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A HAMILTONIAN APPROXIMATION TO SIMULATE SOLITARY WAVES OF THE KORTEWEG-DE VRIES EQUATION

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ABSTRACT. Given the Hamiltonian nature and conservation laws of the Korteweg-de Vries equation, the simulation of the solitary waves of this equation by numerical methods should be effected in such a way as to maintain the Hamiltonian nature of the problem. A semidiscrete finite element approximation of Petrov-Galerkin type, proposed by R. Winther, is analyzed here. It is shown that this approximation is a finite Hamiltonian system, and as a consequence, the energy integral

$$I(u) = \int_0^1 \left(\frac{u_x^2}{2} + u^3\right) dx$$

is exactly conserved by this method. In addition, there is a discussion of error estimates and superconvergence properties of the method, in which there is no perturbation term but instead a suitable choice of initial data. A single-step fully discrete scheme, and some numerical results, are presented.

1. THE HAMILTONIAN NATURE AND CONSERVATION LAWS

In this paper, we shall consider the following problem for the Korteweg-de Vries equation:

$$\text{(P)} \qquad \begin{aligned} &u_t - 6uu_x + u_{xxx} = 0, \qquad x \in R, \ t \geq 0, \\ &u(x+1, t) = u(x, t), \\ &u(x, 0) = u_0(x) \quad \text{(a prescribed 1-periodic function).} \end{aligned}$$

To study the Hamiltonian nature of problem (P), we introduce the following function space with $I = [0, 1]$,

$$H_p^m = \{v \in H^m(I);\, v^{(i)}(x+1) = v^{(i)}(x),\ i = 0, 1, \dots, m-1\},$$

and the functional

$$H(u) = \int_0^1 \left(\frac{u_x^2}{2} + u^3\right) dx,$$

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where $u^{(i)} = \partial^i u/\partial x^i$. Define

$$\delta/\delta u := \sum_{k=0}^{\infty}(-1)^k (d/dx)^k \partial/\partial u^{(k)};$$

then $\delta H/\delta u = 3u^2 - u_{xx}$, and problem (P) is equivalent to finding a map $u(t)$ from $\mathbf{R}^+$ to H_p^m such that

$$(\mathrm{P}') \qquad u_t = J\delta H/\delta u, \qquad J = \partial/\partial x.$$

Since

$$\int_0^1 \frac{\delta H}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta H}{\delta u}\right) dx = 0, \qquad u \in H_p^m,$$

then for any solution $u = u(t)$ of (P') we have

$$\frac{dH(u)}{dt} = \int_0^1 \frac{\delta H}{\delta u}\frac{\partial u}{\partial t} dx = \int_0^1 \frac{\delta H}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u} dx = 0,$$

i.e., $u = u(t)$ satisfies the energy conservation law: $H(u(t)) = \text{const}$.

For any functionals T and $S: H_p^m \to \mathbf{R}$, define

$$\{T, S\} := \int_0^1 \frac{\delta T}{\delta u}\frac{\partial}{\partial x}\frac{\delta S}{\delta u} dx \quad \text{(Poisson bracket)},$$

which also is a functional defined on H_p^m. It can be verified that the operation $\{\ ,\ \}$ has the following properties:

(i) $\{T, S\} = -\{S, T\}$, $T, S: H_p^m \to \mathbf{R}$;
(ii) $\{H, aT + bS\} = a\{H, T\} + b\{H, S\}$, $a, b \in R$, $H, T, S: H_p^m \to \mathbf{R}$;
(iii) (Jacobi identity) $\{\{T, S\}, H\} + \{\{S, H\}, T\} + \{\{H, T\}, S\} = 0$, $H, T, S: H_p^m \to \mathbf{R}$.

Lemma 1. *The functional $T(u)$ is a first integral of problem (P') if and only if $\{T, H\} = 0$.*

Proof. Since, for any solution $u = u(t)$ of (P'),

$$\frac{dT(u)}{dt} = \int_0^1 \frac{\delta T}{\delta u}\frac{\partial u}{\partial t} dx = \int_0^1 \frac{\delta T}{\delta u}\frac{\partial}{\partial x}\frac{\delta H}{\delta u} dx = \{T, H\},$$

the lemma follows immediately from this identity. □

For a given functional $H: H_p^m \to \mathbf{R}$, a family of mappings G_H^t containing a parameter t can be determined through (P'):

$$u(t) = G_H^t u_0, \qquad u_0 \in H_p^m,$$

which is called the *phase flow* corresponding to H. By Lemma 1 and the Jacobi identity, we have

Theorem 1. *Suppose T and S are two first integrals of* (P'). *Then* $\{T, S\}$ *is also a first integral of* (P'). *Therefore, the set of functionals consisting of all first integrals of* (P'), *equipped with the operation* $\{\ ,\ \}$, *forms a Lie algebra* R_H.

Let $L_u = -\partial^2/\partial x^2 + u$ (Schrödinger's operator). P. D. Lax proved in [5] that every eigenvalue $\lambda = \lambda(u)$ of the Sturm-Liouville problem $L_u f = \lambda f$ is a first integral of (P'), i.e., $\lambda(u) \in R_H$. In fact, (P') has infinitely many first integrals, such as

$$I_0(u) = \int_0^1 u\,dx, \quad I_1(u) = \int_0^1 u^2\,dx, \quad I_2(u) = \int_0^1 \left(\frac{u_x^2}{2} + u^3\right) dx, \ldots .$$

From the form (P') and the properties indicated above we see that problem (P) is of the same nature as a Hamiltonian system of ordinary differential equations (see [1, Chapter 8]), which can be viewed as an infinite-dimensional Hamiltonian system. For a given functional $H: H_p^m \to \mathbf{R}$, we call $J\delta H/\delta u$ the *velocity vector* of the phase flow G_H^t with Hamiltonian function H. For any $I_s \in R_H$, the phase flow determined by the equation $u_t = J\delta I_s/\delta u$ commutes with G_H^t, i.e., $G_H^t G_{I_s}^t = G_{I_s}^t G_H^t$.

2. The Hamiltonian approximation of problem (P)

In this paper we seek to develop a numerical method for simulating the solitary waves of the Korteweg-de Vries equation which maintains the Hamiltonian nature of this equation. We believe that such a method will be able to preserve as much as possible the global properties of the original problem, for example, the energy conservation property

$$\frac{dH(u)}{dt} = \frac{d}{dt}\int_0^1 \left(\frac{u_x^2}{2} + u^3\right) dx = 0,$$

which we consider to be particularly important. As is known, the conventional finite difference method (see [7]) and the Galerkin finite element method (see [8]) do not preserve the energy. In this section, we shall show that the Petrov-Galerkin finite element discretization is an appropriate way to derive a numerical method for problem (P) which faithfully preserves the Hamiltonian nature and the energy conservation property of the continuous problem.

Let $L_h: 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of the interval $I = [0, 1]$, $I_j = [x_{j-1}, x_j]$, and $h = \max_{1\le j\le N}(x_j - x_{j-1})$. For a given integer $r \ge 2$, we introduce the spaces

$$V_h = \{v \in H_p^1;\ v|_{I_j} \in P_r(I_j),\ j = 1, 2, \ldots, N\},$$

$$H_h = \{w \in H_p^2;\ w|_{I_j} \in P_{r+1}(I_j),\ j = 1, 2, \ldots, N\},$$

where $P_r(I_j)$ represents the set of all polynomials on I_j with degree $< r$. It is easy to see that $\dim V_h = \dim H_h = (r-1)N$.

Based on the chosen pair of spaces V_h and H_h, the Petrov-Galerkin finite element approximation of problem (P) is defined as follows: find a map $u^h(t)$ from $\mathbf{R}^+$ to V_h such that

$$(\mathrm{P}_h) \qquad (u_t^h, w^h) + 3((u^h)^2, w_x^h) + (u_x^h, w_{xx}^h) = 0 \quad \forall w^h \in H_h.$$

Here and hereafter, $(\ ,\ )$ and $\|\cdot\|$ stand for the inner product and the norm in $L_2(I)$, respectively.

For the purpose of the subsequent analysis, we introduce a linear integration operator $G: H_p^m \to H_p^{m+1}$ uniquely determined by

$$(2.1) \qquad (Gf)_x = f - f^0, \quad (Gf)^0 = f^0, \qquad f \in H_p^m,$$

where $f^0 = (f, 1)$ is the mean value of f on the interval I. In fact, Gf has the following explicit form:

$$(Gf)(x) = \int_0^x f(s)\,ds - f^0 x + \frac{3}{2} f^0 - \int_0^1 \int_0^x f(s)\,ds\,dx.$$

From the definition of G, we see that

$$(2.2) \qquad (Gf_1, f_2) = (Gf_1, (Gf_2)_x) + f_1^0 f_2^0,$$

$$(2.3) \qquad (Gf, f) = (f^0)^2.$$

Moreover, with $\overset{\circ}{H}{}_p^m = \{v \in H_p^m;\ v^0 = (v, 1) = 0\}$ and $\overset{\circ}{V}_h = V_h \cap \overset{\circ}{H}{}_p^1$, we have

$$(2.4) \qquad (Gf_1, f_2) = (Gf_1, (Gf_2)_x) = -(f_1, Gf_2) \quad \text{for any } f_1, f_2 \in \overset{\circ}{H}{}_p^m,$$

i.e., G is a skewsymmetric operator on $\overset{\circ}{H}{}_p^m$. It can be verified that G is a one-to-one map from $\overset{\circ}{H}{}_p^m$ onto $\overset{\circ}{H}{}_p^{m+1}$, and its inverse is precisely the differential operator $J = \partial/\partial x$.

Theorem 2. *The solution* $u^h = u^h(t)$ *of the semidiscrete problem* (P_h) *satisfies the following conservation laws*:

$$\text{(i)} \quad I_0(u^h(t)) = \int_0^1 u^h\,dx = \text{const} \quad \textit{for } t \geq 0,$$

$$\text{(ii)} \quad I_2(u^h(t)) = \int_0^1 \left(\frac{(u_x^h)^2}{2} + (u^h)^3 \right) dx = \text{const} \quad \textit{for } t \geq 0.$$

Proof. Since $1 \in H_h$, by choosing $w^h = 1$ in (P_h), we have

$$\frac{d}{dt} \int_0^1 u^h\,dx = \frac{d}{dt}(u^h, 1) = (u_t^h, 1) = 0,$$

so that (i) holds. To verify (ii), we choose $w^h = Gu_t^h \in H_h$; then

$$(u_t^h, Gu_t^h) + 3((u^h)^2, (Gu_t^h)_x) + (u_x^h, (Gu_t^h)_{xx}) = 0.$$

Since $(u_t^h)^0 = (u_t^h, 1) = 0$, $(u_t^h, Gu_t^h) = 0$, and $(Gu_t^h)_x = u_t^h$, because of (2.1) and (2.4), we obtain from the above equation

$$\frac{d}{dt}I_2(u^h(t)) = \frac{d}{dt}\left\{((u^h)^3, 1) + \frac{1}{2}(u_x^h, u_x^h)\right\} = 0,$$

i.e., (ii) holds, and the theorem is proved. □

Theorem 2 tells us that the conservation laws $I_0 = \text{const}$ and $I_2 = \text{const}$ of problem (P) mentioned in §1 are faithfully preserved by the Petrov-Galerkin finite element approximation (P_h), where $I_2 = H$ represents the energy of the continuous system (P).

It is not difficult to see that the discrete problem (P_h) is a system of ordinary differential equations. After some careful manipulations, we find that (P_h) is precisely a finite Hamiltonian system. To show this, we introduce a kind of second-order discrete derivative $d_{xx}^h u^h \in V_h$ for any given function u^h in V_h, which is uniquely determined by

$$(d_{xx}^h u^h, v^h) = -(u_x^h, v_x^h) \quad \forall v^h \in V_h.$$

By choosing $v^h = 1$, we see that $(d_{xx}^h u^h, 1) = 0$, i.e., $d_{xx}^h u^h \in \mathring{V}_h = V_h \cap \mathring{H}_p^1$. Now let $u^h = u^h(t)$ be a solution of problem (P_h). Since $d_{xx}^h u^h, u_t^h \in \mathring{V}_h$, by using (2.1) and (2.2), equation (P_h) can be rewritten in the form

$$(2.5) \qquad (Gu_t^h, v^h) - 3((u^h)^2, v^h) + (d_{xx}^h u^h, v^h) = 0, \qquad v^h \in \mathring{V}_h.$$

In addition, let P_0 be the L_2 projector from $L_2(I)$ into its subspace $\mathring{V}_h$, and let $G_h := P_0 G$; then for any $f^h, g^h \in \mathring{V}_h$,

$$(G_h f^h, g^h) = (P_0 G f^h, g^h) = (G f^h, g^h) = -(f^h, G g^h) = -(f^h, G_h g^h),$$

which shows that G_h is a skewsymmetric operator on $\mathring{V}_h$. In terms of these notations, we find that (2.5) is equivalent to

$$G_h(u^h)_t = 3P_0(u^h)^2 - d_{xx}^h u^h.$$

It can be verified by calculation that $3P_0(u^h) - d_{xx}^h u^h = \delta H(u^h)/\delta u^h$. Therefore, the solution $u^h(t)$ of (P_h) satisfies

$$(2.6) \qquad G_h(u^h)_t = \delta H(u^h)/\delta u^h.$$

Assume that $P_0 \mathring{H}_h = \mathring{V}_h$; then G_h restricted to $\mathring{V}_h$ is a one-to-one mapping, and the inverse $G_h^{-1} = J_h$ exists, which also is a skewsymmetric operator on $\mathring{V}_h$. We thus obtain a new version of (P_h),

$$(2.7) \qquad (u^h)_t = J_h \delta H(u^h)/\delta u^h.$$

For any two functionals $T, S: V_h \to R$, a discrete analogue of the Poisson bracket, introduced in §1, can be defined by

$$\{T, S\} := \int_0^1 \frac{\delta T}{\delta u^h} J_h \frac{\delta S}{\delta u^h} dx,$$

and most of the analysis and conclusions in [5] can be carried over to the approximation problem (P_h). Comparing the form (2.7) of problem (P_h) with (P'), we see that the Hamiltonian nature of problem (P) is maintained in the discrete approximation (P_h). For this reason, we shall call (P_h) a Hamiltonian approximation of problem (P).

3. Error estimates and superconvergence of the approximate solution

The discrete approximation (P_h) is identical to one of the methods proposed in [9], where H^0 and H^1 estimates for the error $e = u - u^h$ and its time derivative e_t were derived. However, in the bound obtained for e_t there exists an unknown term $\|Gw_t^h(0)\|_2$. In order to achieve superconvergence, D. N. Arnold and R. Winther in [2] altered the discrete equation by a perturbation term. In this section, we obtain superconvergence properties of the unperturbed equation (P_h) by suitable choices of the initial data.

Since $G(H_p^1) = H_p^2$ and $G(V_h) = H_h$, problem (P_h) can be formulated as follows: find a map $u^h(t): [0, T] \to V_h$ such that

$$-(Gu_t^h, v^h) + 3((u^h)^2, v^h) + a_0(u^h, v^h) = 0 \quad \forall v^h \in \mathring{V}_h, \tag{3.1}$$

where $a_0(u, v) = (u_x, v_x)$ and $u^h(0)$ assumes a prescribed value in V_h. In order to be sure that the problem has a unique solution, we assume $P_0\mathring{H}_h = \mathring{V}_h$; then the coefficient matrix in front of the time derivative in (3.1) is nonsingular.

An elliptic projector $P_1: H_p^1 \to V_h$ is defined by

$$a_0(P_1\phi - \phi, v^h) = 0 \quad \text{for any } v^h \in V_h,$$
$$(P_1\phi, 1) = (\phi, 1).$$

Let $u(t) = u(x, t)$ be the exact solution of (P), which is assumed to be sufficiently smooth. From standard results for the Galerkin finite element method for elliptic equations, we know that

$$\|(P_1u - u)^{(k)}(t)\|_s \le C(u)h^{r-s}, \qquad -(r-2) \le s \le 1, \; k \ge 0, \tag{3.2}$$

$$\|(P_1u - u)(t)\|_{L_\infty(I)} \le C(u)h^r, \tag{3.3}$$

where $\phi^{(k)}(t) = (\frac{d}{dt})^k\phi(t)$. Moreover, the following superconvergence estimate at nodes holds (see [6]):

$$|(P_1u - u)(x_i, t)| \le C(u)h^{2r-2} \quad \text{when } r > 2. \tag{3.4}$$

Here and hereafter, $\|\cdot\|_s$ represents the norm in the Sobolev space $H^s(I)$, $s \geq 0$, and

$$\|\cdot\|_{-s} = \sup_{0 \neq v \in H^s} \frac{(\cdot, v)}{\|v\|_s}.$$

In the subsequent analysis, we shall use the inverse properties of $\{V_h\}$, such as

$$\|v^h\|_1 \leq Ch^{-1}\|v^h\| \quad \forall v^h \in V_h.$$

It is well known that such properties can be guaranteed by assuming the family $\{L_h, \ h > 0\}$ of partitions to be quasi-uniform, i.e., there is a constant $c > 0$ such that $h_j = x_j - x_{j-1} \geq ch$ for $1 \leq j \leq N$.

To begin with, we discuss the case $u^h(0) = P_1 u(0)$ and prove the following pointwise error estimates.

Theorem 3. *Suppose that* (P) *has a unique solution $u(t)$ for $0 \leq t \leq T$, $u(t)$ is sufficiently smooth, and $\{L_h, h > 0\}$ is quasi-uniform. Assume $u^h(0) = P_1 u(0)$. Then for small $h > 0$, the discrete problem* (P_h) *has a unique solution $u^h(t)$, $0 \leq t \leq T$, which satisfies*

$$\|u(t) - u^h(t)\|_{L_\infty(I)} \leq C(u)h^r, \tag{3.5}$$

$$|u(x_i, t) - u^h(x_i, t)| \leq C(u)h^{r+d}, \qquad i = 1, 2, \ldots, N, \tag{3.6}$$

where $d = 0$ for $r = 2$, and $d = 1$ for $r > 2$.

Proof. Set $z(t) = u(t) - P_1 u(t)$ and $w^h(t) = P_1 u(t) - u^h(t)$. Then $e(t) = u(t) - u^h(t) = z(t) + w^h(t)$, where $w^h(t) \in \mathring{V}_h$ satisfies

$$-(Gw_t^h, v^h) + a_0(w^h, v^h) = (Gz_t, v^h) + 3((u^h)^2 - u^2, v^h) \quad \forall v^h \in \mathring{V}_h. \tag{3.7}$$

Since $(Gw_t^h, w_t^h) = 0$, choosing $v^h = w_t^h$ in (3.7) yields

$$\frac{1}{2}\frac{d}{dt}\|w_x^h\|^2 = (Gz_t, w_t^h) + 3((u^h)^2 - u^2, w_t^h).$$

Noting that $(u^h)^2 - u^2 = (w^h)^2 - 2(P_1 u)w^h - (P_1 u + u)z$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_x^h\|^2 &= \frac{d}{dt}(Gz_t, w^h) - (Gz_{tt}, w^h) \\ &\quad + \frac{d}{dt}[((w^h)^3, 1) - 3((P_1 u)w^h, w^h) - 3((P_1 u + u)z, w^h)] \\ &\quad + 3((P_1 u_t)w^h, w^h) + 3((P_1 u + u)z_t + (P_1 u_t + u_t)z, w^h). \end{aligned} \tag{3.8}$$

Without loss of generality we may assume

$$\|w^h(t)\|_1 \leq 1 \quad \text{for } 0 \leq t \leq T.$$

In fact, this assumption can be removed by the later estimates combined with the inverse inequalities in V_h (see [8]). By the smoothness of $u(t)$ and estimate

(3.2), $\|P_1 u\|_1$ and $\|P_1 u_t\|_1$ are uniformly bounded for $0 < h < h_0$ in $0 \le t \le T$. Note that $w^h(0) = 0$ by the choice of $u^h(0)$. Integrating (3.8) from 0 to t, we obtain in the usual way

$$\begin{aligned} \|w_x^h(t)\|^2 \le C \Big\{ & \|z(t)\|_{-1}^2 + \|z^{(1)}(t)\|_{-1}^2 + \|Gw^h(t)\|_1^2 \\ & + \int_0^t [\|z(s)\|_{-1}^2 + \|z^{(1)}(s)\|_{-1}^2 + \|z^{(2)}(s)\|_{-1}^2 \\ & + \|Gw^h(s)\|_2^2]\, ds \Big\}, \end{aligned} \tag{3.9}$$

where C is a constant which does not depend on h, but depends on u and its derivatives.

To derive an estimate for $Gw^h(t)$, we choose $v^h = P_0 Gw^h$ in (3.7) and obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|P_0 Gw^h\|^2 &= a_0(w^h, P_0 Gw^h) - (Gz_t, P_0 Gw^h) - 3((u^h)^2 - u^2, P_0 Gw^h) \\ &\le C(\|z\|_{-1}^2 + \|z^{(1)}\|_{-1}^2 + \|Gw^h\|_2^2). \end{aligned}$$

Thus, by integration we have

$$\|P_0 Gw^h(t)\|^2 \le C \int_0^t [\|z(s)\|_{-1}^2 + \|z^{(1)}(s)\|_{-1}^2 + \|Gw(s)\|_2^2]\, ds$$

and

$$\begin{aligned} \|Gw^h(t)\|^2 &\le 2\|P_0 Gw^h(t)\|^2 + 2\|(I - P_0)Gw^h(t)\|^2 \\ &\le C\Big\{ h^4 \|Gw^h(t)\|_2^2 + \int_0^t [\|z(s)\|_{-1}^2 + \|z^{(1)}(s)\|_{-1}^2 \\ &\qquad + \|Gw^h(s)\|_2^2]\, ds \Big\}. \end{aligned} \tag{3.10}$$

Since $\|w^h\|^2 \le \|Gw^h\|\,\|w_x^h\| + \|Gw^h\|^2$, combining (3.9) and (3.10) and applying Gronwall's lemma, we find for $h > 0$ small enough,

$$\begin{aligned} \|Gw^h(t)\|_2^2 \le C\Big\{ & \|z(t)\|_{-1}^2 + \|z^{(1)}(t)\|_{-1}^2 \\ & + \int_0^t [\|z(s)\|_{-1}^2 + \|z^{(1)}(s)\|_{-1}^2 + \|z^{(2)}(s)\|_{-1}^2]\, ds \Big\}, \end{aligned}$$

which shows by (3.2) that

$$\|Gw^h(t)\|_2 \le C(u) h^{r+d}, \tag{3.11}$$

where $d = 0$ for $r = 2$, and $d = 1$ for $r > 2$. In view of

$$\|w^h(t)\|_{L_\infty(I)} \le C\|Gw^h(t)\|_2,$$

the desired estimates (3.5) and (3.6) can be derived from (3.11) combined with (3.3) and (3.4), respectively. □

From estimate (3.6), we see that the approximate solution has a superconvergence property at the nodes, with one order higher when $r > 2$. Following a referee's suggestion, we now improve this result. We shall use the technique of quasi-projection, introduced in [3] for linear second-order parabolic and hyperbolic equations. In [2], quasi-projection was used for the Korteweg-de Vries equation. Since we intend to conserve the energy integral and the Hamiltonian nature, we use this technique only for choosing a suitable initial data, unlike [2], where the discrete equation is altered.

Set $V(t) = P_1 u(t)$, $Z_0(t) = u(t) - V(t)$, and $W_0^h(t) = V(t) - u^h(t)$. The quasi-projections $Z_j(t): [0, T] \to \mathring{V}_h$, $j = 1, 2, \ldots$, are defined inductively by

$$a_0(Z_j, v^h) = (GZ_{j-1}^{(1)} - 6uZ_{j-1}, v^h) \quad \forall v^h \in \mathring{V}_h, \; 0 \le t \le T.$$

We shall use the sum $Z_1(0) + Z_2(0) + \cdots + Z_m(0)$ to modify the previous initial data $V(0) = P_1 u(0)$, i.e., we choose $u^h(0) = V(0) - [Z_1(0) + Z_2(0) + \cdots + Z_m(0)]$, where $m = [(r-1)/2]$.

The improved superconvergence result is then as follows:

Theorem 4. *Assume* (P) *and* $\{L_h, h > 0\}$ *to be as in Theorem* 3 *and* $u^h(0) = V(0) - [Z_1(0) + Z_2(0) + \cdots + Z_m(0)]$, $m = [(r-1)/2]$. *Then for* $h > 0$ *small enough, the approximate solution* $u^h(t)$ *satisfies*

$$|u(x_i, t) - u^h(x_i, t)| \le C(u) h^{2r-2}, \qquad i = 1, 2, \ldots, N. \tag{3.12}$$

To illustrate, let $r = 4$; then $m = 1$ and $u^h(0) = V(0) - Z_1(0)$. The calculation of $u^h(0)$ requires three projections $V(0)$, $(Z_0)_t(0)$, and $Z_1(0)$, where $(Z_0)_t(0) = u_t(0) - V_t(0)$ and $V_t(0)$ is a solution of

$$a_0(V_t(0), v^h) = (Gu_{tt}(0) - 6u(0)u_t(0), v^h), \qquad v^h \in \mathring{V}_h.$$

The extra cost spent on calculating $V_t(0)$ and $Z_1(0)$ will be compensated by a convergence rate of order $O(h^6)$.

Now we sketch the proof of Theorem 4.

Let $Z(t) = \sum_{j=0}^m Z_j(t)$ and $W^h(t) = W_0^h(t) - \sum_{j=1}^m Z_j(t)$. Then

$$e(t) = u(t) - u^h(t) = Z_0(t) + W_0^h(t) = Z(t) + W^h(t),$$

where $W_0^h(t), W^h(t) \in \mathring{V}_h$. It is not difficult to see that $W_0^h(t)$ and the sum of $Z_j(t)$, $j = 1, 2, \ldots, m$, satisfy respectively the following two equations,

$$-(G(W_0^h)^{(1)} - 6uW_0^h, v^h) + a_0(W_0^h, v^h) = (GZ_0^{(1)} - 6uZ_0 + 3e^2, v^h)$$

and

$$-\left(G\left(\sum_{j=1}^m Z_j\right)^{(1)} - 6u\sum_{j=1}^m Z_j, v^h\right) + a_0\left(\sum_{j=1}^m Z_j, v^h\right)$$
$$= (GZ_0^{(1)} - 6uZ_0, v^h) - (GZ_m^{(1)} - 6uZ_m, v^h).$$

Thus, by subtraction we derive an equation for $W^h(t)$,

(3.13) $-(G(W^h)^{(1)} - 6uW^h, v^h) + a_0(W^h, v^h) = (GZ_m^{(1)} - 6uZ_m + 3e^2, v^h).$

By the assumption on $u^h(0)$, we have $W^h(0) = 0$.

The proof of (3.12) consists of estimating $Z(t)$ and $W^h(t)$.

Lemma 2. *Let $s \geq -1$ and $k, j \geq 0$ be integers such that $2j+s \leq r-2$. Then*

$$\|Z_j^{(k)}(t)\|_{-s} \leq C(u)h^{r+2j+s}, \qquad 0 \leq t \leq T, \tag{3.14}$$

$$|Z_j(x_i, t)| \leq C(u)h^{2r-2}, \qquad j = 1, 2, \dots, m;\ i = 1, 2, \dots, N. \tag{3.15}$$

These estimates may be proved by an argument as in [2] or [3], with some obvious changes.

The next step is to show

$$\|W^h(t)\|_1 \leq C(u)h^{2r-2}, \qquad 0 \leq t \leq T. \tag{3.16}$$

Then the proof of (3.12) will be completed by (3.4), (3.15), and (3.16). We first choose $v^h = (W^h)_t$ in (3.13) to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|W_x^h\|^2 = & -3\frac{d}{dt}(uW^h, W^h) + 3(u_tW^h, W^h) \\ & + \frac{d}{dt}(GZ_m^{(1)} - 6uZ_m, W^h) \\ & - (GZ_m^{(2)} - 6uZ_m^{(1)} - 6u_tZ_m, W^h) + 3(e^2, (W^h)_t). \end{aligned} \tag{3.17}$$

In addition to (3.17), by choosing $v^h = P_0GW^h$ in (3.13) and integrating this equation from 0 to t, we get

$$\begin{aligned} &\|P_0GW^h(t)\| \\ &\leq C\int_0^t \left(\|W^h(s)\|_1^2 + \sum_{k=0}^{1} \|Z_m^{(k)}(s)\|_{-1}^2 + \|e(s)\|^2\|e(s)\|_1^2 \right) ds. \end{aligned} \tag{3.18}$$

For lack of available bounds for $(W^h)_t$ and e_t, we treat the nonlinear term $3(e^2, (W^h)_t)$ of (3.17) in the following way:

$$\begin{aligned} 3(e^2, (W^h)_t) &= 3(Z^2 + 2ZW^h + (W^h)^2, (W^h)_t) \\ &= \frac{d}{dt}[3(Z^2, W^h) + 3(ZW^h, W^h) + ((W^h)^3, 1)] \\ &\quad - 6(ZZ_t, W^h) - 3(Z_tW^h, W^h). \end{aligned}$$

As in the proof of Theorem 3, we may assume $\|W^h(t)\|_1 \leq 1$, $0 \leq t \leq T$; then

$|((W^h(t))^3, 1)| \le C\|W^h(t)\|^2$. Integrating (3.17), we obtain

$$\|W_x^h(t)\|^2 \le C\Big\{\|W^h(t)\|^2 + \sum_{k=0}^{1}\|Z_m^{(k)}(t)\|_{-1}^2 + \|Z(t)\|^2\|Z(t)\|_1^2 + \int_0^t \Big[\|W^h(s)\|_1^2 + \sum_{k=0}^{2}\|Z_m^{(k)}(s)\|_{-1}^2 + \|Z(s)\|^2\|Z^{(1)}(s)\|^2\Big]\,ds\Big\}, \tag{3.19}$$

where $|(Z^{(k)}W^h, W^h)| \le C\|W^h\|^2$, $k = 0, 1$, are implicitly used. Lemma 2 tells us that $\|Z_m^{(k)}(t)\|_{-1} \le Ch^{2r-2}$ and $\|Z^{(k)}(t)\|_s \le Ch^{r-s}$, for $k = 0, 1, 2$, $s = 0, 1$, and $0 \le t \le T$. Thus, by (3.19),

$$\|W_x^h(t)\|^2 \le C\left\{h^{2(2r-2)} + \|W^h(t)\|^2 + \int_0^t \|W^h(s)\|_1^2\,ds\right\}. \tag{3.20}$$

Since [9] $\|e(t)\|_s \le Ch^{r-s}$, $s = 0, 1$, and $\|(I - P_0)GW^h(t)\| \le Ch^2\|W^h(t)\|_1$, we have by (3.18)

$$\|GW^h(t)\|^2 \le C\left\{h^4\|W^h(t)\|_1^2 + h^{2(2r-2)} + \int_0^t \|W^h(s)\|_1^2\,ds\right\}. \tag{3.21}$$

Similar to the proof of (3.11), when $h > 0$ is small enough, the desired estimate (3.16) can be derived from (3.20), (3.21), and Gronwall's lemma. Thus, the proof of Theorem 4 is complete.

4. Numerical results of simulating 1-solitary waves

A numerical experiment is performed for the following solitary wave of (P) with initial data:

$$u_0(x) = -(3d^2)^{-1}[1 + q(x)], \qquad 0 \le x \le 1,$$
$$q(x) = q_0 + a\,\mathrm{sech}^2(a/6d^2)^{1/2}(x - 0.5),$$
$$q_0 = -2d(6a)^{1/2}\tanh(a/24d^2)^{1/2},$$

where $a = 0.2$ and $d = 10^{-2}$. Here, $u_0(x)$ is extended as a 1-periodic function to the whole real axis, and we denote the corresponding solution of (P) by $u(x, t)$; then $q(x, s) = -1 - 3d^2u(x, \frac{1}{2}d^2s)$ solves the following equation:

$$q_s + (1 + q)q_x + \tfrac{1}{2}d^2q_{xxx} = 0.$$

The solitary wave $u(x, t)$ is simulated by means of the method (P_h) with $r = 2$ and uniform mesh $x_j = jh$, $h = 1/47$, while the approximate solution $u^h(t)$ is a piecewise linear function. Let $\{q_j(x); j = 1, 2, \ldots, 47\}$ be the basis of the subspace V_h, and

$$u^h(x, t) = \sum_{j=1}^{47} u_j(t)q_j(x).$$

Then it can be seen that $\{u_j(t)\,;\ j = 1, 2, \ldots, 47\}$ is the solution of the system of ordinary differential equations

$$(4.1)\qquad \begin{aligned} &\sum_{j=1}^{47} a_{ij}\frac{du_j}{dt} - \frac{1}{h^3}(u_{i-1} - 2u_i + u_{i+1}) + \frac{1}{4h}(u_{i-1}^2 + 6u_i^2 + u_{i+1}^2) \\ &\quad + \frac{1}{2h}(u_{i-1}u_i + u_iu_{i+1}) - \sum_{j=1}^{47}(u_j^2 + u_ju_{j+1} + u_{j+1}^2) = 0, \end{aligned}$$

where $a_{ij} = (q_j, Gq_i)/h^2$, and by the periodicity, $u_0 = u_{47}$, $u_1 = u_{48}$.

We choose the time step $\Delta t = 3.125 \times 10^{-7}$ and discretize (4.1) in the time variable by the midpoint rule; then a fully discrete scheme for (P) is obtained, namely

$$(4.2)\qquad \sum_{j=1}^{47} a_{ij}\frac{u_j^{n+1} - u_j^n}{\Delta t} = F_i\left(\frac{u^{n+1} + u^n}{2}\right), \qquad i = 1, 2, \ldots, 47; \quad n = 0, 1, \ldots,$$

where

$$\begin{aligned} F_i(v) = &\frac{1}{h^3}(v_{i-1} - 2v_i + v_{i+1}) - \frac{1}{4h}(v_{i-1}^2 + 6v_i^2 + v_{i+1}^2) \\ &- \frac{1}{2h}(v_{i-1}v_i + v_iv_{i+1}) + \sum_{j=1}^{47}(v_j^2 + v_jv_{j+1} + v_{j+1}^2). \end{aligned}$$

As pointed out by Feng Kang in [4], the midpoint rule (i.e., the centered implicit Euler scheme) is a symplectic scheme, which behaves very well as far as preserving conservation laws is concerned.

Table 1 indicates the ability of scheme (4.2) to preserve the conservation laws $I_i = \text{const}$, $i = 0, 1, 2$, when this scheme is used to simulate the solitary waves of (P).

Figures 1–3 exhibit the shapes of solitary waves $q(x, s)$ calculated by scheme (4.2) at time steps $n = 0, 30, 60$, respectively.

TABLE 1

Values of I_i, $i = 0, 1, 2$, at various time steps

n	$I_0(u)$	$I_1(u)$	$I_2(u)$
0	−3333.33333	11137605.2	-373082079×10^2
30	−3333.33333	11137605.0	-373082041×10^2
60	−3333.33448	11137624.0	-373082365×10^2
90	−3333.33206	11138221.0	-373081466×10^2
140	−3333.33251	11138159.6	-373081527×10^2
190	−3333.33141	11138299.3	-373081693×10^2

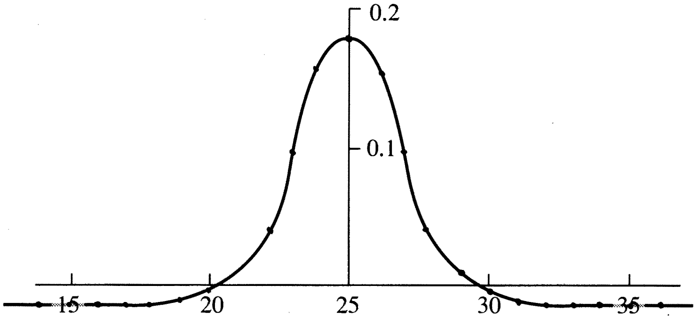

FIGURE 1
The shape of solitary wave $q(x, s)$ *at time step* $n = 0$

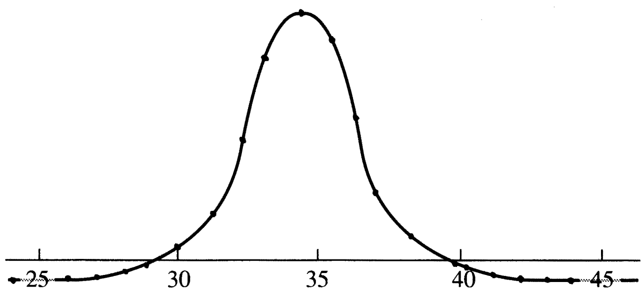

FIGURE 2
The shape of solitary wave $q(x, s)$ *at time step* $n = 30$

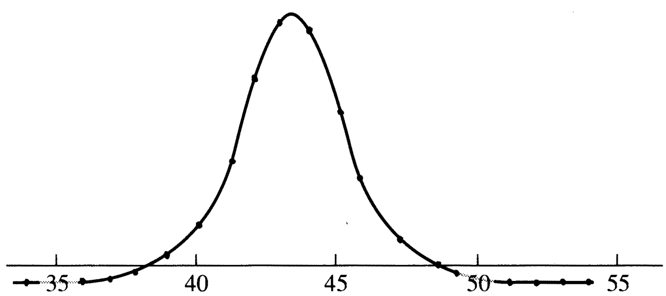

FIGURE 3
The shape of solitary wave $q(x, s)$ *at time step* $n = 60$

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