The book under review treats mathematical properties of some function classes which are particularly useful for curve and surface *modelling*. It is a translation of a book originally published in Chinese some ten years earlier. The book can be read by anyone acquainted with calculus and linear algebra (but of course will be better appreciated by those with some knowledge of numerical analysis and approximation theory). It should be of special interest to both researchers and practitioners working with CAD.

The first chapter of the book is an introduction to the field. Except for one short chapter on surfaces, the remainder of the book deals with planar curves. The second chapter introduces splines, concentrating on the cubic and quartic cases. Chapters 3 and 4 discuss parametric and Bézier methods for fitting and modelling curves in the plane. Bicubic splines and Coon's and Bézier patch methods are treated in Chapter 5 on surfaces. The remaining three chapters deal with nonlinear splines (including the mechanical spline), with curve and net fairing, and with affine invariants of parametric curves.

Despite the fact that the book was written more than ten years ago, I think it will be a very useful supplement to more recent books on splines, curve and surface fitting, and computer-aided geometric design. In particular, there is a great deal of information about affine invariants and the characterization of inflection points and singular points (leading to loops and cusps) which is not treated in other books. In general, the authors have adhered to modern notation and terminology (referring to the control polygon as a characteristic polygon is one exception), and the translator has done an excellent job of rendering the text in very readable smooth English. The book has very few obvious misprints, and benefits from a large number of well-drawn figures.

L. L. S.

11[41-02, 41A60].—R. Wong, *Asymptotic Approximation of Integrals*, Computer Science and Scientific Computing, Academic Press, Boston, 1989, xiii+ 543 pp., $23\frac{1}{2}$ cm. Price $69.95.

J. E. Littlewood once remarked on the fading—with the growing emphasis on rigor—of the "aroma of paradox and audacity" that had pervaded the subject of divergent series—those curious expansions that were usually divergent but nevertheless from which information could be obtained. It is true that the subject now resides on a firm theoretical foundation, but for most of us the scent still clings to what for Abel was "the invention of the devil." Some of this mystery is shared by asymptotic expansions—typically a kind of divergent series—and probably constitutes a large part of their appeal: they represent a let-it-all-hang-out approach to mathematics, a swift uppercut to the fatiguing demands of rigor. Asymptotics constitutes a collection of powerful tools, and if we are sometimes a little careless in devising and using them, perhaps they provide us with what we secretly want: to be both naughty and effective.

There are still vital matters to be resolved in asymptotic analysis. At least one widely quoted theory, the asymptotic theory of irregular difference equations expounded by G. D. Birkhoff and W. R. Trjitzinsky [5, 6] in the early 1930's, is vast in scope; but there is now substantial doubt that the theory is correct in all its particulars. The computations involved in the algebraic theory alone (that is, the theory that purports to show there are a sufficient number of solutions which formally satisfy the difference equation in question) are truly mindboggling. (For some modern comments on this venerable work, see [26].) A large part of the literature dealing with the uses of asymptotic expansions, particularly the literature from the physical sciences, dispenses with rigor altogether. The mere fact that one or two terms of the expansion can somehow be produced is to be accepted as sufficient justification that the rest of the expansion exists and represents asymptotically the quantity in question. This approach, and the results it displayed, is sometimes called *formal*, although mathematicians whose philosophical view of mathematics puts them in the camp of formalism would strenuously object to the term. A more accurate description would be: lacking in rigor. Suffice it to say that very much of what we know about the physical world comes to us through the use or maybe even misuse of asymptotics. Asymptotics are—to paraphrase Bismarck's observation about laws and sausages—always in demand, but we'll sometimes sleep better if we don't know how they're made.

The expression *asymptotic analysis* often implies the process of obtaining a single term describing the behavior of the quantity in question. Obtaining an expression for the coefficients in the complete asymptotic expansion if, indeed, one exists, is in most physical problems simply too difficult.

Wong's book provides ingenious examples to remind us that a totally casual approach to asymptotics will not work as a general policy. Strange, even illogical, results may be a consequence of the most straightforward methods. Quixotic results can occur when one is dealing with asymptotic scales more general than the Poincaré scale, for instance. Sometime ago I gave the following example [53]. Let

$$\Gamma(1+t) = \sum_{n=0}^{\infty}(-1)^n a_n t^n,$$

the Taylor series converging for $|t| < 1$. It can be shown that

$$a_n - \sum_{j=0}^{k-1} \frac{(-1)^j}{j!(j+1)^{n+1}} = O[(k+1)^{-n}], \qquad n \to \infty,$$

in other words,

$$a_n \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{n+1}}, \qquad n \to \infty.$$

This is a perfectly valid asymptotic series; but not with the ordinary Poincaré scale $\{n^{-k}\}$, rather with scale $\{(k+1)^{-n}\}$. It is also a convergent series. *But it does not converge to* a_n.[1]

[1]This phenomenon almost never occurs with the Poincaré scale, since a sequence having a convergent asymptotic series in this scale may usually be considered a restriction to integer arguments of a function analytic at ∞.

Nonsense can result when one applies standard methods in a facile way, neglecting the error term. In his book Wong gives an example where two different widely used methods give incorrect results, both applied to the function

$$S(x) = \int_0^\infty \frac{1}{(x+t)(1+t)^{1/3}}\,dt, \qquad x > 0.$$

Repeated integration by parts gives us

$$S(x) \sim -\sum_{n=1}^{\infty} \frac{3^n(n-1)!}{2\cdot 5\cdot\cdots\cdot(3n-1)} x^{-n}, \qquad x \to \infty,$$

which is obviously incorrect, since $S(x)$ is positive. And termwise integration using the series

$$(1+t)^{-1/3} = \sum_{s=0}^{\infty} \binom{-1/3}{s} t^{-s-1/3},$$

also gives an incorrect (though less obviously so) answer

$$S(x) \sim \frac{2\pi}{\sqrt{3}} \sum_{s=0}^{\infty} (-1)^s \binom{-1/3}{s} x^{-s-1/3}.$$

The correct expansion is the *sum* of the two expansions above, and can be obtained by the use of distribution theory. (Another way of dealing with this integral is to let $t = u/(1-u)$ to make the identification

$$S(x) = \frac{3}{x}\,{}_2F_1\left(\begin{matrix}1,1\\4/3\end{matrix};\ 1-\frac{1}{x}\right),$$

and then use connecting formulas for the Gaussian hypergeometric function.)

Asymptotics are as invaluable in analysis as in the physical sciences. Almost all of what we know about the theory of polynomials orthogonal on the real line is due to our detailed asymptotic knowledge of the three-term recurrence that such polynomials satisfy, see [37, 51]. Analytic number theory is indebted to and has the debt of asymptotics; one of the most elegant methods—the circle method of Rademacher [2], for instance—was introduced to study the growth of arithmetical functions. The asymptotics of nonlinear recurrences are crucial in the resolution of important questions in orthogonal polynomials originally raised by G. Freud. (See [3] and the references given there, and the references in *those* references. This has been a hot topic recently! See also [32].)

Books on the subject have always had a highly personal flavor. Some, like those by the mathematical physicists Jeffreys and Heading [28] and [22], emphasize methods immediately applicable to the physical sciences, like the method of stationary phase, so important in wave mechanics. Copson's book [13], because of his interest in complex variables, stresses the method of steepest descents. The earlier books tended to treat specialized subjects. Ford [20] was concerned with the algebraic properties of asymptotic series, and the asymptotic behavior induced on an entire function when the form of its Taylor coefficients was prescribed.

The number of books seems to be growing exponentially. Obviously, publishers have found them a very marketable item. They have become increasingly specialized. Several recent books emphasize differential equations, either from the point of view of systems theory or turning point problems [19, 43, 47, 48, 52]. Some deal with abstruse and specialized topics, like applications of nonstandard analysis to fields of formal series, or to fiberings over manifolds [4, 27, 31]. Others treat the asymptotics of eigenvalues of Toeplitz matrices [9, 24], or of difference equations [26]. Asymptotics are becoming increasingly important in queuing theory, number theory, and statistics [1, 10, 14, 16, 25, 33]. One of the most exciting recent developments is the application of statistical methods for asymptotic estimation to problems in analysis. A good discussion of these techniques can be found in Van Assche's book [51], where he shows how a remarkably ingenious application of the local limit theorem for lattice distributions (for example, see Petrov [40]) serves to generate the asymptotics for many of the classical orthogonal polynomials. My experience with these methods has been less than completely satisfactory; I have found it difficult to aim the methods in the right direction, in other words, to define the probability space and the associated distribution functions so that the theory generates the desired asymptotics. In general, the method gives you what it chooses to give you, not necessarily what you want.

Matched asymptotic expansions has become a popular topic recently, with an inevitable brood of books [17, 29]. The literature swells when one considers that perturbation theory is nothing more than a disguised form of asymptotics (for ε read $1/n$ or $1/z$), special in its application to differential equations. The books [36, 38] are good examples of some of the recent publications in the field. For numerical applications see [23].

Some recent books so completely reflect an engineer's viewpoint, or are so mired in a specialized set of physical problems, that nonspecialists may well find them meaningless [8, 21, 35, 44, 46, 50, 54]. There is at least one book on the asymptotic analysis of ocean currents [45]! Almost every physical discipline—fluid mechanics, wave propagation, etc.—seems to have spawned its own set of ad hoc techniques for asymptotic analysis, some of them resting on fragile mathematical foundations. I have the feeling that the territories of these disciplines are never breached, and these scientists will go from one rediscovery to another, carrying—like Sisyphus—the same stone again and again up the hill.

Consequently, the need is greater now than ever before for balanced, comprehensive accounts of the progress we have made in asymptotic methods. For the reader interested in a general treatment, there are some superb books to choose among: [11, 15, 18, 22, 28, 30, 34, 39, 41, 49]. The reader will note that the books are getting larger, simply because the field has developed so rapidly. Jeffrey's 1962 book is a mere 140 pages, Heading's only 160; but Olver's, published in 1974, is 560 pages. Each of the books in the above list has its own virtues. Of these I believe Olver's is the most useful because it is current, it covers such an abundance of material and it is well organized and beautifully written.

Now the above are joined by another comprehensive reference, namely, the present work. The title suggests that it is just another treatise on a specialized topic. But that is far from the truth. Many problems in asymptotics having little apparent connection with integrals can be formulated in terms of integrals by the exercise of a little ingenuity, for instance, the asymptotics of the partition function $p(n)$ can be obtained by an application of the circle method to a contour integral representation for $p(n)$. The computations are made possible by the use of a q-function identity. Often an integral transform may be obtained for the quantity in question, and this integral may then be examined by a variety of techniques.

Books on asymptotics differ widely in their approach to rigor. The book by de Bruijn [11] has intermittent lapses of rigor, is occasionally reduced to hand waving and foot shuffling. The book by Erdélyi [18] is fastidiously rigorous. Yet both are compulsively readable. In this subject there seems to be little relationship between rigor and readability. The problems de Bruijn chooses to attack are uncommonly interesting, and his book has been very influential. In few other fields of mathematics, in fact, are the standards of exposition so imposing; and the book by Wong joins the best of these. Because he comes down soundly on the side of rigor, his arguments are more difficult than they would otherwise be, but the added complexity is mostly offset by the great clarity of the writing and organization. Wong's book is not the first to concentrate on the asymptotic estimation of integrals—see [7, 12, 42]—but it is, I think, by far the best.

Some of the material in Wong's book is covered in other books. But there are a number of ways that the book makes a unique and substantial contribution to the literature. The book starts with a chapter on the fundamentals of asymptotics. This chapter deserves high praise. It can be understood by a good calculus student; yet the examples are provocative. The reader will get an idea of the tenor of the writing from the following passage:

"In analysis and applied mathematics, one frequently comes across problems concerning the determination of the behavior of a function as one of its parameters tends to a specific value, or of a sequence as its index tends to infinity. The branch of mathematics that is devoted to the investigation of these types of problems is called asymptotics. Thus, for instance, results such as

$$\log n! \sim (n + 1/2)\log n - n + 1/2 \log 2\pi ,$$

$$H_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \log n ,$$

and

$$L_n \equiv \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt \sim \frac{4}{\pi^2}\log n ,$$

are all part of this subject."

Notice how the author has stated formulas before he has defined what "$\sim$" means. A reader unfamiliar with the field will infer from context what asymptotic equality means, and the introduction provides what is really needed in the beginning of any viable treatment of the subject: examples to motivate and intrigue. A complete array of definitions soon follows. Then the author discusses generalized asymptotic expansions and some basic techniques for deriving them, including integration by parts, the Euler-Maclaurin summation formula, and Watson's lemma.

The second chapter discusses the classical procedures for obtaining asymptotic representations from integrals, including Laplace's method, the principle of stationary phase, and Perron's method. There follows a welcome discussion of Darboux's method and the formula of Hayman, methods for deducing the asymptotics of the Taylor coefficients of a function, and not directly related to problems involving integrals.

In many instances the function whose asymptotics are desired has a representation as an integral transform. Much of what has been accomplished on this problem is due to the author, and the discussion in Chapters 3 and 4 constitutes the most extensive resource in the literature for dealing with Mellin, Fourier and Hankel transforms.

Chapter 5 treats the theory of distributions. Since many applied mathematicians are still unfamiliar with this material, the author provides a self-contained treatment—one of the most readable treatments I have seen. (It could easily be used as a text in a discussion of the subject in a real variable course.) The results obtainable with the distributional approach are many and varied, and every applied mathematician should consider such methods part of his set of tools. In Chapter 6 the author applies the previous theory to Stieltjes transforms, Hilbert transforms, Laplace and Fourier transforms, and fractional integrals. Although Wong makes no special claims for himself in having initiated the use of distributional tools and attributes much of the work in this chapter to writers such as Lighthill, Jones, Durbin and Zayed, he has, in fact, been a very productive researcher in the area, particularly in association with J. P. McClure.

Chapter 7 treats the difficult problem of obtaining uniform asymptotic expansions, for example, in the case where the integrand has coalescing saddle points. The most general kind of an integral which occurs is

$$I(\lambda) = \int_C g(t; \alpha_1, \alpha_2, \dots, \alpha_m) e^{-\lambda f(t; \alpha_1, \alpha_2, \dots, \alpha_m)} \, dt.$$

It is an extremely difficult task to obtain asymptotic expansions in λ which are uniform in the α_i's. The author discusses several simple (but important) situations, namely those in which two or three of the critical points of the integrand (zeros of f') are allowed to approach each other. A number of illuminating examples are given, including applications to Bessel functions and Laguerre polynomials. Despite the usefulness of results due to Chester, Ursell, Friedman, Olver, Erdélyi, and others, it must be admitted that the derivation of uniform expansions is still as much of an art as a science; most situations have to be treated on an ad hoc basis, and the analysis of even simple examples usually involves a major computational effort. The discussion in the present book, along with the relevant portions of Olver's book (where the problem is formulated in terms of differential equations), constitutes a major addition to the literature.

Double and higher-dimensional integrals are treated in Chapters 8 and 9. In Chapter 8 the behavior of double integrals of the form

$$I(\lambda) = \iint_D g(x, y) e^{i\lambda f(x, y)} \, dx \, dy$$

is investigated. One wishes such integrals and all their associated complexities—degenerate cases, boundary critical points, curves consisting entirely of stationary points—would just go away. But unfortunately such integrals are crucial in many disciplines, including, nowadays, combinatorics. Higher-dimensional integrals are discussed in Chapter 9; one of the most frequently occurring cases is that of multiple Fourier transforms. This is the first time the material has been treated fully in a reference work, and it is very welcome. For a number of years I have been hounded by engineer colleagues who wanted asymptotic analyses of multidimensional integrals. And now I have a source of nearly one hundred pages on the subject to refer them to.

The book has many virtues, as I have mentioned. One of the most striking is one of the least expected: the exercises. No other book, past or present, can approach Wong's book in its profusion of carefully selected exercises. The exercises vary from the quick and easy to the research-problem variety. Where the going gets tough (regularization of divergent integrals, tempered distributions) the number of exercises actually increases, rather than decreases. Only a writer profoundly familiar with his field could lavish such pedagogical goodies on his readers.

The book is beautifully bound and printed, in the best standards of Academic Press, and replete with a sixteen-page bibliography, a list of symbols (highly useful), and a thorough index. Every applied mathematician (indeed most physical scientists) will want to own this book. There is simply no other book that does as much as it does as well as it does.

JET WIMP

Department of Mathematics and Computer Science
Drexel University
Philadelphia, Pennsylvania 19104

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This is a Pascal version of the original *Numerical Recipes*, published in 1986 and reviewed in [1]. To quote from the authors' Preface: "Pascal was not,