

DEGENERATE KERNEL METHOD FOR HAMMERSTEIN EQUATIONS

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ABSTRACT. The classical method of the degenerate kernel method is applied to numerically solve the Hammerstein equations. Several numerical examples are given to demonstrate the effectiveness of the current method. A brief discussion of a number of methods to decompose the kernel is also included.

1. INTRODUCTION

This paper is concerned with the problem of finding numerical solutions of the Hammerstein equation

$$(1.1) \quad \varphi(x) - \int_0^1 k(x, y)\psi(y, \varphi(y)) dy = f(x), \quad x \in [0, 1],$$

where k , f , and ψ are given functions and φ is the solution to be determined. Several numerical methods for approximating the solution of (1.1) are known. The classical method of successive approximations was introduced in the 1950's [7]. A variation of Nyström's method was presented in [5]. A new collocation-type method was developed in recent papers [3, 4]. In this paper, we employ the degenerate kernel method, which plays an important role in the study of numerical solutions for the Fredholm integral equations of the second kind. A unified theory to present various degenerate kernel methods within the framework of the projection method was recently presented in [2]. In §2 we present the degenerate kernel method and consider the problem of existence and uniqueness of the solution of the new Hammerstein equation associated with the degenerate kernel. The problem of convergence of the numerical solution is also considered in §2. Some examples are considered in §3. In §4 we approximate a given kernel by several specific degenerate kernels and obtain convergence rates for the corresponding approximate solutions of equation (1.1). The idea of approximating the kernel by Boolean sums is also included in this section. A comparison of the performance of the Boolean sum approximation scheme with that of the tensor product approximation scheme (§3, Example 4) will be given in a future paper.

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In order to guarantee the existence of a unique solution to equation (1.1), we assume throughout this paper that the following conditions (i)–(iv) be satisfied:

- (i) $k(x, y) \in C([0, 1] \times [0, 1])$;
- (ii) $\psi(y, u)$ is continuous in $y \in [0, 1]$ and $u \in (-\infty, \infty)$, and

$$\left\{ \int_0^1 |\psi(y, \varphi(y))|^2 dy \right\}^{1/2} \leq A \|\varphi\|_2,$$

where $\|\cdot\|_2$ denotes the L_2 norm;

- (iii) $\psi(y, u)$ satisfies the Lipschitz condition $|\psi(y, u) - \psi(y, v)| \leq B|u - v|$;
- (iv) k is bounded by $|k(x, y)| < C$ with $BC < 1$.

Under these conditions there exists a unique solution in $L_2[0, 1]$ for equation (1.1). This can be proved by the Banach contraction mapping principle. To obtain a higher-order convergence rate, we need to assume higher-order smoothness conditions on k .

As the first example in §4 shows, the method of degenerate kernel described in this paper may be applied to integral equations with multiple solutions. Hence, conditions (i)–(iv), which guarantee the global uniqueness of the solution, are perhaps too strong for deciding whether our method is applicable. A reader interested in a more local property of the solution can find a related discussion in [3].

2. DEGENERATE KERNEL METHOD

Suppose that $k_n(x, y)$ is an approximation of the kernel $k(x, y)$, and that it is of a degenerate kernel form,

$$(2.1) \quad k_n(x, y) = \sum_{i=1}^n B_i(x)C_i(y),$$

where the set of functions $\{B_i(x)\}$ is assumed to be linearly independent. We also assume that

$$(2.2) \quad \left\{ \int_0^1 \int_0^1 |k(x, y) - k_n(x, y)|^2 dx dy \right\}^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It is natural to expect the solutions of the following equation (2.3) associated with the degenerate kernels $k_n(x, y)$ to converge to the exact solution of equation (1.1),

$$(2.3) \quad \varphi_n(x) - \int_0^1 k_n(x, y)\psi(y, \varphi_n(y)) dy = f(x), \quad x \in [0, 1].$$

To solve equation (2.3), substitute $k_n(x, y)$ into (2.3) and find

$$(2.4) \quad \varphi_n(x) - \sum_{i=1}^n B_i(x) \int_0^1 C_i(y)\psi(y, \varphi_n(y)) dy = f(x), \quad x \in [0, 1].$$

Let

$$(2.5) \quad \alpha_i = \int_0^1 C_i(y)\psi(y, \varphi_n(y)) dy.$$

Then φ_n can be expressed as

$$(2.6) \quad \varphi_n(x) = f(x) + \sum_{i=1}^n \alpha_i B_i(x),$$

where the α_i 's are constants to be determined. Once the α_i 's have been obtained, (2.6) gives the solution of (2.3). Upon substitution of (2.6) into (2.5), we have

$$(2.7) \quad \alpha_j = \int_0^1 C_j(y)\psi\left(y, f(y) + \sum_{i=1}^n \alpha_i B_i(y)\right) dy \quad \text{for } j = 1, 2, \dots, n.$$

Define

$$(2.8) \quad F_j(\alpha_1, \alpha_2, \dots, \alpha_n) = \int_0^1 C_j(y)\psi\left(y, f(y) + \sum_{i=1}^n \alpha_i B_i(y)\right) dy.$$

Then (2.7) becomes a system of nonlinear algebraic equations

$$(2.9) \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} F_1(\alpha_1, \alpha_2, \dots, \alpha_n) \\ F_2(\alpha_1, \alpha_2, \dots, \alpha_n) \\ \vdots \\ F_n(\alpha_1, \alpha_2, \dots, \alpha_n) \end{bmatrix},$$

or, in vector notation,

$$(2.10) \quad \alpha = \mathbf{F}(\alpha),$$

where $\alpha^T = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\mathbf{F}(\alpha)^T = (F_1(\alpha), F_2(\alpha), \dots, F_n(\alpha))$.

We shall show that the unique solution of equation (2.10) corresponds to the unique solution of equation (2.3) for each n under some mild assumptions. Equation (2.10) can be solved by a number of standard numerical methods, such as Newton's iterative method or modified Newton's iterative method.

Now we consider the existence of a unique solution of (2.3). We recall that assumptions (i)–(iv) are standing assumptions throughout the rest of this paper.

Theorem 1. *Let $k_n(x, y) \in C([0, 1] \times [0, 1])$ satisfy condition (2.2). Then there exists an integer $N > 0$ such that, for each $n > N$, equation (2.3) has a unique solution $\varphi_n \in L_2[0, 1]$.*

Proof. In view of assumption (iv) and condition (2.2), there exists an integer N such that for each $n > N$,

$$\left\{ \int_0^1 \int_0^1 |k_n(x, y)|^2 dx dy \right\}^{1/2} \leq C.$$

For each $n > N$, define

$$(T_n \varphi)(x) = \int_0^1 k_n(x, y) \psi(y, \varphi(y)) dy$$

and

$$(\widehat{T}_n \varphi)(x) = (T_n \varphi)(x) + f(x).$$

It is straightforward to verify from (ii) that $\|T_n \varphi\|_2 \leq CA \|\varphi\|_2$ for all $\varphi \in L_2[0, 1]$. Hence, T_n is a bounded nonlinear operator. Also, for $\varphi_1, \varphi_2 \in L_2[0, 1]$,

$$\begin{aligned} \|\widehat{T}_n \varphi_1 - \widehat{T}_n \varphi_2\|_2 &= \|T_n \varphi_1 - T_n \varphi_2\|_2 \\ &\leq \left\{ \int_0^1 \int_0^1 |k_n(x, y)|^2 dx dy \int_0^1 |\psi(y, \varphi_1(y)) - \psi(y, \varphi_2(y))|^2 dy \right\}^{1/2} \\ &\leq CB \|\varphi_1 - \varphi_2\|_2 \quad \text{by (iii) and (iv)}. \end{aligned}$$

Since $0 < CB < 1$ by assumption, \widehat{T}_n is a contraction operator. It follows that there is a unique fixed point of \widehat{T}_n , which is of course the unique solution of equation (2.3), and $\varphi_n \in L_2[0, 1]$. \square

Theorem 2. *Let*

$$(2.11) \quad M = B \left\{ \sum_{i=1}^n \int_0^1 |B_i(x)|^2 dx \right\}^{1/2} \left\{ \sum_{i=1}^n \int_0^1 |C_i(x)|^2 dx \right\}^{1/2}$$

and assume $M < 1$. Then the nonlinear algebraic equations (2.10) have a unique solution $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*)$, and

$$(2.12) \quad \varphi_n(x) = f(x) + \sum_{i=1}^n \alpha_i^* B_i(x)$$

is the unique solution of equation (2.3).

Proof. Define, as usual, the discrete l_2 norm by $\|\alpha\|_{l_2} = \{\sum_{i=1}^n |\alpha_i|^2\}^{1/2}$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in l_2(n)$. For $\alpha^{(1)} = (\alpha_1^{(1)}, \dots, \alpha_n^{(1)})$ and $\alpha^{(2)} = (\alpha_1^{(2)}, \dots, \alpha_n^{(2)})$,

$$\begin{aligned} \|\mathbf{F}(\alpha^{(1)}) - \mathbf{F}(\alpha^{(2)})\|_{l_2} &\leq B \left\{ \sum_{i=1}^n \int_0^1 |B_i(x)|^2 dx \right\}^{1/2} \\ &\quad \times \left\{ \sum_{i=1}^n \int_0^1 |C_i(x)|^2 dx \right\}^{1/2} \|\alpha^{(1)} - \alpha^{(2)}\|_{l_2}. \end{aligned}$$

Consequently, \mathbf{F} is a contraction operator in $l_2(n)$, since $M < 1$. Hence, \mathbf{F} has a unique fixed point α^* , i.e., $\mathbf{F}(\alpha^*) = \alpha^*$. For this α^* , it is obvious that $\varphi_n(x)$ defined by (2.12) is a solution of (2.3), and by Theorem 1, $\varphi_n(x)$ is the unique solution of (2.3). \square

Theorem 3. For

$$(2.13) \quad \|K - K_n\|_2 \equiv \left\{ \int_0^1 \int_0^1 |k(x, y) - k_n(x, y)|^2 dx dy \right\}^{1/2}$$

assume that condition (2.2) holds. Then,

$$(2.14) \quad \|\varphi - \varphi_n\|_2 \leq \frac{A\|\varphi\|_2}{1 - CB} \|K - K_n\|_2,$$

$$(2.15) \quad \|\varphi - \varphi_n\|_2 \leq \frac{A\|\varphi_n\|_2}{1 - CB} \|K - K_n\|_2.$$

Therefore, φ_n converges to φ in $L_2[0, 1]$.

Proof. Since

$$\begin{aligned} \varphi(x) - \varphi_n(x) &= \int_0^1 [k(x, y) - k_n(x, y)]\psi(y, \varphi(y)) dy \\ &\quad + \int_0^1 k_n(x, y)[\psi(y, \varphi(y)) - \psi(y, \varphi_n(y))] dy, \end{aligned}$$

we have

$$\|\varphi - \varphi_n\|_2 \leq A\|K - K_n\|_2\|\varphi\|_2 + CB\|\varphi - \varphi_n\|_2.$$

By virtue of the condition $0 < CB < 1$,

$$\|\varphi - \varphi_n\|_2 \leq \frac{A\|\varphi\|_2}{1 - CB} \|K - K_n\|_2.$$

The proof of (2.15) is quite similar. Since $\|K - K_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$, $\|\varphi - \varphi_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. \square

The inequalities (2.14) and (2.15) can be viewed as a priori and a posteriori estimates, respectively, of the error of the approximate solution.

3. EXAMPLES

Example 1. First we substantiate the claim made earlier that uniqueness of the solution is not necessary for the current method to be applicable. Consider [5]

$$\varphi(x) - \int_0^1 xy\varphi^2(y) dy = \frac{3}{4}x, \quad x \in [0, 1].$$

The actual solutions are $\varphi_1(x) = x$ and $\varphi_2(x) = 3x$. The kernel function is already degenerate and we obviously take $B_1(x) = x$ and $C_1(y) = y$. Upon substituting $\varphi_n(x) = \frac{3}{4}x + \alpha x$ into (2.5), we obtain $\alpha^2 - \frac{5}{2}\alpha + \frac{9}{16} = 0$. Solving this equation, the exact solutions are obtained.

Example 2. Consider

$$\varphi(x) - \int_0^1 (1 + xy)\varphi^2(y) dy = \frac{3}{4}x - \frac{1}{3}, \quad x \in [0, 1].$$

Again, the kernel is degenerate. We take $B_1(x) = 1$ and $B_2(x) = x$. The C_i 's are chosen in the same way. Upon substituting $\varphi_n = (\frac{3}{4}x - \frac{1}{3}) + \alpha_1 + \alpha_2 x$ into

(2.5) and solving the resulting equations, we obtain $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{1}{4}$. The exact solution $\varphi(x) = x$ is obtained by substituting these values of α_1 and α_2 in the expression φ_n above.

Example 3. Consider [5]

$$\varphi(x) - \int_0^1 e^{xy} e^{-\varphi^2(y)} dy = \sqrt{x} - \frac{e^{x-1} - 1}{x-1}, \quad x \in [0, 1].$$

The exact solution is $\varphi(x) = \sqrt{x}$. We approximate e^{xy} by $1 + xy + (xy)^2/2 + \dots + (xy)^n/n!$. An approximate solution

$$\varphi_n(x) = \sqrt{x} - \frac{e^{x-1} - 1}{x-1} + \alpha_1 + \alpha_2 x + \alpha_3 \frac{x^2}{\sqrt{2}} + \dots + \alpha_{n+1} \frac{x^n}{\sqrt{n!}}$$

is obtained by solving (2.5). For example, for $n = 2$, we obtain $\alpha_1 = .634038$, $\alpha_2 = .266290$, $\alpha_3 = .114898$. The errors e_n for different values of n are listed below. The e_n 's are approximated by the composite Simpson's formula.

n	$e_n = \ \varphi - \varphi_n\ _2$
2	7.218E - 2
3	9.847E - 3
4	2.013E - 3

Example 4. Consider the same equation as in Example 3. Now the kernel is approximated by $\sum_{i=0}^{2^n} \sum_{j=0}^{2^n} e^{x_i y_j} B_i(x) C_j(y)$, where the B_i 's and C_j 's are linear B -splines with knots $0 = x_0 < x_1 < \dots < x_{2^n} = 1$, where $x_i = i/2^n$ for $i = 0, 1, \dots, 2^n$. The approximation

$$\varphi_n(x) = \sqrt{x} - \frac{e^{x-1} - 1}{x-1} + \sum_{i=0}^{2^n} \alpha_i B_i(x)$$

is obtained upon solving (2.5) and gives rise to errors e_n shown below.

n	$e_n = \ \varphi - \varphi_n\ _2$
2	6.203E - 2
3	1.553E - 2
4	3.768E - 3
5	9.432E - 4

4. APPROXIMATIONS OF THE KERNEL

The examples in §3 are mainly concerned with the method of approximating the kernel $k(x, y)$ by degenerate kernels having the form of tensor products of

univariate functions. In this section, we present another approach, a variation of tensor products, to decompose $k(x, y)$.

Let L_x be a projection of $C([0, 1] \times [0, 1])$ onto $X_n \times C[0, 1]$, where X_n is an n -dimensional subspace of $C[0, 1]$. Similarly, let M_y be a projection of $C([0, 1] \times [0, 1])$ onto $C[0, 1] \times Y_m$, where Y_m is an m -dimensional subspace of $C[0, 1]$. Define

$$(4.1) \quad R_x k(x, y) = k(x, y) - L_x k(x, y)$$

and

$$(4.2) \quad R_y k(x, y) = k(x, y) - M_y k(x, y).$$

Then $L_x k$ and $M_y k$ provide two degenerate kernels that approximate partially $k(x, y)$ with respect to x and y , with respective errors $R_x k$ and $R_y k$. Then $L_x M_y k$ gives a complete approximation of $k(x, y)$ in both x and y , with error term

$$(4.3) \quad R(L_x M_y)k = R_x k + R_y k - R_x R_y k.$$

Thus, the rate of convergence depends on the approximation powers of L_x and M_y . In order to enhance the speed of convergence, we define the Boolean sum of L_x and M_y by

$$(4.4) \quad L_x \otimes M_y = L_x + M_y - L_x M_y.$$

Then $(L_x \otimes M_y)k$ approximates k with the error

$$(4.5) \quad R(L_x \otimes M_y)k = R_x R_y k.$$

To see (4.5), we have by the definition of $L_x \otimes M_y$,

$$\begin{aligned} k - (L_x \otimes M_y)k &= (I - L_x)k + (I - M_y)k - (I - L_x M_y)k \\ &= R_x k + R_y k - (R_x k + R_y k - R_x R_y k) = R_x R_y k. \end{aligned}$$

Thus, this new approximation of k by $(L_x \otimes M_y)k$ enhances the rate of convergence. More specifically, let X_n be the space of spline functions of degree $n_1 - 1$ with knots at $0 = x_0 < x_1 < \dots < x_{k_1} < x_{k_1+1} = 1$, where $n_1 + k_1 =: n$. Similarly, let Y_m be the space of spline functions of degree $m_1 - 1$ with knots at $0 = y_0 < y_1 < \dots < y_{k_2} < y_{k_2+1} = 1$, where $m_1 + k_2 =: m$. Chapters 4 and 12 of [6] are particularly useful in relation to our current discussion.

Let L_x and M_y be the interpolatory operators onto X_n and Y_m , respectively. Assuming $k \in C^{n_1}[0, 1] \times C[0, 1]$, we have

$$(4.6) \quad \sup |R_x k(x, y)| \leq |C_1(y)| h_x^{n_1} \leq C_1 h_x^{n_1}, \quad C_1 \text{ a constant,}$$

and similarly, when $k \in C[0, 1] \times C^{m_1}[0, 1]$, then

$$(4.7) \quad \sup |R_y k(x, y)| \leq |C_2(x)| h_y^{m_1} \leq C_2 h_y^{m_1}, \quad C_2 \text{ a constant,}$$

where $h_x = \max\{x_i - x_{i-1} : i = 1, 2, \dots, k_1 + 1\}$ and $h_y = \max\{y_i - y_{i-1} : i = 1, 2, \dots, k_2 + 1\}$.

The following theorem is an immediate consequence of the preceding discussion.

Theorem 4. (i) Let $k \in C^{n_1}[0, 1] \times C[0, 1]$, and let φ_n be the solution of (2.3) with $k_n(x, y) := L_x k(x, y)$. Then $\|\varphi - \varphi_n\|_2 \leq \tilde{C}_1 h_x^{n_1}$.

(ii) Let $k \in C[0, 1] \times C^{m_1}[0, 1]$, and let φ_m be the solution of (2.3) with $k_m(x, y) := M_y k(x, y)$. Then $\|\varphi - \varphi_m\|_2 \leq \tilde{C}_2 h_y^{m_1}$.

(iii) Let $k \in C^{n_1}[0, 1] \times C^{m_1}[0, 1]$, and let φ_n be the solution of (2.3) with $k_n(x, y) := L_x M_y k(x, y)$. Then $\|\varphi - \varphi_n\|_2 \leq \tilde{C}_3 h^{n_2}$, where $h = \max\{h_x, h_y\}$ and $n_2 = \min\{n_1, m_1\}$.

(iv) Let $k \in C^{n_1}[0, 1] \times C^{m_1}[0, 1]$, and let φ_n be the solution of (2.3) with $k_n(x, y) = (L_x \otimes M_y)k(x, y)$. Then $\|\varphi - \varphi_n\|_2 \leq \tilde{C}_4 h^{n_1+m_1}$.

Proof. Proofs of (i)–(iv) of this theorem are quite obvious from the discussion in this section and Theorem 3. We demonstrate the proof for (iv) only. By (4.5), (4.6), and (4.7),

$$\|k - (L_x \otimes M_y)k\| \leq C_4 h_x^{n_1} h_y^{m_1}.$$

By Theorem 3,

$$\|\varphi - \varphi_n\|_2 \leq \frac{A\|\varphi_n\|_2}{1-CB} C_4 h_x^{n_1} h_y^{m_1} \leq \tilde{C}_4 h^{n_1+m_1}, \quad \text{where } \tilde{C}_4 = \frac{AC_4\|\varphi_n\|_2}{1-CB}. \quad \square$$

There exists only a small number of papers which use the Boolean sum approximation technique. These papers are mainly in the area of approximation of eigenvalues of certain operators (see, e.g., [1] and references cited therein).

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