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THE LOCAL HURWITZ CONSTANT AND DIOPHANTINE APPROXIMATION ON HECKE GROUPS

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ABSTRACT. Define the Hecke group by

$$G_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

$\lambda_q = 2\cos\pi/q$, $q = 3, 4, \ldots$. We call $G_q(\infty)$ the G_q-rationals, and $\mathbb{R} - G_q(\infty)$ the G_q-irrationals. The problem we treat here is the approximation of G_q-irrationals by G_q-rationals. Let $M(\alpha)$ be the upper bound of numbers c for which $|\alpha - k/m| < 1/cm^2$ for all G_q-irrationals and infinitely many $k/m \in G_q(\infty)$. Set $h'_q = \inf_\alpha M(\alpha)$. We call h'_q the Hurwitz constant for G_q. It is known that $h'_q = 2$, q even; $h'_q = 2(1 + (1 - \lambda_q/2)^2)^{1/2}$, q odd. In this paper we prove this result by using λ_q-continued fractions, as developed previously by D. Rosen. Write

$$\alpha - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n}{m_{n-1}(\alpha)Q_{n-1}^2},$$

where $\varepsilon_i = \pm 1$ and P_i/Q_i are the convergents of the λ_q-continued fraction for α. Then $M(\alpha) = \overline{\lim}_n\, m_n(\alpha)$. We call $m_n(\alpha)$ the local Hurwitz constant. In the final section we prove some results on the local Hurwitz constant. For example (Theorem 4), it is shown that if q is odd and $\varepsilon_{n+1} = \varepsilon_{n+2} = +1$, then $m_i \geq (\lambda_q^2 + 4)^{1/2} > h'_q$ for at least one of $i = n-1, n, n+1$.

1. INTRODUCTION

Let the Hecke group

$$G_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \qquad \lambda_q = 2\cos\frac{\pi}{q},\ q \geq 3,$$

act on the upper half-plane $\operatorname{Im} z > 0$ by Möbius transformations $z \to (kz + l)/(mz + n)$, $\left(\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right) \in G_q$. G_q is a horocyclic group with cusp set $G_q(\infty)$, which are called G_q-rationals. The points of $\mathbb{R} - G_q(\infty)$ are the G_q-irrationals. In [4] we considered the problem of approximating a G_q-irrational by G_q-rationals.

When $q = 3$, G_q becomes the classical modular group $\mathrm{PSL}(2, \mathbb{Z})$ and we are considering classical Diophantine approximation of rationals by irrationals.

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A. Hurwitz showed that when α is irrational, there exist infinitely many reduced fractions k/m for which

$$\left|\alpha - \frac{k}{m}\right| < \frac{1}{\sqrt{5}m^2},$$

where $\sqrt{5}$ is the best constant possible. From now on we consider only $q \geq 4$.

Let α be G_q-irrational and suppose

$$(1.1)\qquad \left|\alpha - \frac{k}{m}\right| < \frac{1}{cm^2}, \qquad \frac{k}{m} \in G_q(\infty),\ m > 0.$$

We denote by $M(\alpha)$ the upper bound of numbers c for which (1.1) holds for infinitely many k/m and put

$$(1.2)\qquad h'_q = \inf_{\alpha} M(\alpha), \qquad \alpha\ G_q\text{-irrational.}$$

We call h'_q the Hurwitz constant for G_q. In [4] we proved that $h'_q = 2$ when q is even and gave bounds for h'_q when q is odd. In [3] A. Haas and C. Series found the exact value of h'_q. So we now know that $h'_q = h_q$, where h_q is defined by

$$(1.3)\qquad h_q = \begin{cases} 2, & q \text{ even},\ \geq 4, \\ 2(1 + (1 - \lambda_q/2)^2)^{1/2}, & q \text{ odd.} \end{cases}$$

(Note that the notation of [3] differs from ours—their h_q is the reciprocal of ours—and the methods of the two papers are quite different.)

From now on we write G for G_q, and λ for λ_q. In [4] we made use of a type of continued fraction expansion of the limit set of G_q, i.e., of $\mathbb{R}$, developed by D. Rosen [5]. (This limit set was also studied by Thea Pignataro in her Princeton thesis (1984, unpublished).) This expansion is called a (reduced) λ-fraction and represents every real number α uniquely:

$$(1.4)\qquad \alpha \equiv \alpha_0 = r_0\lambda + \frac{\varepsilon_1}{r_1\lambda + \cdots} = \left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \ldots\right].$$

Here $\varepsilon_i = \pm 1$, $r_0 = r_0(\alpha_0)$ is an integer, $r_i = r_i(\alpha_0)$, $i \geq 1$, are positive integers, and certain conditions are placed on the ε_i and r_i. The above expansion, referred to as λCF α_0, is finite if and only if α_0 is G-rational. Denote the convergents of (1.4) by

$$(1.5)\qquad \frac{P_n}{Q_n} = \left[r_0\lambda, \ldots, \frac{\varepsilon_n}{r_n\lambda}\right], \qquad Q_0 = 1.$$

Our general plan of attack follows Hurwitz and was described in [4] at the beginning of §3. Hurwitz first shows that if (1.1) is satisfied by any rational number P/Q in lowest terms, then P/Q must be a convergent in the expansion of α as a regular continued fraction. The problem is thus reduced to studying the approximation of α by its convergents.

Here we follow a similar plan. By a preliminary theorem [4, Theorem 3] the approximation of a G-irrational α_0 by G-rationals was reduced to the approximation of α_0 by the convergents P_n/Q_n of λCF α_0. Thus the inequality (1.1) was replaced by an inequality derived from

$$(1.6) \qquad \alpha_0 - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n}{m_{n-1}Q_{n-1}^2}, \qquad m_{n-1} = m_{n-1}(\alpha_0),$$

and the object of study was $m_{n-1}(\alpha_0)$. Clearly,

$$(1.7) \qquad M(\alpha_0) = \varlimsup_{n\to\infty} m_{n-1}(\alpha_0), \qquad h_q = \inf_{\alpha_0} M(\alpha_0).$$

We call $m_n(\alpha_0)$ a *local Hurwitz constant.*

Two λCF α and β are said to be equivalent, and we write $\alpha \sim \beta$, if their expansions agree from a certain point on. It is easy to check that $\alpha \sim \beta$ if and only if $\alpha = \pm V\beta$ for a $V \in G$. It is clear that

$$(1.8) \qquad \alpha \sim \beta \Rightarrow M(\alpha) = M(\beta).$$

The object of the present paper is to provide inequalities for the local Hurwitz constants. First, however, we shall prove that the Hurwitz constant h_q' has the value h_q in (1.3), using the method of λ-fractions. The result follows from

Theorem 1. *Let α_0 be a G-irrational given by (1.4). When q is odd,*

$$M(\alpha_0) \geq 2(1 + (1 - \lambda/2)^2)^{1/2},$$

with equality if and only if

$$\alpha_0 \sim 1 - \lambda/2 + (1 + (1 - \lambda/2)^2)^{1/2}.$$

When q is even, $M(\alpha_0) \geq 2$, with equality if and only if $\alpha_0 \sim 1$.

Of course, knowledge of the value of h_q, q odd, given in [3], was of the greatest value in constructing the proof.

The local Hurwitz constants are also discussed. Let $m_{n-1} \equiv m_{n-1}(\alpha)$ be defined by (1.6).

Theorem 2. *If $\varepsilon_{n+1} = 1$, then $m_{n-1} > 2$, $m_n < 2$, or $m_{n-1} < 2$, $m_n > 2$.*

Theorem 3. *Let q be odd. If $r_n \geq 2$ and $\varepsilon_{n-1} = 1$, then $m_{n-1} \geq h_q$.*

Theorem 4. *Let q be odd. If $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$, then $m_i \geq (\lambda^2 + 4)^{1/2} > h_q$ for at least one of $i = n-1, n, n+1$.*

2. Definitions and basic lemmas

In this section we gather together definitions and theorems needed in the sequel; most of these can be found in [5 and 4]. Let $q \geq 4$. With the notations of (1.4), (1.5) we have

$$(2.1) \qquad \begin{aligned} P_n &= r_n\lambda P_{n-1} + \varepsilon_n P_{n-2}, \\ Q_n &= r_n\lambda Q_{n-1} + \varepsilon_n Q_{n-2}, \end{aligned} \qquad n \geq 1,$$

where

$$P_{-1} = 1, \quad P_0 = r_0\lambda, \quad Q_{-1} = 0, \quad Q_0 = 1,$$
$$P_nQ_{n-1} - P_{n-1}Q_n = (-1)^{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n, \qquad n \geq 1, \tag{2.2}$$

$$\alpha - \frac{P_{n-1}}{Q_{n-1}} = (-1)^{n-1}\frac{\varepsilon_1\cdots\varepsilon_n}{m_{n-1}(\alpha)Q_{n-1}^2}. \tag{2.3}$$

Here,

$$m_{n-1}(\alpha) \equiv m_{n-1} = \alpha_n + \varepsilon_n/\alpha'_{n-1}, \qquad n \geq 3, \tag{2.4}$$

$$\alpha_n = \left[r_n\lambda, \frac{\varepsilon_{n+1}}{r_{n+1}\lambda}, \dots\right], \qquad n \geq 0;$$
$$\alpha'_{n-1} = \left[r_{n-1}\lambda, \frac{\varepsilon_{n-1}}{r_{n-2}\lambda}, \dots, \frac{\varepsilon_2}{r_1\lambda}\right]. \tag{2.5}$$

As we shall see later, $Q_n \geq 1$ and $m_{n-1}(\alpha) > 0$. Note that P_n/Q_n is a strictly decreasing sequence when all $\varepsilon_i = -1$. The periodic λCF of period p,

$$\alpha = \left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \dots, \frac{\varepsilon_{p-1}}{r_{p-1}\lambda}, \frac{\varepsilon_p}{r_0\lambda}, \frac{\varepsilon_1}{r_1\lambda}, \dots\right]$$

can be written as

$$\alpha = \left[\overline{r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \dots, \frac{\varepsilon_{p-1}}{r_{p-1}\lambda}, \frac{\varepsilon_p}{r_0\lambda}}\right],$$

or as

$$\alpha = \left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \dots, \frac{\varepsilon_{p-1}}{r_{p-1}\lambda}, \frac{\varepsilon_p}{\alpha}\right].$$

The following lemma is slightly more general than [5, p. 556].

Lemma 1. *Let*

$$\alpha_{n\nu} = [b_n, \varepsilon_{n+1}/b_{n+1}, \dots, \varepsilon_\nu/b_\nu]$$

and

$$\alpha'_{n\nu} = [b'_n, -1/b'_{n+1}, \dots, -1/b'_\nu]$$

have $b_\mu, b'_\mu > 0$, $0 \leq n \leq \mu < \nu$. *If* $b_\mu \geq b'_\mu$, *then* $\alpha_{n\nu} \geq \alpha'_{n\nu}$, *and* $\alpha_{n\nu} > \alpha'_{n\nu}$ *if some* $b_\mu > b'_\mu$. *If*

$$\alpha_n = [b_n, \varepsilon_{n+1}/b_{n+1}, \dots]$$

and

$$\alpha'_n = [b'_n, -1/b'_{n+1}, \dots]$$

are convergent fractions, and $b_\mu \geq b'_\mu$, $\mu \geq n$, *then* $\alpha_n \geq \alpha'_n$.

For q odd, write $q = 2l - 1$, $l \geq 3$; for q even, write $q = 2l$, $l \geq 2$. Let

$$s = [(q-3)/2] = l - 2, \qquad l \geq 2. \tag{2.5a}$$

The notation $(-1/r\lambda)^n$ means a block of n consecutive terms $-1/r\lambda$. We shall frequently need the λCF

$$B(n) = [\lambda, (-1/\lambda)^{n-1}], \quad n \geq 2, \qquad B(1) = \lambda,$$

with n partial quotients. Thus [5, p. 556],

$$(2.6)\qquad \begin{aligned} &B(n+1) = \lambda - 1/B(n), \qquad 1 \leq n \leq s+1, \\ &B(n) \text{ is strictly decreasing}, \\ &B(s) = 1/(\lambda - 1), \quad B(s+1) = 1, \quad q \text{ odd}, \\ &B(s) = \lambda/(\lambda^2 - 2), \quad B(s+1) = 2/\lambda, \quad q \text{ even}. \end{aligned}$$

Also let

$$C(n) = [2\lambda, (-1/2\lambda)^{n-1}], \quad n \geq 2, \qquad C(1) = 2\lambda.$$

Then

$$(2.7)\qquad \begin{aligned} &C(n+1) = 2\lambda - 1/C(n), \qquad n \geq 1, \\ &C(n) \text{ is strictly decreasing}, \\ &\lim_{n\to\infty} C(n) = \lambda + (\lambda^2 - 1)^{1/2}. \end{aligned}$$

We have

$$(2.8)\qquad \left[C(n+1), -\frac{1}{T}\right] > \left[C(n), -\frac{1}{T}\right], \qquad n \geq 1,\ 0 < T < \lambda + (\lambda^2 - 1)^{1/2}.$$

Indeed, by Lemma 1,

$$\left[C(n+1), -\frac{1}{T}\right] = \left[C(n), -\frac{1}{2\lambda - 1/T}\right] > \left[C(n), -\frac{1}{T}\right],$$

since $T + 1/T < \lambda + (\lambda^2 - 1)^{1/2} + \lambda - (\lambda^2 - 1)^{1/2} = 2\lambda$. Similarly,

$$(2.9)\qquad [B(k), -1/T] > [B(k+1), -1/T], \qquad k \leq s,\ T > 0.$$

In fact,

$$\begin{aligned} \left[B(k), -\frac{1}{T}\right] &> \left[B(k), -\frac{1}{\lambda - 1/T}\right] \\ &= \left[B(k) - \frac{1}{\lambda}, -\frac{1}{T}\right] = \left[B(k+1), -\frac{1}{T}\right], \end{aligned}$$

since $T + 1/T \geq 2 > \lambda$.

When λCF α is reduced (see §§3 and 5 for the definition), we have

$$(2.10)\qquad \alpha_{n\nu} \geq 2/\lambda, \quad \nu \geq n; \qquad \alpha_n \geq 2/\lambda \quad \text{if } r_0 \geq 1 \text{ [5, Lemma 2]},$$

where

$$(2.11)\qquad \begin{aligned} &\alpha_{n\nu} = \left[r_n\lambda, \frac{\varepsilon_{n+1}}{r_{n+1}\lambda}, \dots, \frac{\varepsilon_\nu}{r_\nu\lambda}\right], \quad \nu > n; \qquad \alpha_{nn} = r_n\lambda, \\ &\alpha_n = \left[r_n\lambda, \frac{\varepsilon_{n+1}}{r_{n+1}\lambda}, \dots\right], \\ &Q_n \geq Q_{n-1}, \qquad n \geq 1 \text{ [5, Theorem 3]}. \end{aligned}$$

Using these inequalities in (2.4) and (2.1), we get

$$m_{n-1}(\alpha) \geq \frac{2}{\lambda} - 1 > 0, \quad n \geq 3; \qquad Q_n \geq 1, \quad n \geq 0,$$

as stated earlier.

3. Evaluation of the Hurwitz constant

In this section our object is to prove Theorem 1. The result for even q having been established in [4, Theorem 1], *we now assume q odd.*

A λCF $\alpha_0 = [r_0\lambda, \varepsilon_1/r_1\lambda, \dots]$ is said to be *reduced* [5, p. 555] if

(3.1) The inequality $r_i\lambda + \varepsilon_{i+1} < 1$ (i.e., $r_i = 1$, $\varepsilon_{i+1} = -1$) is satisfied for no more than s consecutive values $i = j, j+1, \dots, j+s-1$, $j \geq 1$. Here s is defined in (2.5a).

(3.2) If $r_i\lambda + \varepsilon_{i+1} < 1$ is satisfied for s consecutive values $i = j, \dots, j+s-1$, then $r_{j+s} \geq 2$.

(3.3) If $[B(s), -1/2\lambda, -1/B(s)]$ occurs, the succeeding ε is $+1$.

(3.4) If λCF terminates with $\varepsilon/B(s+1)$, then $\varepsilon = +1$.

A reduced λCF has the following properties, in addition to (2.9) and (2.10):

(3.5) An infinite reduced λCF converges.

(3.6) Every real number α can be expanded uniquely by the "nearest integer algorithm" in a reduced λCF. If the fraction is infinite, it converges to α.

(3.7) $$Q_n \to \infty, \qquad n \to \infty.$$

From now on, λCF shall mean reduced λCF. Bear in mind that at this point we are interested in $\overline{\lim}\, m_{n-1}(\alpha_0)$ rather than $m_{n-1}(\alpha_0)$ itself, because of (1.7).

We first consider the λCF α_0 with all $\varepsilon_\nu = -1$. In α_0, some terms $-1/r\lambda$, $r \geq 2$, must occur by (3.1); in fact, there is at least one such term in every block of length $s+1$. We shall make a series of transformations in λCF α_0, each having the effect of decreasing α_0 while leaving it reduced. The first transformation is to replace each $r_\nu > 2$ by $r_\nu = 2$, which by Lemma 1 decreases α_0. For convenience let $r_0 = 2$, so that now

(3.8) $$\alpha_0 = [C(t_1), -1/B(u_1), -1/C(t_2), \dots], \qquad t_i \geq 1, \ 1 \leq u_i \leq s,$$

by (3.1). By (2.8) we can assume further that $t = 1$ or 2.

The case $q = 5$ is simpler to treat than the higher values of q. Let $\lambda = \lambda_5$; then $s = 1$, so $u_i = 1$. Moreover, $t_i \geq 2$ for all $i \geq 2$, otherwise (3.3) is violated. Thus, we decrease α_0 by assuming $t_i = 2$, and we shall temporarily assume $t_1 = 1$. Hence,

$$\alpha_0 \geq \left[2\lambda, \overline{-\frac{1}{\lambda}, -\frac{1}{2\lambda}, -\frac{1}{2\lambda}}; -\frac{1}{\lambda}\right] =: \tau_0 = \tau_{3n}, \tag{3.9}$$

a periodic λCF of period 3. If any t_j is greater than 2, we have strict inequality.

The reverse α'_{3n-1} can be extended to a periodic fraction with a decrease in value. This fraction, still denoted by α'_{3n-1}, obviously satisfies

$$\alpha'_{3n-1} \geq \left[2\lambda, -\frac{1}{\lambda}, \left(-\frac{1}{2\lambda}\right)^2, \dots\right] = \tau_0.$$

Therefore,

$$m_{3n-1} \geq \tau_0 - 1/\tau_0.$$

By similar calculations we can show that

$$m_{3n-2} \geq \tau_2 - \frac{1}{\tau'_{3n+1}} = 2\lambda - \frac{1}{\tau_0} + \tau_0 - 2\lambda = \tau_0 - \frac{1}{\tau_0},$$

where we used $\tau'_{3n+1} = \tau_{3n+1} = \tau_1$, $-1/\tau_1 = \tau_0 - 2\lambda$. Thus, m_{3n-1} and m_{3n-2} are both bounded below by $\tau_0 - 1/\tau_0$. On the other hand,

$$m_{3n} = \tau_{3n+1} - 1/\tau'_{3n} < \tau_{3n+1} - \tau_1 = \lambda - \cdots < \lambda. \tag{3.10}$$

It remains to evaluate $\tau_0 - 1/\tau_0$. Now, $\tau_0 = 2\lambda - 1/\tau_1$, and it was shown in [4, p. 126] that τ_1 satisfies

$$\tau_1^2 - \lambda\tau_1 + \frac{2\lambda - 1}{5} = 0,$$

where we used $\lambda^2 - \lambda - 1 = 0$. From this we calculate that

$$\tau_0^2 + (2 - 3\lambda)\tau_0 + 1 = 0, \tag{3.11}$$

or

$$\tau_0 \sim 1 - \lambda/2 + (1 + (1 - \lambda/2)^2)^{1/2}. \tag{3.12}$$

Let τ_0^* be the other root, $\tau_0\tau_0^* = 1$. Then,

$$\begin{aligned} m_{3n-1}, m_{3n-2} &\geq \tau_0 - \frac{1}{\tau_0} = \tau_0 - \tau_0^* = (9\lambda^2 - 12\lambda)^{1/2} \\ &= (9 - 3\lambda)^{1/2} = 2\left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2} = h_5. \end{aligned} \tag{3.13}$$

From (3.10), (3.12), and (3.13) it follows that $M(\tau_0) = h_5$ when τ_0 satisfies (3.12), and this is the only case of equality. Theorem 1 is now proved for $q = 5$.

We next assume $q \geq 7$. The case $t_i = 2$ for some i in (3.8) is not difficult. Suppose $B(u_1)$, $-1/2\lambda$, $-1/2\lambda$, $-1/B(u_2)$ occurs. Setting $[2\lambda, -1/B(u_2), \ldots] = [r_n\lambda, \ldots]$, we have $\alpha_n \geq 2\lambda - \lambda/2 = 3\lambda/2$ by (2.10). Also, $\alpha'_{n-1} = [2\lambda, -1/B(u_1), \ldots] \geq 3\lambda/2$, since α'_{n-1} is reduced. Hence,

$$m_{n-1} \geq 3\lambda/2 - 2/3\lambda > h_q + 0.3, \qquad q \geq 7, \tag{3.14}$$

as a calculation shows. It follows that

$$M(\alpha_0) \geq h_q + 0.3, \qquad q \geq 7, \tag{3.15}$$

for α_0 in this class.

We may now assume all $t_i = 1$. Define two periodic λCF of period $p = 2s + 1$:

$$\beta_0 = [2\lambda, -1/B(s), -1/2\lambda, -1/B(s-1), \beta_0] = \beta_p, \tag{3.16}$$

$$\gamma_0 = [2\lambda, -1/B(s-1), -1/2\lambda, -1/B(s), -1/\gamma_0] = \gamma_p. \tag{3.17}$$

Note that $\gamma_0 = \beta_{s+1}$, so that $\beta_0 \sim \gamma_0$.

Let

$$\beta_0 = \lambda + \delta_0, \qquad \delta_0 = \left[\lambda, -\frac{1}{B(s)}, -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{\beta_0}\right],$$

and let P_i/Q_i, $i \geq 0$, be the convergents of δ_0. P_i and Q_i satisfy the recurrence (2.1), and we calculate certain convergents explicitly. Recall $q = 2l - 1$. When $2 \leq j \leq s$, the recurrence (2.1) has constant coefficients and we solve for

$$Q_j = A\zeta^j + B\zeta^{-j}, \quad \text{where } \zeta = 2^{-1}(\lambda + (\lambda^2 - 4)^{1/2}) = e^{\pi i/q}.$$

Hence,

$$Q_0 = A + B = 1, \qquad Q_1 = A\zeta + B\zeta^{-1} = \lambda,$$

yielding

$$\begin{gathered} A = -\zeta/(\zeta^{-1} - \zeta), \qquad B = \zeta^{-1}(\zeta^{-1} - \zeta), \\ (\zeta^{-1} - \zeta)Q_j = -\zeta^{j+1} + \zeta^{-j-1} = -2i \sin \pi(j+1)/q. \end{gathered} \tag{3.18}$$

In particular, put $j = s - 2 = l - 4$:

$$(\zeta^{-1} - \zeta)Q_{s-2} = -2i \sin \pi \frac{l-3}{2l-1} = -2i \cos \frac{5\pi}{2q}.$$

Let $\omega = e^{\pi i/2q}$; note $\zeta = \omega^2$, $\omega + \omega^{-1} = 2\cos \pi/2q$, $\omega^2 + \omega^{-2} = \lambda$, $\omega^4 + \omega^{-4} = \lambda^2 - 2$. Hence,

$$\begin{aligned} 2\cos 5\pi/q = \omega^5 + \omega^{-5} &= (\omega + \omega^{-1})(\omega^4 - \omega^2 + 1 - \omega^{-2} + \omega^{-4}) \\ &= (\omega + \omega^{-1})(\lambda^2 - \lambda - 1). \end{aligned} \tag{3.19}$$

Also, $\zeta^{-1} - \zeta = -2i \sin \pi/q$. Therefore,

$$Q_{s-2} = (\lambda^2 - \lambda - 1)\Omega, \tag{3.20}$$

with the abbreviation

$$\Omega = \frac{\omega + \omega^{-1}}{2\sin \pi/q} = \frac{1}{2\sin \pi/2q}.$$

This illustrates the calculation. Similarly we find

$$Q_{s-1} = (\lambda - 1)\Omega, \tag{3.21}$$

and by applying the recurrence (2.1) we derive further

$$\begin{gathered} Q_s = \Omega, \quad Q_{s+1} = (\lambda + 1)\Omega, \quad Q_{s+2} = (\lambda^2 + \lambda - 1)\Omega, \\ Q_{s+3} = (\lambda^3 + \lambda^2 - 2\lambda - 1)\Omega. \end{gathered} \tag{3.22}$$

Next we consider Q_j for $s + 2 \le j \le 2s$. Write $Q'_j = Q_{s+j+2}$, so now $0 \le j \le s - 2$. Q'_j satisfies the same recurrence as Q_j with initial values $Q'_0 = Q_{s+2}$, $Q'_1 = Q_{s+3}$. Solving, we find

$$(\zeta^{-1} - \zeta)Q'_j = -Q_{s+3}(\zeta^j - \zeta^{-j}) + Q_{s+2}(\zeta^{j-1} - \zeta^{-j+1}), \tag{3.23}$$

$0 \le j \le s - 2$. For example, set $j = s - 2 = l - 4$. Then, $\zeta^{s-2} - \zeta^{-s+2} = 2i\cos 7\pi/2q$ and $\zeta^{s-3} - \zeta^{-s+3} = 2i\cos 9\pi/2q$. The values of the cosines are calculated as in (3.19). Using $\zeta^{-1} - \zeta = -2i\sin \pi/q$ and the values (3.22), we get

$$\begin{aligned} Q_{2s} &= Q'_{s-1} \\ &= \frac{\Omega(\omega + \omega^{-1})}{2\sin \pi/q}\{(\lambda^3 + \lambda^2 - 2\lambda - 1)(\lambda^3 - \lambda^2 - 2\lambda + 1) \\ &\qquad - (\lambda^2 + \lambda - 1)(\lambda^4 - \lambda^3 - 3\lambda^2 + 2\lambda + 1)\} \\ &= \frac{(\omega + \omega^{-1})^2}{4\sin^2 \pi/q}\lambda = \frac{(\lambda + 2)\lambda}{4 - \lambda^2} = \frac{\lambda}{2 - \lambda}. \end{aligned} \tag{3.24}$$

In this same way one can derive $Q_{2s-1} = (\lambda^2 - 2)/(2 - \lambda)$.

To calculate P_j, we note that

$$P_i = Q_{i+1}, \qquad 0 \le i \le s - 1.$$

$P_s, \dots, P_{s+3}$ are calculated by the recurrence (2.1). We now use the analogues of (3.23), (3.24) to get P_{2s}, P_{2s-1}.

In summary, we now have

$$\begin{aligned} &Q_{s-2} = (\lambda^2 - \lambda - 1)\Omega, \quad Q_{s-1} = (\lambda - 1)\Omega, \quad Q_s = \Omega, \\ &P_{s-2} = Q_{s-1}, \quad P_{s-1} = \Omega, \quad P_s = \Omega, \quad \text{where } \Omega = 1/(2\sin(\pi/2q)); \\ &P_{2s-1} = (\lambda^3 - 2\lambda^2 + \lambda)\omega, \qquad P_{2s} = (\lambda^2 - 2\lambda + 2)\omega, \\ &Q_{2s-1} = (\lambda^2 - 2)\omega, \qquad Q_{2s} = \lambda\omega, \quad \text{where } \omega = 1/(2 - \lambda). \end{aligned} \tag{3.25}$$

This gives

$$\beta_0 - \lambda = \delta_0 = \frac{P_{2s+1}}{Q_{2s+1}} = \frac{\beta_0 P_{2s} - P_{2s-1}}{\beta_0 Q_{2s} - Q_{2s-1}},$$

$$\beta_0^2 - (3\lambda - 2)\beta_0 + 2\lambda^2 - 2\lambda - 1 = 0, \tag{3.26}$$

$$\beta_0 = \frac{3\lambda}{2} - 1 + \left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2};$$

we take the plus sign for the square root, since $\beta_0 \geq 2/\lambda > 1$ from (3.9).

The evaluation of γ_0 is similar:

$$\gamma_0 - \lambda = \left[B(s), -\frac{1}{\beta_0}\right] = \frac{\beta_0 - (\lambda - 1)}{\beta_0(\lambda - 1) - (\lambda^2 - \lambda - 1)}. \tag{3.27}$$

At this point, it is convenient to introduce

$$\rho, \rho^* = 1 - \frac{\lambda}{2} \pm \left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2}, \tag{3.28}$$

so that $\rho\rho^* = -1$. Then,

$$\beta_0 = \lambda - \rho^*.$$

Substituting in (3.27),

$$\gamma_0 = \lambda + \frac{\rho^* - 1}{\rho^*(\lambda - 1) - 1} = \lambda + \rho.$$

The reverse β'_{p-1} can be extended to a periodic λCF of period p with a decrease in value. We denote this fraction by β'_{p-1} also. Hence

$$\beta'_{tp-1} = \left[\overline{B(s-1) - \frac{1}{2\lambda}, -\frac{1}{B(s)}, -\frac{1}{2\lambda}}; -\frac{1}{B(s-1)}\right] = \frac{1}{2\lambda - \gamma_0}, \tag{3.29}$$

$$\gamma'_{tp-1} = \frac{1}{2\lambda - \beta_0}.$$

These values enable us to calculate (see (2.4))

$$\begin{aligned} m_{tp-1}(\beta_0) &= \beta_{tp} - \frac{1}{\beta'_{tp-1}} = \beta_0 - (2\lambda - \gamma_0) \\ &= \lambda - \rho^* + \lambda + \rho - 2\lambda \\ &= 2\left(1 + \left(1 - \frac{\lambda}{2}\right)^2\right)^{1/2} = h_q, \end{aligned} \tag{3.30}$$

$$m_{tp-1}(\gamma_0) = \gamma_0 - (2\lambda - \beta_0) = h_q. \tag{3.31}$$

On the other hand, if $v \not\equiv 0 \pmod{p}$, $m_{v-1}(\beta_0) < \beta_v = \lambda - \cdots < \lambda < h_q$. So,

$$\overline{\lim_{n\to\infty}}\, m_{n-1}(\beta_0) = h_q = \overline{\lim_{n\to\infty}}\, m_{n-1}(\gamma_0),$$

that is,

$$M(\beta_0) = M(\gamma_0) = h. \tag{3.32}$$

Next, we wish to show that β_0 and γ_0 are unique up to G-equivalence. Recall that $q \geq 7$, so $s \geq 2$. Define

$$\gamma_0^* = \left[2\lambda, -\frac{1}{B(s-1)}, -\frac{1}{T}\right], \qquad T = \left[2\lambda, -\frac{1}{B(l_1)}, \dots\right], \tag{3.33}$$

$$\beta_0^* = \left[2\lambda, -\frac{1}{B(s)}, -\frac{1}{T}\right], \qquad T = \left[2\lambda, -\frac{1}{B(k_1)}, \dots\right]. \tag{3.34}$$

We shall show that every $\alpha_0 \not\sim \beta_0$ can be replaced by β_0^* or γ_0^* with a decrease in $M(\alpha_0)$.

Consider γ_0^*. Since it is reduced, we have $l_j \leq s$, $l_j + l_{j+1} \leq 2s - 1$, $j \geq 1$, by conditions (3.1) and (3.5). Replace $l_j \leq s-1$ by $l_j = s-1$; this decreases γ_0^*. We say the sequence $\{l_j\}$ is alternating if the entries $s-1$ and s occur in succession. Clearly, if λCF γ_0^* ends in an infinite alternating sequence, then $\gamma_0^* \sim \gamma_0$.

Suppose, on the contrary, that for some odd t the sequence $l_1 = s$, $l_2, \dots,$ $l_{t+1} = s-1$ is alternating, but $(l_{t+2}, l_{t+3}, \dots, l_{t+k+3}) = (s, s-1, \dots, s-1, s)$. There are k entries $s-1$. If k is odd, we can replace every other $s-1$ by s to obtain an alternating sequence. Suppose k is even, $k = 4$, say. Then $(l_{t+3}, \dots, l_{t+6})$ can be replaced by $(s-1, s, s-1, s-1)$. Thus, the sequence we must treat is $(s, s-1, s-1, s)$, and we wish to replace it by $(s, s-1, s, s-1)$. This applies to any even k.

What we must prove is that

$$\left[B(s-1), -\frac{1}{2\lambda}, -\frac{1}{B(s)}, -\frac{1}{U}\right] > \left[B(s), -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{U}\right], \tag{$*$}$$

where $U = [2\lambda, -1/B(l_1), -1/V] = \lambda + [\lambda, \dots] \geq \lambda + 2/\lambda > \lambda + 1$, $U < 2\lambda$. By writing the left member as $[\lambda, -1/B(s-2), \dots]$, and similarly for the right member, and repeating the process, we eventually bring $(*)$ to the form

$$\left[\lambda, -\frac{1}{B(s)}, -\frac{1}{U}\right] > \left[0, -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{U}\right]. \tag{$**$}$$

Since $U > \lambda + 1$,

$$\left[B(s), -\frac{1}{U}\right] > \left[B(s), -\frac{1}{\lambda+1}\right] > \frac{2}{\lambda},$$

from which it follows that the left member of $(**)$ is positive. But

$$[2\lambda, -1/B(s-1), -1/U] > \lambda + [B(s), -1/U] > 0,$$

so the right member of $(**)$ is negative. This establishes $(*)$. We have shown that $\gamma_0^* \geq \gamma_0$. Similarly, $\beta_0^* \geq \beta_0$.

If n is an index for which $\gamma_n^* = [2\lambda, -1/B(s-1), \dots]$, then by the previous reasoning

$$\gamma_n^* \geq \gamma_0 = \lambda + \rho, \qquad \beta_n^* \geq \beta_0 = \lambda - \rho^*.$$

Now,

$$\begin{aligned}(\gamma_{n-1}^*)' &= \left[B(l_k), -\frac{1}{2\lambda}, -\frac{1}{B(l_{k-1})}, -\frac{1}{W}\right] \\ &\geq \left[B(s), -\frac{1}{2\lambda}, -\frac{1}{B(s-1)}, -\frac{1}{W}\right],\end{aligned}$$

where $W = [2\lambda, -1/B(l_{k-2}), \dots]$, with a finite alternating sequence l_{k-2}, $l_{k-3}, \dots, l_1$. This can be extended to an infinite alternating sequence with a decrease in the value of W. Hence,

$$(\gamma_{\lambda-1}^*)' \geq \beta_1^* = \frac{1}{2\lambda - \beta_0^*}.$$

It follows that

$$\begin{aligned}m_{n-1}(\gamma_n^*) &= \gamma_n^* - \frac{1}{(\gamma_{n-1}^*)'} \geq \gamma_0 - (2\lambda - \beta_0^*) \\ &\geq \gamma_0 + \beta_0 - 2\lambda = h_q;\end{aligned} \tag{3.35}$$

see (3.30). Similarly,

$$m_{n-1}(\beta_n^*) \geq h_q. \tag{3.36}$$

On the other hand, if $r_n = 1$, we have $m_{n-1}(\beta_n) < \lambda - \cdots < h_q$. We have proved

$$M(\beta_0^*) = \varlimsup_{\nu\to\infty} m_\nu(\beta_0^*) \geq M(\beta_0) = h_q, \qquad M(\gamma_0^*) \geq h_q, \tag{3.37}$$

equality occurring if and only if $\beta_0^*, \gamma_0^* \sim \beta_0$.

Putting (3.13), (3.15), (3.35), and (3.36) together, we get

Lemma 2. *If α_0 has all $\varepsilon_\nu = -1$ and $r_n = 2$, then*

$$m_{n-1}(\alpha_0) \geq h_q. \tag{3.38}$$

Hence,

$$M(\alpha_0) \geq h_q, \qquad q \geq 5,$$

with equality if and only if $\alpha_0 \sim \rho$.

The last statement follows since $r_n = 2$ must occur infinitely often.

To complete the proof of Theorem 1, we proceed as follows. If $\varepsilon_\mu = 1$ occurs in α_0 only a finite number of times, we may assume it never occurs; then by (3.2), $r_\mu \geq 2$ infinitely often. Hence $M(\alpha_0) \geq h_q$ by Lemma 2, with the cases of equality mentioned there. So we now assume $\varepsilon_\mu = 1$ occurs infinitely often but not always. We look for the largest block of terms with $\varepsilon = -1$, i.e., bounded by $\varepsilon = +1$ at both ends. Denote this block by

$$\alpha_{\mu\nu} = \left[r_\mu\lambda, -\frac{1}{r_{\mu+1}\lambda}, \dots, \frac{1}{r_\nu\lambda}\right], \qquad \varepsilon_\mu = \varepsilon_{\nu+1} = 1.$$

The terms with $r_t = 1$ yield only $m_{t-1} < \lambda < h_q$, $\mu + 1 \le t \le \nu$. So let $r_n = 2$ for an n with $\mu + 1 \le n \le \nu$. If $\alpha_{\mu\nu}$ does not end in $B(s)$, $-1/2\lambda$, $-1/B(s)$, we can adjoin U with all $\varepsilon = -1$ so that $[\alpha_{\mu\nu}, -1/U]$ is reduced: for example, we could take a periodic $U = [2\lambda, -1/2\lambda, \dots]$. Then by (2.2), $\alpha_n > \alpha_{n\nu} > [\alpha_{n\nu}, -1/U]$. Similarly, $\alpha'_{n-1} > \alpha'_{n-1,\mu} > [\alpha'_{n-1,\mu}, -1/V]$, where $V = [r'_{\mu-1}\lambda, -1/r'_{\mu-2}\lambda, \dots, -1/r'_1\lambda]$ is chosen so that $[\alpha'_{n-1,\mu}, -1/V]$ is reduced. Then, $\delta_0 := [V', -1/\alpha_{\mu\nu}, -1/U]$ has all $\varepsilon = -1$ and is reduced. By Lemma 2,

$$m_{n-1}(\alpha_0) \ge m_{n-1}(\delta_0) \ge h_q, \qquad r_n \ge 2. \tag{3.39}$$

It follows that

$$M(\alpha_0) \ge M(\delta_0) \ge h_q. \tag{3.40}$$

When $\alpha_{\mu\nu}$ ends with $B(s)$, $-1/2\lambda$, $-1/B(s)$, there is no U satisfying the required conditions because of (3.3). We derive successively, using the values (3.25):

$$\alpha_0 = [\dots, -1/B(s), 1/T],$$

$$\left[B(s), \frac{1}{T}\right] = \frac{TP_{s-1} + P_{s-2}}{TQ_{s-1} + Q_{s-2}} > \frac{1}{\lambda - 1},$$

$$\left[2\lambda, -\frac{1}{B(s)}, \frac{1}{T}\right] > \lambda + 1,$$

$$\left[B(s), -\frac{1}{2\lambda}, -\frac{1}{B(s)}, \frac{1}{T}\right] > \left[B(s), -\frac{1}{\lambda+1}\right] = \frac{2}{\lambda},$$

$$\left[2\lambda, -\frac{1}{B(s)}, \dots, \frac{1}{T}\right] > \left[2\lambda, -\frac{\lambda}{2}\right] = \frac{3\lambda}{2},$$

and finally

$$\alpha_n \ge \eta_n := \left[2\lambda, -\frac{1}{B(l_1)}, \dots, -\frac{1}{B(l_k)}, -\frac{1}{3\lambda/2}\right].$$

We assign $l_j = s$ or $s-1$ in alternation, so that η_n is of the form β_0^* in (3.34) or γ_0^* in (3.33); then from (3.37), (3.36) we again get (3.39), (3.40).

The final case is: all $\varepsilon = +1$. If $1/r_n\lambda$ occurs with $r_n \ge 2$, then $m_{n-1} \ge \alpha_n > 2\lambda > h_q$. When $1/r_n\lambda$ occurs infinitely often, we get

$$M(\alpha) = \varlimsup_{n\to\infty} m_{n-1}(\alpha) \ge 2\lambda > h_q.$$

Otherwise, we may assume $1/r\lambda$, $r \ge 2$, never occurs and

$$\alpha_n = \left[\lambda, \frac{1}{\lambda}, \dots\right] = \left[\lambda, \frac{1}{\alpha_n}\right] = \frac{1}{2}(\lambda + (\lambda^2+4)^{1/2}) =: \mu.$$

So,

$$(3.41) \qquad m_{n-1} = \mu + \frac{1}{\mu} = (\lambda^2 + 4)^{1/2} > h_q,$$

as a small calculation shows, and this implies

$$M(\alpha_0) > h_q.$$

In all cases, then, $M(\alpha_0)$ is bounded below by h_q, with the cases of equality stated in (3.12), (3.37), (3.38). This completes the proof of Theorem 1.

4. The local Hurwitz constant

In this section we shall consider the local Hurwitz constant, i.e., $m_i(\alpha_0)$. Our object is to compare m_i with h_q.

We first use a geometric method. Let α be G-irrational. The Ford circle C_n is defined by

$$C_n: \left| z - \left(\frac{P_n}{Q_n} + \frac{i}{2Q_n^2} \right) \right| = \frac{1}{2Q_n^2},$$

where P_n/Q_n are the convergents of α. Different C_n do not overlap; C_n and C_m are tangent externally if and only if $m = n+1$ or $n-1$. These assertions follow easily from the determinant condition (2.2). Also from (2.3), (2.12), with $\alpha = [r_0\lambda, \varepsilon_1/r_1\lambda, \dots]$, we have

$$\operatorname{sgn}\left(\alpha - \frac{P_{n-1}}{Q_{n-1}}\right) = \pm\operatorname{sgn}\left(\alpha - \frac{P_n}{Q_n}\right)$$

according as $\varepsilon_{n+1} = -1$ or $+1$.

Suppose $\varepsilon_{n+1} = -1$. Then α is on the same side of both P_n/Q_n and P_{n-1}/Q_{n-1}. It follows that

$$(4.1) \qquad \left| \alpha - \frac{P_i}{Q_i} \right| > \frac{1}{Q_i^2} \quad \text{for } i = n-1 \text{ or } n.$$

Equality is impossible because P_i/Q_i is G-rational, but α is G-irrational.

Next suppose $\varepsilon_{n+1} = 1$. Then α lies between P_n/Q_n and P_{n-1}/Q_{n-1}. Let (i, j) be a permutation of $(n-1, n)$. Then,

$$(4.2) \qquad \left| \alpha - \frac{P_i}{Q_i} \right| < \frac{1}{2Q_i^2}, \qquad \left| \alpha - \frac{P_j}{Q_j} \right| > \frac{1}{2Q_j^2}.$$

Equality can occur only if α coincides with the real projection of the point of tangency of the Ford circles, which is impossible because α is G-irrational. Hence,

Theorem 2. *If* $\varepsilon_{n+1} = 1$, *we have* $m_{n-1} > 2$, $m_n < 2$, *or* $m_{n-1} < 2$, $m_n > 2$.

An elegant algebraic proof of this theorem in the rational case ($q = 3$) was given by K. Th. Vahlen [6].

Theorem 2 holds for all $q \geq 4$, even or odd. Since $h_q = 2$ when q is even, it provides an estimate of the desired type for even q. We now concentrate on odd q.

Theorem 3. *Let q be odd. If $r_n \geq 2$ and $\varepsilon_{n-1} = 1$, then $m_{n-1} \geq h_q$.*

Theorem 3 is a special case of (3.39).

If we drop the assumption $r_n \geq 2$, we can have two consecutive $m_i < h_q$, as we see from the following example: let

$$\begin{gathered} \lambda = \lambda_7 = 1.80, \ldots, r_{n-1} = 4, \\ \alpha_0 = [\ldots, -1/4\lambda, 1/\lambda, -1/\lambda, -1/\lambda, \ldots], \end{gathered} \tag{4.3}$$

for which $m_n < \lambda$, $m_{n-1} < 1.97 < 2$. We make further assumptions on the ε_i.

Theorem 4. *Let q be odd. If $\varepsilon_{n+1} = \varepsilon_{n+2} = 1$, then $m_i \geq (\lambda^2+4)^{1/2} > h_q$ for at least one of $i = n-1, n, n+1$.*

The proof is modelled after one by M. Fujiwara [2]; see also F. Bagemihl and J. R. McLaughlin [1]. In contradiction to the conclusion

$$m_i(\alpha) \geq (\lambda^2+4)^{1/2}, \tag{4.4}$$

we can assert that

$$\left|\alpha - \frac{P_j}{Q_j}\right| > \frac{1}{(\lambda^2+4)^{1/2} Q_j^2}, \qquad n-1 \leq j \leq n+1. \tag{4.5}$$

We observe from (2.3) that $\alpha - P_{n-1}/Q_{n-1}$ and $\alpha - P_n/Q_n$ have opposite signs, in view of $\varepsilon_{n+1} = 1$. Hence,

$$\frac{1}{(\lambda^2+4)^{1/2}}\left(\frac{1}{Q_{n-1}^2} + \frac{1}{Q_n^2}\right) < \left|\alpha - \frac{P_{n-1}}{Q_{n-1}}\right| + \left|\alpha - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n Q_{n-1}}.$$

Write

$$(\lambda^2+4)^{1/2} = u + \frac{1}{u}, \qquad u > \lambda;$$

then

$$\frac{Q_n^2}{Q_{n-1}^2} - (\lambda^2+4)^{1/2}\frac{Q_n}{Q_{n-1}} + 1 = \left(\frac{Q_n}{Q_{n-1}} - \frac{1}{u}\right)\left(\frac{Q_n}{Q_{n-1}} - u\right) < 0.$$

Now $Q_n/Q_{n-1} - 1/u > 1 - 1 = 0$, so $Q_n/Q_{n-1} - u < 0$, that is,

$$\frac{Q_n}{Q_{n-1}} < u. \tag{4.6}$$

Hence,

$$\frac{Q_{n-1}}{Q_n} > \frac{1}{u}. \tag{4.7}$$

Replacing n by $n+1$ in (4.6)—recall $\varepsilon_{n+2} = 1$—we get

$$\frac{Q_{n+1}}{Q_n} < u.$$

Therefore,

$$u > \frac{Q_{n+1}}{Q_n} = r_{n+1}\lambda + \varepsilon_{n+1}\frac{Q_{n-1}}{Q_n} \geq \lambda + \frac{Q_{n-1}}{Q_n},$$

yielding

$$\frac{Q_{n-1}}{Q_n} < u - \lambda.$$

But

$$\left(u - \frac{1}{u}\right)^2 = \left(u + \frac{1}{u}\right)^2 - 4 = \lambda^2,$$

and so

$$\frac{Q_{n-1}}{Q_n} < \frac{1}{u},$$

contradicting (4.7). This completes the proof of Theorem 4.

Note added in proof. In a recent letter Thomas A. Schmidt has pointed out an error in [4] that carries over to the present paper. It can be corrected as follows. Replace the two paragraphs following (1.5) by the following:

We now consider the approximation of a G-irrational α_0 by the convergents P_n/Q_n of its λCF (1.4). Note that in (1.1) the fraction $k/m \in G(\infty)$ determines k and m uniquely up to sign, since

$$\begin{pmatrix} k & \cdot \\ m & \cdot \end{pmatrix}^{-1} \begin{pmatrix} k_1 & \cdot \\ m_1 & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ 0 & \cdot \end{pmatrix} = \pm \begin{pmatrix} 1 & \cdot \\ 0 & 1 \end{pmatrix}$$

when $k/m = k_1/m_1$. Thus we write

$$(1.6) \qquad \alpha_0 - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}\varepsilon_1\varepsilon_2\cdots\varepsilon_n}{m_{n-1}Q_{n-1}^2}, \qquad m_{n-1} = m_{n-1}(\alpha_0),$$

and study $m_{n-1}(\alpha_0)$. Clearly,

$$(1.7) \qquad M(\alpha_0) = \varlimsup_{n\to\infty} m_{n-1}(\alpha_0), \qquad h'_q = \inf_{\alpha_0} M(\alpha_0).$$

We call $m_n(\alpha_0)$ a *local Hurwitz constant.*

Similar changes are required in [4]. In particular, Theorem 3 should be eliminated.

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