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SUMS OF CUBES IN POLYNOMIAL RINGS

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ABSTRACT. For any associative ring A with 1 of prime characteristic $\neq 0, 2, 3$, every element of A is the sum of three cubes in A.

For any ring A, let $w_3(A)$ denote the least integer $s \geq 0$ such that every sum of cubes in A is a sum of s cubes in A. If no such s exists, $w_3(A) = \infty$ by definition.

For example, when $A = \mathbb{Z}$, the integers, it is known [2] that $4 \leq w_3(\mathbb{Z}) \leq 5$.

In this paper, we study $w_3(A)$ for $A = F[x]$, the polynomial ring in one variable x with coefficients in a field F. It is easy to see [4] that every polynomial in $F[x]$ is a sum of cubes, if and only if $\mathrm{char}(F) \neq 3$ and $\mathrm{card}(F) \neq 2, 4$. If this is the case, $w_3(F[x])$ coincides with the least s such that x is the sum of s cubes in $F[x]$. Moreover, in this case, every element of any associative F-algebra A with 1 is the sum of $w_3(F[x])$ cubes in A; in particular, $w_3(A) \leq w_3(F[x])$.

The formula

$$(x+1)^3 - 2x^3 + (x-1)^3 = 6x$$

shows that $w_3(A) \leq 4$ for any associative ring $A = 6A$ with 1. In particular, $w_3(F[x]) \leq 4$ for any field F with $\mathrm{char}(F) \neq 2, 3$. By [4], $w_3(F[x]) \leq 4$ also in the case when $\mathrm{char}(F) = 2$ and $\mathrm{card}(F) \neq 2, 4$. When $\mathrm{card}(F) = 2, 4$, formulas on p. 63 of [3] show that $w_3(F[x]) \leq 5$. These formulas together with the formulas

$$(xy+s)^3 + (xy+s+1)^3 + (x+sy^2)^3 + (x+(1+s)y^2)^3 = x(y+y^4),$$

$$\begin{aligned}
&(x^2y^2 + x(y+y^4) + 1 + s + y + y^8 + y^{10})^3 \\
&\quad + (x^2(1+y^2) + x(y+y^4) + s + y + y^2 + y^{10})^3 \\
&\quad + (x^2y^2 + x(y+y^4) + s + y + y^4 + y^8 + y^{10})^3 \\
&\quad + (x^2(1+y^2) + x(y+y^4) + s + y + y^2 + y^4 + y^{10})^3 \\
&= x(y+y^4) + y + y^2,
\end{aligned}$$

where $2 = 0$ and $s^2 + s + 1 = 0$, show that $w_3(F[x]) \leq 4$ in the case when $\mathrm{card}(F) = 4$.

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On the other hand, it is clear that $w_3(F[x]) = 1$ when $\operatorname{char}(F) = 3$, and that $w_3(F[x]) \geq 3$ when $\operatorname{char}(F) \neq 3$.

Our first easy result (which was obtained by the author in July of 1987 and was known to J.-P. Serre since April of 1982) is the following

Lemma 1. *If* $\operatorname{char}(F) \neq 3$ *and there are nonzero* $\alpha, \beta, \gamma \in F$ *such that* $\alpha^3 + \beta^3 + \gamma^3 = 0$, *then* $w_3(F[x]) = 3$.

Proof. The formula

$$(\alpha x + \beta^2)^3 + (\beta x - \alpha^2)^3 + (\gamma x)^3 = \beta^6 - \alpha^6 - 3\alpha\beta\gamma^3 x,$$

where x can be replaced by an arbitrary polynomial in $F[x]$, shows that $w_3(F[x]) \leq 3$. □

Corollary 2. *If* $\operatorname{char}(F) \neq 0, 3$ *and* $\operatorname{card}(F) \neq 2, 4, 7, 13, 16$, *then* $w_3(F[x]) = 3$.

By more complicated computations, we will prove the following

Proposition 3. *If* $\operatorname{char}(F) \neq 3$, *then the condition of Lemma* 1 *is equivalent to the following*: x *is the sum of cubes of three polynomials in* $F[x]$ *of degree* ≤ 4.

J.-P. Serre knew this for polynomials of degree ≤ 2 since April of 1982. The case of degree ≤ 3 was done by August 1987 independently by the author and David Hayes. In his letter of August 11, 1987, to Serre, Hayes wrote that the case of degree ≤ 4 defeated him. The author obtained Proposition 3 with $2F \neq F$ in September of 1987. He thanks Serre for providing copies of relevant letters and useful suggestions.

This result leads one to wonder whether the converse of Lemma 1 is true. However J.-P. Serre wrote to the author on September 9, 1987, that he did not dare to conjecture anything himself even for $F = \mathbb{Q}$, the rational numbers, or $F = \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/13\mathbb{Z}$.

Computations with polynomials of degree ≥ 5 are very complicated, and it is only after many hours of computations with computers, that the author obtained the following result.

Theorem 4. *If* $\operatorname{char}(F) \neq 0, 3$ *and* $\operatorname{card}(F) \neq 2, 4, 16$, *then* $w_3(F[x]) = 3$. *So every element of every associative* F*-algebra* A *is the sum of three cubes in* A.

Proof. By Corollary 2, we have only the cases $\operatorname{char}(F) = 7, 13$ to deal with. Modulo 13, we have

$$(-4x + 5x^2 + 6x^3 + x^4)^3 + (1 + 4x + x^2 - 5x^3 - 3x^4 + 5x^5)^3 + (-1 + 5x - 5x^2 + 5x^3 + 3x^4 - 5x^5)^3 = x.$$

Modulo 7, we have

$$(x - 3x^3 - x^5)^3 + (-1 - x - x^3 + 3x^4 - x^6)^3 + (1 - x - x^3 - 3x^4 + x^6)^3 = x. \quad \square$$

Remarks. The cases $F = \mathbb{Q}$ and $\text{card}(F) = 16$ remain unresolved. The above equalities were found using *Mathematica* on Macintosh IIx. We work in one of the following cases: $F = \mathbb{Q}, \mathbb{Z}/7\mathbb{Z}$, or $\mathbb{Z}/13\mathbb{Z}$. In degree 5, we proceed as follows. First, we set $d = a^3 + b^3 + c^3$ with

$$a = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$
$$b = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5,$$
$$c = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5.$$

Note that $d = d_0+d_1x+\cdots+d_{15}x^{15}$, where d_j are polynomials in a_i, b_i, c_i. We want $d_j = 0$ for $i \geq 2$ and $d \neq 0$. (For proving Theorem 4, we want $d_1 \neq 0$.) We assume that $\text{char}(F) \neq 2, 3$. By Proposition 3, we can assume that one of the leading coefficients a_5, b_5, c_5 is not 0. Since the condition of Lemma 1 does not hold, one of these coefficients must be 0. Say, $a_5 \neq 0$ and $c_5 = 0$. Then $b_5 = -a_5$, because $d_{15} = 0$. The condition $d \neq 0$ forces $c_4 \neq 0$. By a linear invertible change of the variable x and multiplying both sides of $d = a^3 + b^3 + c^3$ by a nonzero constant, we are reduced to the case when $a_5 = 1 = -b_5$, $a_4 = 0$, $c_4 = 1$. Now $d_{15} = 0 \neq d$.

The polynomial equations $d_i = 0$ $(7 \leq i \leq 15)$ allow us to exclude all unknown coefficients but a_3, c_0, c_1, c_2, c_3. The conditions $d_6 = 0$, $d_5 = 0$, $d_4 = 0$, $d_3 = 0$, $d_2 = 0$ give a system of five polynomial equations for five unknowns a_3, c_0, c_1, c_2, c_3. Namely,

$$\begin{aligned} d_6 = &-1/108 + 5a_3^3 + 3a_3c_0 + 3c_1^2/4 - 18a_3^2c_2 - 3c_0c_2 + 21a_3c_2^2 \\ &- 8c_2^3 - 12a_3c_1c_3 + 15c_1c_2c_3 + 63a_3^2c_3^2 + 3c_0c_3^2 - 159a_3c_2c_3^2 \\ &+ 99c_2^2c_3^2 - 3c_1c_3^3/2 + 12a_3c_3^4 - 21c_2c_3^4 - 37c_3^6/4 = 0; \end{aligned}$$

$$\begin{aligned} d_5 = &\,9a_3^2c_1/2 + 3c_0c_1/2 - 12a_3c_1c_2 + 15c_1c_2^2/2 - c_3/12 \\ &- 30a_3^3c_3 - 3a_3c_0c_3 + 3c_1^2c_3/4 + 117a_3^2c_2c_3 + 6c_0c_2c_3 \\ &- 150a_3c_2^2c_3 + 63c_2^3c_3 - 6a_3c_1c_3^2 + 9c_1c_2c_3^2/2 \\ &+ 57a_3^2c_3^3/2 + 15c_0c_3^3/2 - 51a_3c_2c_3^3 + 33c_2^2c_3^3/2 \\ &- 15c_1c_3^4/2 + 57a_3c_3^5 - 165c_2c_3^5/2 - 69c_3^7/4 = 0; \end{aligned}$$

$$\begin{aligned} d_4 = &\,a_3/18 + 15a_3^4/4 + 3a_3^2c_0/2 + 3c_0^2/4 - c_2/12 \\ &- 12a_3^3c_2 + 3c_1^2c_2/4 + 21a_3^2c_2^2/2 - 3c_0c_2^2/2 \\ &- 9c_2^4/4 - 6a_3^2c_1c_3 + 3c_0c_1c_3/2 + 9c_1c_2^2c_3 \\ &- c_3^2/3 + 42a_3^3c_3^2 - 159a_3^2c_2c_3^2/2 \\ &+ 9c_0c_2c_3^2/2 + 81c_2^3c_3^2/2 - 3c_1c_2c_3^3 \\ &+ 6a_3^2c_3^4 + 3c_0c_3^4/2 - 69c_2^2c_3^4/4 - 3c_1c_3^5/2 - 51c_2c_3^6/4 - 3c_3^8/2 = 0; \end{aligned}$$

$$\begin{aligned}
d_3 = {} & 21a_3^3c_1/2 + 9a_3c_0c_1/2 + c_1^3 - 33a_3^2c_1c_2 - 3c_0c_1c_2 \\
& + 63a_3c_1c_2^2/2 - 9c_1c_2^3 + a_3c_3/3 - 135a_3^4c_3/4 \\
& - 6a_3^2c_0c_3 + 3c_0^2c_3/4 - 18a_3c_1^2c_3 - c_2c_3/2 \\
& + 129a_3^3c_2c_3 + 9a_3c_0c_2c_3 + 45c_1^2c_2c_3/2 \\
& - 321a_3^2c_2^2c_3/2 + 63a_3c_2^3c_3 + 9c_2^4c_3/4 \\
& + 219a_3^2c_1c_3^2/2 + 9c_0c_1c_3^2/2 - 549a_3c_1c_2c_3^2/2 \\
& + 333c_1c_2^2c_3^2/2 - 3c_3^3/4 - 209a_3^3c_3^3/2 \\
& + 3a_3c_0c_3^3/2 - 9c_1^2c_3^3/4 + 393a_3^2c_2c_3^3 \\
& + 9c_0c_2c_3^3/2 - 933a_3c_2^2c_3^3/2 + 171c_2^3c_3^3 \\
& + 12a_3c_1c_3^4 - 63c_1c_2c_3^4/2 + 42a_3^2c_3^5 - 87a_3c_2c_3^5/2 \\
& - 81c_2^2c_3^5/4 - 27c_1c_3^6/2 + 6a_3c_3^7 - 45c_2c_3^7/2 - 9c_3^9/4 = 0; \\
d_2 = {} & -a_3^2/9 + 9a_3^5/2 + 9a_3^3c_0 + 9a_3c_0^2/2 + 27a_3^2c_1^2/4 + 3c_0c_1^2 \\
& + a_3c_2/3 - 99a_3^4c_2/4 - 63a_3^2c_0c_2/2 - 15c_0^2c_2/4 - 18a_3c_1^2c_2 \\
& - c_2^2/4 + 54a_3^3c_2^2 + 36a_3c_0c_2^2 + 45c_1^2c_2^2/4 - 117a_3^2c_2^3/2 \\
& - 27c_0c_2^3/2 + 63a_3c_2^4/2 - 27c_2^5/4 - 135a_3^3c_1c_3/2 - 27a_3c_0c_1c_3 \\
& + 252a_3^2c_1c_2c_3 + 36c_0c_1c_2c_3 - 621a_3c_1c_2^2c_3/2 + 126c_1c_2^3c_3 \\
& + 5a_3c_3^2/6 + 549a_3^4c_3^2/4 + 108a_3^2c_0c_3^2 + 9c_0^2c_3^2/2 + 9a_3c_1^2c_3^2 \\
& - 5c_2c_3^2/4 - 639a_3^3c_2c_3^2 - 270a_3c_0c_2c_3^2 - 63c_1^2c_2c_3^2/4 \\
& + 1107a_3^2c_2^2c_3^2 + 333c_0c_2^2c_3^2/2 - 846a_3c_2^3 + 963c_2^4c_3^2/4 \\
& - 153a_3^2c_1c_3^3/2 + 9c_0c_1c_3^3/2 + 228a_3c_1c_2c_3^3 - 333c_1c_2^2c_3^3/2 \\
& - c_3^4 + 357a_3^3c_3^4/2 + 12a_3c_0c_3^4 - 9c_1^2c_3^4 - 720a_3^2c_2c_3^4 \\
& - 27c_0c_2c_3^4 + 951a_3c_2^2c_3^4 - 1647c_2^3c_3^4/4 + 165a_3c_1c_3^5/2 \\
& - 108c_1c_2c_3^5 - 519a_3^2c_3^6/4 - 27c_0c_3^6/2 + 294a_3c_2c_3^6 - 639c_2^2c_3^6/4 \\
& - 27c_1c_3^7/2 + 21a_3c_3^8/2 - 9c_2c_3^8/4 + 9c_3^{10}/4 = 0.
\end{aligned}$$

Further exclusion of variables required too much computer memory. So a complete search was used instead for $F = \mathbb{Z}/7\mathbb{Z}$ and $F = \mathbb{Z}/13\mathbb{Z}$. The author thanks A. Ocneanu for help with programming. The computer search showed that there was no solution for this system of five polynomial equations in five variables in the case $F = \mathbb{Z}/7\mathbb{Z}$. Thus, the equation $d = a^3 + b^3 + c^3$ has no solutions in $(\mathbb{Z}/7\mathbb{Z})[x]$ such that $5 \geq \deg(a) \geq \deg(b) \geq \deg(c)$, $\deg(a) \neq 0$, and $\deg(d) = 0$ or 1. In the case $F = \mathbb{Z}/13\mathbb{Z}$, the computer took 57128 seconds to try all 13^5 possible solutions and found all 12 solutions. The first of them is

$$(1+3x+10x^2+x^5)^3-(12+12x+6x^2+x^5)^3+(2+4x+8x^2+9x^3+x^4)^3 = 10+8x.$$

It was written above in the proof of Theorem 4 after a linear change of variable. Here are five more solutions (mod 13):

$$(3 + x + 10x^2 + x^5)^3 - (10 + 4x + 6x^2 + x^5)^3 + (5 + 4x + 11x^2 + 3x^3 + x^4)^3 = 10 + 11x;$$

$$(9 + 9x + 10x^2 + x^5)^3 - (4 + 10x + 6x^2 + x^5)^3 + (6 + 4x + 7x^2 + x^3 + x^4)^3 = 10 + 7x;$$

$$(5 + 7x + 7x^2 + 7x^3 + x^5)^3 + (7 + 3x + 7x^2 + 3x^3 + x^4)^3 - (12 + 10x + 3x^2 + 7x^3 + x^5)^3 = 1;$$

$$(2 + 11x + 7x^2 + 8x^3 + x^5)^3 - (10 + 12x + 3x^2 + 8x^3 + x^5)^3 + (11 + 3x + 8x^2 + x^3 + x^4)^3 = 1;$$

$$(6 + 8x + 7x^2 + 11x^3 + x^5)^3 - (4 + 4x + 3x^2 + 11x^3 + x^5)^3 + (8 + 3x + 11x^2 + 9x^3 + x^4)^3 = 1.$$

The other six equalities can be obtained from the above six solutions by switching a and b and replacing x by $-x$.

The degree-6 case was treated similarly. The following lemma was used to restrict possible values for the coefficients of a, b, c.

Lemma 5. *Let F be a field such that there is no α, β, γ in F with $\alpha^3+\beta^3+\gamma^3 = 0 \neq \alpha\beta\gamma$. Let $x = a^3 + b^3 + c^3$ with $a, b, c \in F[x]$, $N = \deg(a) \geq \deg(b) \geq \deg(c) \geq 1$, $a = \sum a_i x^i$, $b = \sum b_i x^i$, $c = \sum c_i x^i$, and $a_i, b_i, c_i \in F$. Then $c_0 = 0$ and a_0/a_N is a cube in F.*

Proof. By the condition, $c_N = 0$, hence $(-b_N/a_N)^3 = 1$. Replacing b by bb_N/a_N, we can assume that $b_N = -a_N \neq 0$.

Let us show that the assumption that $c_0 \neq 0$ leads to a contradiction. Indeed, in this case $a_0 b_0 = 0$. Say, $b_0 = 0$. Replacing c by ca_0/c_0 we can assume that $c_0 = a_0$.

Let α_i be the zeros of $b+c$ in a field extension of F. Then $a(\alpha_i)^3 = \alpha_i$ for all i, hence $a_0/a_N = (-1)^N \prod \alpha_i = ((-1)^N \prod a(\alpha_i))^3$ is a cube in F. Applying this argument to the zeros β_j of $a^2 - ab + b^2$, we obtain that $a_0^2/(3a_N^2) = (\prod c(\beta_j))^3$ is also a cube in F. So 9 is a cube (in fact, a 12th power) in F. In the case $2F = F$, this leads to a contradiction, because $9 = 8 + 1$ is also the sum of two cubes.

In the case $2F = 0$, assume first that F contains an element ε such that $\varepsilon^2 = \varepsilon + 1$. Using the zeros of $a + \varepsilon b$, we conclude that $a_0/(a_N(1 + \varepsilon))$ is a cube in F. Using the zeros of $a + \varepsilon c$, we conclude that $a_0(1 + \varepsilon)/a_N$ is a cube in F. So $(1 + \varepsilon)^2 = \varepsilon$ is a cube in F. This leads to a contradiction, because $\varepsilon = 1 + 1/\varepsilon$ is also the sum of two cubes.

Assume now that $2F = 0$ and F does not contain any ε as above. Then we consider a field extension $F[\varepsilon]$ with ε as above. Using the zeros of $a + \varepsilon b$, we conclude that $a_0/(a_N(1+\varepsilon))$ is a cube in $F[\varepsilon]$. Using the zeros of $a+\varepsilon c$, we conclude that $a_0(1+\varepsilon)/a_N$ is a cube in $F[\varepsilon]$. So $(1+\varepsilon)^2 = \varepsilon$ is a cube in $F[\varepsilon]$. We write $\varepsilon = (u+v\varepsilon)^3$ with $u, v \in F$. Then

$$0 = u^3 + uv^2 + v^3 = (u+v)^3 - u^2 v \quad \text{and} \quad 1 = u^2 v + uv^2 = uv(u+v),$$

hence $u^5 v^4 = 1$. Combining this with

$$0 = (1 + u^2 v + uv^2)^4 = 1 + u^8 v^4 + u^4 v^8,$$

we obtain that $1 + u^3 + u^{-6} = 0$, which contradicts the condition of the lemma.

Thus, $c_0 = 0$ in all cases. Now we use the zeros γ_k of $a+c$ to conclude that $a_0/a_N = ((-1)^N \prod b(\gamma_k))^3$ is a cube in F. □

Proof of Proposition 3. It was shown above that the condition of Lemma 1 implies that x is the sum of cubes of three polynomials of degree 1.

Assume now that there is no α, β, γ in $F = 6F$ such that $\alpha^3 + \beta^3 + \gamma^3 = 0 \neq \alpha\beta\gamma$. We want to prove that $\deg(d) \neq 1$ for any

$$d = a^3 + b^3 + c^3, \tag{6}$$

where $a, b, c \in F[x]$, provided that a, b, c are all of degree ≤ 4.

Let $\deg(a) \geq \deg(b) \geq \deg(c)$. Assume that $\deg(d) = 1$. Then it is clear that all three polynomials a, b, c cannot have the same degree. So $\deg(a) = \deg(b) > \deg(c)$. Dividing both sides of (6) by the cube of the leading coefficient of a, we can assume that a and $-b$ are monic, i.e., their leading coefficients are 1. Then the next two coefficients must also be equal. This leads immediately to a contradiction when the degree of a is ≤ 2.

Consider now the case when $\deg(a) = 3$ or 4. If $2F = F$, we set $u = a + b$ and $v = a - b$. The equation takes the form

$$u(u^2 + 3v^2)/4 + c^3 = d.$$

When $\deg(a) = 3$, we see that $\deg(c) = 2$, $\deg(u) = 0$, and $\deg(v) = 3$. We rewrite the equation as $v^2/4 + c^3/3u = d/3u - u^2/3 = d'$ with $\deg(d') = 1$. We write $v/2 = x^3 + v_2 x^2 + v_1 x + v_0$ and $c^3/3u = -(x^2 + c_1 x + c_0)^3$. Our equation takes the form

$$(x^3 + v_2 x^2 + v_1 x + v_0)^2 - (x^2 + c_1 x + c_0)^3 = d'.$$

Then we replace x by $x - v_2/3$ to make $v_2 = 0$. Looking at terms of degree 5, we conclude that $c_1 = 0$. Looking at terms of degree 3, we conclude that $v_0 = 0$. Looking at terms of degree 1, we obtain a contradiction. (The "*abc*-theorem" yields that $V^2 - C^3$ cannot be a nonzero constant for nonconstant polynomials V and C, if $\mathrm{char}(F) = 0$. It is not difficult to show that the necessary and sufficient condition on a field F for this conclusion to be true is that $6F = F$.)

Let now $\deg(a) = 4$ (and still $2F = F$). Since $\deg(v) = 4$, it is clear that $\deg(u) = 1$ and $\deg(c) = 3$. Now we set $y = 1/u$. After we divide both sides of (6) by u^9, it takes the form

$$3/4(v/u^4)^2 + (c/u^3)^3 = -y^6/4 + x/u^9.$$

Note that c/u^3 is a polynomial of degree 3 in y with a constant term $e' \neq 0$. After dividing the equation by e'^3, it takes the form

$$(1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4)^2 - (1 + c_1y + c_2y^2 + c + 3y^3)^3 = -y^6/4e'^3 + e''y^8 + e'''y^9 \quad \text{with } e', e'', e''' \text{ in } F \text{ and } e'e'' \neq 0. \tag{7}$$

Now we see that, modulo $y^6F[y]$,

$$(1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4)^2 \equiv (1 + c_1y + c_2y^2 + c_3y^3)^3,$$

hence

$$1 + v_1y + v_2y^2 + v_3y^3 + v_4y^4 \equiv (1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5)^3$$

and

$$1 + c_1y + c_2y^2 + c_3y^3 \equiv (1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5)^2$$

for some f_i in F. Set $f = 1 + f_1y + f_2y^2 + f_3y^3 + f_4y^4 + f_5y^5$.

Then, modulo $y^6F[y]$,

$$f^2 \equiv 1 + 2f_1y + (2f_2 + f_1^2)y^2 + (2f_3 + 2f_1f_2)y^3 + (2f_4 + 2f_1f_3 + f_2^2)y^4 + (2f_5 + 2f_1f_4 + 2f_2f_3)y^5$$

and

$$f^3 \equiv 1 + 3f_1y + (3f_2 + 3f_1^2)y^2 + (3f_3 + 6f_1f_2 + f_1^3)y^3 + (3f_4 + 3f_1^2f_2 + 6f_1f_3 + 3f_2^2)y^4 + (3f_5 + 6f_1f_4 + 6f_2f_3 + 3f_1^2f_3 + 3f_1f_2^2)y^5.$$

So

$$0 = c_4 = 2f_4 + 2f_1f_3 + f_2^2,$$
$$0 = c_5 = 2f_5 + 2f_1f_4 + 2f_2f_3,$$
$$0 = v_5 = 3f_5 + 3f_1f_2^2 + 6f_1f_4 + 6f_2f_3 + 3f_1^2f_3,$$

hence

$$f_4 + f_2^2/2 + f_3f_1 = 0, \quad f_5 - f_2^2f_1 - f_3f_1^2 = 0, \quad f_3f_2 + f_2^2f_1/2 = 0.$$

We set $C = 1 + 2f_1y + (2f_2 + f_1^2)y^2 + (2f_3 + 2f_1f_2)y^3$,

$$V = 1 + 3f_1y + (3f_2 + 3f_1^2)y^2 + (3f_3 + 6f_1f_2 + f_1^3)y^3 + (3f_4 + 3f_1^2f_2 + 6f_1f_3 + 3f_2^2)y^4,$$

and $S = V^2 - C^3$. Then (7) takes the form

$$S = V^2 - C^3 = S_5 y^5 + S_6 y^6 + S_7 y^7 + S_8 y^8 + S_9 y^9$$
$$= -y^6/4e'^3 + e'' y^8 + e''' y^9,$$

where the coefficients S_i of S are polynomials in f_j. Substituting $f_4 = -f_2^2/2 - f_{33} f_1$ and $f_5 = f_2^2 f_1 + f_3^3 f_1^2$ into S_i, we obtain:

$$S_9 = -8f_3^3 - 24f_3^2 f_2 f_1 - 24 f_3 f_2^2 f_1^2 - 8 f_2^3 f_1^3 = e''',$$
$$S_8 = -24 f_3^2 f_2 + 9f_2^4/4 - 39 f_3 f_2^2 f_1 - 3 f_3^2 f_1^2$$
$$- 15 f_2^3 f_1^2 - 6 f_3 f_2 f_1^3 - 3 f_2^2 f_1^4 = e'' \neq 0,$$
$$S_7 = -15 f_3 f_2^2 - 6 f_3^2 f_1 - 6 f_2^3 f_1 - 18 f_3 f_2 f_1^2 - 9 f_2^2 f_1^3 = 0,$$
$$S_6 = -3f_3^2 + f_2^3 - 18 f_3 f_2 f_1 - 9 f_2^2 f_1^2 = -1/4e'^3 \neq 0,$$
$$S_5 = -6 f_3 f_2 - 3 f_2^2 f_1 = 0.$$

Since $S_5 = 0$, we see that either $f_2 = 0$, or $2f_3 + f_1 f_2 = 0$.
In the first case,

$$(S_9, S_8, S_7, S_6) = (-8f_3^3, -3f_3^2 f_1^2, -6f_3^2 f_1, -3f_3^2),$$

and $S_7 = -6f_3^2 f_1 = 0$ implies that $S_8 = -3f_3^2 f_1^2 = 0$, so this case is impossible.
Assume now that $f_2 \neq 0 = 2f_3 + f_1 f_2$. Then

$$S_9 = -f_2^3 f_1^3, \qquad S_8 = 9f_2^4/4 - 3f_2^3 f_1^2/2 - 3f_2^2 f_1^4/4,$$
$$S_7 = 3f_2^3 f_1/2 - 3f_2^2 f_1^3/2, \qquad S_6 = f_2^3 - 3f_2^2 f_1^2/4.$$

From $S_7 = 3f_2^3 f_1/2 - 3f_2^2 f_1^3/2 = 0$, we obtain that either $f_1 = 0$, or $f_2 = f_1^2$. The latter case is impossible because then the coefficient S_7 vanishes. So let $f_1 = 0$.

Then the coefficients (S_9, S_8, S_7, S_6) become $(0, 9f_2^4/4, 0, f_2^3)$. So $-e'^3/4 = f_2^3$, hence 2 is a cube in F. Since 2 is also the sum of two cubes, we are done.

The case $2F = 0$, $\deg(a) = 3$ or 4 was done using the computer. When $\deg(a) = 3$, we consider (6) with $a = x^3 + a_2 x^2 + a_1 x + a_0$, $b = a + f$, $c = c_2 x^2 + c_1 x + c_0$, $d = \sum d_i x^i \neq 0$, $d_i = 0$ for $i \geq 2$.

It is clear that $\deg(f) = 0$. Replacing x by $x + c_2$, we make $c_2 = 0$. From $d_6 = f + c_2^3 = 0$, we get $f = c_2^3$. Substituting this into d, we get that $d_5 = c_1 c_2^2 = 0$. If $c_1 = 0$, then $d_3 = c_2^6 = 0$. Thus, $c_2 = 0$, hence $d_2 = c_0 c_1^2$ and $d_1 = c_0^2 c_1$. So $d_2 = 0$ implies that $d_1 = 0$.

The reader is spared from longer computations which were done in the case when $2F = 0$ and $\deg(a) = 4$. □

Remark. The identity

$$x = ((x^3 - 1/27)^3 + (x^2 + x/3)^3 + (-x^3 + x/3 + 1/27)^3)/(x^2 + x/3 + 1/9)^3$$

shows that $w_3(F(x)) \leq 3$ and $w_3(F) \leq 3$ for any field F with $\operatorname{char}(F) \neq 3$; if $\operatorname{char}(F) = 3$, then $w_3(F(x)) = w_3(F) = 1$.

Remarks. All the solutions of (6) with $\deg(a) = \deg(b) = 3$, $\deg(c) = 2$, $\deg(d) = 0$ can be obtained from

$$(1 + 6x^3)^3 + (1 - 6x^3)^3 + (-6x^2)^3 = 2$$

by a linear change of variable and scaling. There are no such solutions when $2F = 0$.

All the solutions of (6) with $\deg(a) = \deg(b) = 4$, $\deg(c) = 3$, $\deg(d) = 0$ can be obtained from

$$(9x^4)^3 + (3x - 9x^4)^3 + (1 - 9x^3)^3 = 1$$

by a linear change of variable, switching a and b, and scaling. P. Erdös pointed out to the author that the last two formulas can be found in [1].

All solutions of (6) with $\deg(a) = \deg(b) = 5$, $\deg(c) = 4$, $\deg(d) = 0$ or 1 were described above when $\operatorname{char}(F) = 13$. They do not exist when $\operatorname{card}(F) = 5, 7$. When $\operatorname{char}(F) = 11$, all such solutions can be obtained from the equalities

$$(2 + 2x^2 + 3x^3 + x^5)^3 - (2 + 5x + 6x^2 + 3x^3 + x^5)^3 + (x + 10x^2 + 5x^3 + x^4)^3 = 6x,$$

$$(1 + 4x + x^2 + 5x^3 + x^5)^3 - (9 + 6x + 5x^2 + 5x^3 + x^5)^3 + (8 + x + 2x^3 + x^4)^3 = 4,$$

$$(4 + 4x + 3x^2 + 8x^3 + x^5)^3 - (8 + 6x + 7x^2 + 8x^3 + x^5)^3 + (5 + 4x + 9x^2 + 2x^3 + x^4)^3 = 7$$

by a linear change of variable, switching a and b, and scaling.

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