda_k^{-1/2} |||v|||_{2,k} \quad \forall v \in V_k.
 \end{aligned}$$

Let  $\rho(B)$  be the spectral radius of an operator  $B$ . In contrast to nested multigrid methods, (i)  $I_k Q_{k-1}$  may not be an  $a(\cdot, \cdot)$ -projection operator; (ii)  $\rho(I - I_k Q_{k-1}) \leq 1$  may not hold; and (iii)

$$(4.3) \quad \rho(I_k Q_{k-1}) \leq 1$$

may fail. In particular, violation of (4.3) means  $(I - I_k Q_{k-1})$  is not a non-negative operator. To study such operators, we need the following lemma. By expanding functions in  $V_k$  as Fourier series of the eigenfunctions of  $A_k$ , it is straightforward to verify the next lemma.

**Lemma 4.2.** *Let  $B: V_k \rightarrow V_k$  be a selfadjoint operator in  $(V_k, a(\cdot, \cdot))$ ; then the following are equivalent:*

- (1)  $|a(Bv, v)| \leq \gamma a(v, v) \quad \forall v \in V_k$ ;
- (2)  $|||Bv||| \leq \gamma |||v||| \quad \forall v \in V_k$ ;
- (3)  $\rho(B) \leq \gamma$ .  $\square$

Let  $B_k$  be the  $V$ -cycle ( $p = 1$  in Definition 1.1) multigrid reducing operator associated with the iteration defined in Definition 4.1; i.e., after one cycle of the iteration, the iterative error changes from  $e$  to  $B_k e$ . Then

$$(4.4) \quad B_k = (I - \lambda_k^{-1} A_k)^{m_k} (I - (I_k - I_k B_{k-1}) Q_{k-1}) (I - \lambda_k^{-1} A_k)^{m_k}, \quad k = 2, 3, \dots$$

By (4.1), it follows that all  $B_k$  are, inductively, selfadjoint with respect to  $a(\cdot, \cdot)$ .

**Lemma 4.3.** *If  $\rho(B_{k-1}) \leq \gamma \leq 1$ , then  $\rho(B_k) \leq \gamma + c_0/\sqrt{2m_k}$ .*

*Proof.* Let  $v = \sum c_i \phi_{k,i} \in V_k$ , where  $\phi_{k,i}$  is an eigenfunction of  $A_k$  associated with  $\lambda_{k,i}$ . Let  $\tilde{v} = (I - A_k/\lambda_k)^{m_k} v$ . By Lemma 4.2, (4.1) and (4.4), we have

$$\begin{aligned}
 |a(B_k v, v)| & \leq |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + |a(B_{k-1} Q_{k-1} \tilde{v}, Q_{k-1} \tilde{v})| \\
 & \leq |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + \gamma a(I_k Q_{k-1} \tilde{v}, \tilde{v}) \\
 & \leq (1 + \gamma) |a(\tilde{v} - I_k Q_{k-1} \tilde{v}, \tilde{v})| + \gamma a(\tilde{v}, \tilde{v}) \\
 & \leq (1 + \gamma) |||\tilde{v} - I_k Q_{k-1} \tilde{v}||| |||\tilde{v}||| + \gamma |||\tilde{v}|||^2.
 \end{aligned}$$

By (4.2), the Schwarz inequality and  $\gamma \leq 1$ , it follows that

$$\begin{aligned} |a(B_k v, v)| &\leq (1 + \gamma)c_0 \lambda_k^{-1/2} \|\tilde{v}\|_{2,k} \|\tilde{v}\| + \gamma \|\tilde{v}\|^2 \\ &\leq (1 + \gamma)c_0 \left( \frac{\sqrt{2m_k} \|\tilde{v}\|_{2,k}^2}{2\lambda_k} + \frac{\|\tilde{v}\|^2}{2\sqrt{2m_k}} \right) + \gamma \|\tilde{v}\|^2 \\ &= \sum \left( \frac{(1 + \gamma)c_0}{2\sqrt{2m_k}} \left( 2m_k \frac{\lambda_{k,i}}{\lambda_k} + 1 \right) + \gamma \right) \left( 1 - \frac{\lambda_{k,i}}{\lambda_k} \right)^{2m_k} c_i^2 \lambda_{k,i} \\ &\leq \left( \frac{(1 + \gamma)c_0}{2\sqrt{2m_k}} + \gamma \right) \sup_{0 \leq x \leq 1} (2m_k x + 1)(1 - x)^{2m_k} \sum c_i^2 \lambda_{k,i} \\ &\leq (c_0 / \sqrt{2m_k} + \gamma) \|v\|^2. \quad \square \end{aligned}$$

By Lemma 4.3, we can derive trivially the constant rate of convergence of symmetric  $W$ -cycle methods (see Theorem 3.5). The following two theorems, too, are direct corollaries of Lemma 4.3. However, we cannot show the optimal order of computations of  $V$ -cycle methods from either one. With additional assumptions on the multiple meshes (e.g., assuming (4.3)), it is possible to obtain somewhat better results (cf. [3]). But in view of a numerical example in [11], the number of smoothings,  $m_k$ , has to be sufficiently large, depending on couplings of meshes.

**Theorem 4.4** ( $V$ -cycle methods). *Let  $m_j = m \geq c_0^2/2$  for  $j = 2, 3, \dots$ ; then*

$$\rho(B_k) \leq kc_0 / \sqrt{2m}, \quad 2 \leq k \leq (\sqrt{2m}/c_0) + 1,$$

*provided that  $\rho(B_1) \leq c_0 / \sqrt{2m}$ .  $\square$*

**Theorem 4.5** (Variable  $V$ -cycle methods). *For any  $0 < \gamma < 1$ , let  $m_j = 2^{2j-1}c_0^2/\gamma^2$  for  $j = 2, 3, \dots$ ; then*

$$\rho(B_k) \leq \gamma, \quad k = 2, 3, \dots,$$

*provided  $\rho(B_1) \leq \gamma/2$ .  $\square$*

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