

AN AUTOMATIC QUADRATURE FOR CAUCHY PRINCIPAL VALUE INTEGRALS

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ABSTRACT. An automatic quadrature is presented for computing Cauchy principal value integrals $Q(f; c) = \int_a^b f(t)/(t - c) dt$, $a < c < b$, for smooth functions $f(t)$. After subtracting out the singularity, we approximate the function $f(t)$ by a sum of Chebyshev polynomials whose coefficients are computed using the FFT. The evaluations of $Q(f; c)$ for a set of values of c in (a, b) are efficiently accomplished with the same number of function evaluations. Numerical examples are also given.

1. INTRODUCTION

We present an automatic quadrature scheme for approximating principal value integrals

$$(1.1) \quad Q(f; c) = \int_{-1}^1 \frac{f(t)}{t - c} dt, \quad -1 < c < 1,$$

where $f(t)$ are assumed to be smooth functions. Piessens et al. [17] give an automatic quadrature program for evaluating $Q(f; c)$ in (1.1) for a single value of c .

In this paper, for a set of values of c in $(-1, 1)$ we efficiently compute a set of approximations $\{Q_N(f; c)\}$ to the integrals (1.1) satisfying the prescribed tolerance ε_a . To this end, it is required to construct quadrature rules which have error estimates independent of the values of c for smooth functions $f(t)$.

Our method is an extension of the Clenshaw-Curtis method [4] (henceforth abbreviated to CC method) for the integral $\int_{-1}^1 f(t) dt$ to the problem (1.1) [1], [2], [3], [14]. In the CC method, the function $f(t)$ is approximated by a sum of Chebyshev polynomials $T_k(t)$,

$$(1.2) \quad p_N(t) = \sum_{k=0}^N a_k^N T_k(t), \quad -1 \leq t \leq 1,$$

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interpolating $f(t)$ at the abscissae $t_j^N = \cos(\pi j/N)$ ($0 \leq j \leq N$), which are the zeros of the polynomial $\omega_{N+1}(t)$ defined by

$$(1.3) \quad \omega_{N+1}(t) = T_{N+1}(t) - T_{N-1}(t) = 2(t^2 - 1)U_{N-1}(t), \quad N \geq 1,$$

where $U_k(t)$ is the Chebyshev polynomial of the second kind defined by $U_k(t) = \sin(k+1)\theta / \sin \theta$ ($t = \cos \theta$). In (1.2), the double prime denotes the summation where the first and last terms are halved. The truncated Chebyshev series (1.2) converges rapidly as N increases if $f(t)$ is a smooth function.

Chawla and Kumar [3] substituted $p_N(t)$ (1.2) for $f(t)$ in (1.1) to obtain an approximation $Q_N^{CK}(f; c)$ to $Q(f; c)$ as follows:

$$(1.4) \quad Q_N^{CK}(f; c) = \sum_{k=0}^N {}'' a_k^N Q(T_k; c),$$

where the modified moment $Q(T_k; c) = \int_{-1}^1 T_k(t)/(t-c) dt$ can be computed by means of a three-term recurrence relation [1]. However, this method is not suitable for our purpose because the error $Q(f; c) - Q_N^{CK}(f; c)$ cannot be bounded independently of the value of c [3].

On the other hand, subtracting out the singularity [5, p.184], [7, p.104], [18], [19], one can write $Q(f; c)$ (1.1) in the form

$$(1.5) \quad Q(f; c) = \int_{-1}^1 g_c(t) dt + f(c) \log\left(\frac{1-c}{1+c}\right),$$

where $g_c(t)$ is defined by

$$(1.6) \quad g_c(t) = \{f(t) - f(c)\}/(t-c).$$

Chawla and Jayarajan [2], and subsequently Kumar [14], made use of the approximate polynomial $p_N(t)$ (1.2) to interpolate $g_c(t)$ instead of $f(t)$ at t_j^N and obtained the quadrature formulae

$$(1.7) \quad Q_N^{CJ}(f; c) = \sum_{j=0}^N {}'' A_j^N g_c(t_j^N) + f(c) \log\left(\frac{1-c}{1+c}\right),$$

when $t_j^N \neq c$ for all j . In the above, A_j^N are given by

$$A_j^N = \frac{4}{N} \sum_{k=0}^{N/2} {}'' T_{2k}(t_j^N)/(1-4k^2), \quad 0 \leq j \leq N,$$

where here and henceforth we conveniently assume that N is even.

It is known [14] that the quadrature formulae (1.7) can yield, in general, better approximate values for (1.1) than the formulae (1.4), but in the computation of $g_c(t_j^N)$, we have severe numerical cancellation if a node t_j^N happens to be very close to c [9], [15]. This instability requires special care in programming the function g_c .

We now show that we can avoid this instability by approximating $f(t)$ and $f(c)$ in (1.6) by $p_N(t)$ and $p_N(c)$ (1.2), respectively; the approximation $Q_N(f; c)$ to the integral $Q(f; c)$ then becomes

$$(1.8) \quad Q_N(f; c) = \int_{-1}^1 \frac{p_N(t) - p_N(c)}{t - c} dt + f(c) \log\left(\frac{1 - c}{1 + c}\right).$$

Expanding the integrand in (1.8) in Chebyshev polynomials,

$$(1.9) \quad \frac{p_N(t) - p_N(c)}{t - c} = \sum_{k=0}^{N-1} 'd_k T_k(t),$$

and integrating term by term, yields a new integration formula

$$(1.10) \quad Q_N(f; c) = 2 \sum_{k=0}^{N/2-1} 'd_{2k} / (1 - 4k^2) + f(c) \log\left(\frac{1 - c}{1 + c}\right),$$

where the prime denotes the summation whose first term is halved. The coefficients d_k in (1.9) can be stably computed by using the recurrence relation

$$(1.11) \quad d_{k+1} - 2c d_k + d_{k-1} = 2a_k^N, \quad k = N, N - 1, \dots, 1,$$

in the backward direction with the starting values $d_N = d_{N+1} = 0$, where we take $a_N^N/2$ instead of a_N^N . We have omitted the dependence of d_k on N and c .

It is well known that the Fast Fourier Transform (FFT) is useful for efficiently computing the coefficients $\{a_k^N\}$ in (1.2); see also (2.1) below, [1] and [10], where by doubling N the computation can be repeated, reusing the previous values until an error criterion is satisfied. It is advantageous to have more chances of checking the stopping criterion than by doubling N , in order to enhance the efficiency of automatic quadrature. In [12], we allowed N to take the forms 3×2^n and 5×2^n as well as 2^n , that is,

$$(1.12) \quad N = 3, 4, 5, \dots, 3 \times 2^n, 4 \times 2^n, 5 \times 2^n, \dots \quad (n = 1, 2, \dots).$$

In §2 we briefly review how to generate recursively the sequence of the interpolating polynomials $\{p_N(t)\}$ by increasing N as in (1.12) and by using the FFT. The set of the $N + 1$ nodes u_j^N ($0 \leq j \leq N$) for $p_N(t)$ is chosen to be a subset of $\{\cos \pi j / 2^m\}$ ($0 \leq j \leq 2^m$) used in the CC method, where m is the smallest integer such that $N \leq 2^m$.

We remark that the present quadrature rule $Q_N(f; c)$ (1.8) or (1.10) is not of interpolatory type because the degree of exactness in the present rule, using $N + 2$ abscissae, u_j^N ($0 \leq j \leq N$) and c , is N , not $N + 1$. As will be shown in §3, however, since the function value $f(c)$ is used in the quadrature rule $Q_N(f; c)$ (1.8), but not in interpolating $f(t)$, the error of $Q_N(f; c)$ can be bounded independently of the value of c for smooth functions $f(t)$. See (3.8), (3.10), and (3.11) below. This fact enables us to use the polynomial $p_N(t)$ common to the set of the approximations $\{Q_N(f; c)\}$ for a set of c -values

in $(-1, 1)$. In §4 numerical comparisons with other automatic quadrature methods are shown.

2. COMPUTATION OF THE CHEBYSHEV COEFFICIENTS

We will outline the iterative procedure for computing the sequence $\{p_N(t)\}$ (1.2) of the truncated Chebyshev series by increasing N as in (1.12). For details, see [12].

We begin with the sample points for $p_N(t)$ to interpolate $f(t)$. If the sample points are carefully chosen, the interpolating polynomial converges [13, p. 254]. We gave in [11] and [12] a sequence $\{\beta_j\}$ which is a modification of the van der Corput sequence and satisfies the recurrence relation:

$$\beta_{2j} = \beta_j/2, \quad \beta_{2j+1} = \beta_{2j} + 1/2, \quad j = 1, 2, \dots,$$

with the starting value $\beta_1 = 3/4$. The set of the sample points $\{\cos 2\pi\beta_j\}$ ($j = -1, 0, 1, \dots$), where we put $\beta_{-1} = 0$ and $\beta_0 = 1/2$, is a sequence of Chebyshev points [13, p. 254], which makes the sequence of interpolating polynomials converge uniformly on $[-1, 1]$ for functions analytic on $[-1, 1]$. The polynomial $p_N(t)$ is determined so as to interpolate $f(t)$ at the first $N+1$ points of the sequence $\{\cos 2\pi\beta_j\}$ ($j = -1, 0, 1, \dots$).

Let $N = 2^n$ ($n = 2, 3, \dots$); then the set of the $N+1$ abscissae $\{\cos 2\pi\beta_j\}$ ($-1 \leq j < N$) coincides with the zeros of $\omega_{N+1}(t)$ (1.3), that is, $\{\cos \pi j/N\}$ ($0 \leq j \leq N$) used in the CC method. Therefore, the interpolation condition

$$p_N(\cos \pi j/N) = f(\cos \pi j/N), \quad 0 \leq j \leq N,$$

determines the coefficients a_k^N for $p_N(t)$ (1.2) as follows:

$$(2.1) \quad a_k^N = \frac{2}{N} \sum_{j=0}^N f(\cos \pi j/N) \cos(\pi k j/N), \quad 0 \leq k \leq N.$$

It is known that the right-hand side of (2.1) can be efficiently computed by means of the FFT for real data [10].

We represent the polynomials $p_{5N/4}(t)$ and $p_{3N/2}(t)$ interpolating $f(t)$ at the nodes $\{\cos 2\pi\beta_j\}$, where $-1 \leq j < N + N/4$ for $p_{5N/4}(t)$ and $-1 \leq j < N + N/2$ for $p_{3N/2}(t)$, respectively, in the form

$$(2.2) \quad \begin{aligned} p_{5N/4}(t) - p_N(t) &= -\omega_{N+1}(t) \sum_{k=1}^{N/4} b_k^N U_{k-1}(t) \\ &= \sum_{k=1}^{N/4} b_k^N \{T_{N-k}(t) - T_{N+k}(t)\}, \end{aligned}$$

$$\begin{aligned}
 (2.3) \quad p_{3N/2}(t) - p_N(t) &= -\omega_{N+1}(t) \sum_{k=1}^{N/2} B_k^N U_{k-1}(t) \\
 &= \sum_{k=1}^{N/2} B_k^N \{T_{N-k}(t) - T_{N+k}(t)\}.
 \end{aligned}$$

Then, the coefficients $\{b_k^N\}$ and $\{B_k^N\}$ are determined to satisfy the conditions

$$\begin{aligned}
 p_{5N/4}(v_j^N) &= f(v_j^N), & 0 \leq j < N/4, \\
 p_{3N/2}(w_j^N) &= f(w_j^N), & 0 \leq j < N/2,
 \end{aligned}$$

where the sample points v_j^N and w_j^N are defined by

$$\begin{aligned}
 (2.4) \quad v_j^N &= \cos 8\pi(j + \beta_4)/N \quad \text{or} \quad T_{N/4}(v_j^N) - \cos 2\pi\beta_4 = 0, \\
 (2.5) \quad w_j^N &= \cos 4\pi(j + \beta_2)/N \quad \text{or} \quad T_{N/2}(w_j^N) - \cos 2\pi\beta_2 = 0,
 \end{aligned}$$

respectively. This is because the set of the additional $N/4$ ($N/2$) abscissae $\{\cos 2\pi\beta_j\}$, $N \leq j < N/4$ ($N \leq j < N/2$) for $p_{5N/4}(t)$ ($p_{3N/2}(t)$) coincides with $\{v_j^N\}$, $0 \leq j < N/4$ ($\{w_j^N\}$, $0 \leq j < N/2$) [12]. If the set of $N/2$ sample points $\{\cos 4\pi(j + \beta_3)/N\}$ ($0 \leq j < N/2$), which agrees with $\{\cos 2\pi\beta_j\}$ ($3N/2 \leq j < 2N$), is added to the set of abscissae for $p_{3N/2}(t)$, we have $2N + 1$ abscissae $\{\cos \pi j/(2N)\}$ ($0 \leq j \leq 2N$) for $p_{2N}(t)$. Thus the sequence of the interpolating polynomials $\{p_{3m}(t), p_{4m}(t), p_{5m}(t), \dots\}$ ($m = 2^n$, $n = 1, 2, \dots$) is recursively generated. The FFT [12] is used to efficiently compute the coefficients $\{b_k^N\}$ and $\{B_k^N\}$.

3. ERROR ESTIMATES

Assume that $N = 2^n$ ($n = 2, 3, \dots$) and define A_k^N by

$$(3.1) \quad A_k^N = \begin{cases} a_k^N, & 0 \leq k < N - N/4, \\ a_k^N + b_{N-k}^N, & N - N/4 \leq k < N, \\ a_N^N/2, & k = N, \\ -b_{k-N}^N, & N < k \leq N + N/4. \end{cases}$$

Then, the approximate quadrature $Q_{5N/4}(f; c)$ depending on the polynomial $p_{5N/4}(t)$ (2.2) is given by the right-hand side of (1.10), where the sum ranges from 0 to $N/2 + N/8 - 1$, and by (1.11) with a_k^N replaced by A_k^N (3.1). Similarly, one can obtain the approximation $Q_{3N/2}(f; c)$ depending on the polynomial $p_{3N/2}(t)$ (2.3).

Now, we will give error estimates for the approximations $Q_N(f; c)$, $Q_{5N/4}(f; c)$, and $Q_{3N/2}(f; c)$, especially for analytic functions f . Let

ε_ρ denote the ellipse in the complex plane $z = x + iy$ with foci $(x, y) = (-1, 0), (1, 0)$ and semimajor axis $a = (\rho + \rho^{-1})/2$ and semiminor axis $b = (\rho - \rho^{-1})/2$ for a constant $\rho > 1$.

Assume that $f(z)$ is single-valued and analytic inside and on ε_ρ . Then, the error of the interpolating polynomial $p_N(t)$ can be expressed in terms of a contour integral [6], [7, p. 105], [8], which is also expanded in a Chebyshev series [11]:

$$(3.2) \quad f(t) - p_N(t) = \frac{1}{2\pi i} \oint_{\varepsilon_\rho} \frac{\omega_{N+1}(t) f(z) dz}{(z-t)\omega_{N+1}(z)} = \omega_{N+1}(t) \sum_{k=0}^{\infty} V_k^N(f) T_k(t),$$

where the coefficients $V_k^N(f)$ are given by

$$(3.3) \quad V_k^N(f) = \frac{1}{\pi^2 i} \oint_{\varepsilon_\rho} \frac{\tilde{U}_k(z) f(z) dz}{\omega_{N+1}(z)}, \quad k \geq 0.$$

The Chebyshev function of the second kind, $\tilde{U}_k(z)$, is defined by

$$(3.4) \quad \tilde{U}_k(z) = \int_{-1}^1 \frac{T_k(t) dt}{(z-t)\sqrt{1-t^2}} = \frac{\pi}{\sqrt{z^2-1} w^k} = \frac{2\pi}{(w-w^{-1})w^k},$$

where $w = z + \sqrt{z^2-1}$ and $|w| > 1$ for $z \notin [-1, 1]$ [8], [11].

Using (3.2) in (1.5), (1.6) and (1.8) yields the error for the approximate integral $Q_N(f; c)$:

$$(3.5) \quad Q(f; c) - Q_N(f; c) = \sum_{k=0}^{\infty} \Omega_k^N(c) V_k^N(f),$$

where $\Omega_k^N(c)$ is given by

$$(3.6) \quad \Omega_k^N(c) = \int_{-1}^1 \frac{\omega_{N+1}(t) T_k(t) - \omega_{N+1}(c) T_k(c)}{t-c} dt, \quad k \geq 0.$$

In Appendix A we prove the following lemma.

Lemma 3.1. *Let $N = 2^n$, $n = 2, 3, \dots$, and $\Omega_k^N(c)$ be defined by (3.6). Then, $\Omega_k^N(c)$ is bounded independently of the value of c as well as N and k ; indeed,*

$$(3.7) \quad |\Omega_k^N(c)| \leq 8.$$

From (3.5) and (3.7) we have the following theorem.

Theorem 3.2. *Let $N = 2^n$, $n = 2, 3, \dots$, and assume that $f(z)$ is single-valued and analytic inside and on ε_ρ . Then, the error of the approximate integral $Q_N(f; c)$ given by (1.10) is bounded independently of c by*

$$(3.8) \quad |Q(f; c) - Q_N(f; c)| \leq 8 \sum_{k=0}^{\infty} |V_k^N(f)|,$$

where $V_k^N(f)$ is given by (3.3).

Similarly, the errors of the approximate integrals $Q_{5N/4}(f; c)$ and $Q_{3N/2}(f; c)$ are bounded as follows:

Theorem 3.3. *Let $N = 2^n$ ($n = 2, 3, \dots$) and assume that $f(z)$ is single-valued and analytic inside and on ε_ρ . Further, let $V_k^{N+N/\sigma}(f)$ ($\sigma = 2, 4$) be defined by*

$$(3.9) \quad V_k^{N+N/\sigma}(f) = \frac{1}{\pi^2 i} \oint_{\varepsilon_\rho} \frac{\tilde{U}_k(z) f(z) dz}{\omega_{N+1}(z) \{T_{N/\sigma}(z) - \cos 2\pi\beta_\sigma\}},$$

$$k \geq 0, \quad \sigma = 2, 4.$$

Then, we have

$$(3.10) \quad |Q(f; c) - Q_{5N/4}(f; c)| \leq 8(1 + |\cos 2\pi\beta_4|) \sum_{k=0}^{\infty} |V_k^{N+N/4}(f)|$$

$$\sim 11.1 \sum_{k=0}^{\infty} |V_k^{N+N/4}(f)|,$$

$$(3.11) \quad |Q(f; c) - Q_{3N/2}(f; c)| \leq 8(1 + |\cos 2\pi\beta_2|) \sum_{k=0}^{\infty} |V_k^{N+N/2}(f)|$$

$$\sim 13.7 \sum_{k=0}^{\infty} |V_k^{N+N/2}(f)|,$$

where $\beta_4 = 3/16$ and $\beta_2 = 3/8$.

Proof. The error of the interpolating polynomial $p_{N+N/\sigma}(t)$ ($\sigma = 2, 4$) has an expression similar to (3.2):

$$(3.12) \quad f(t) - p_{N+N/\sigma}(t) = \frac{1}{2\pi i} \oint_{\varepsilon_\rho} \frac{\omega_{N+1}(t) \{T_{N/\sigma}(t) - \cos 2\pi\beta_\sigma\} f(z) dz}{(z-t) \omega_{N+1}(z) \{T_{N/\sigma}(z) - \cos 2\pi\beta_\sigma\}}$$

$$= \omega_{N+1}(t) \{T_{N/\sigma}(t) - \cos 2\pi\beta_\sigma\}$$

$$\times \sum_{k=0}^{\infty} V_k^{N+N/\sigma}(f) T_k(t), \quad \sigma = 2, 4,$$

where $V_k^{N+N/\sigma}(f)$ is given by (3.9). If we note in (3.12) that

$$(3.13) \quad 2\omega_{N+1}(t) \{T_{N/\sigma}(t) - \cos 2\pi\beta_\sigma\}$$

$$= \omega_{N+N/\sigma+1}(t) + \omega_{N-N/\sigma+1}(t) - 2 \cos 2\pi\beta_\sigma \omega_{N+1}(t),$$

then the proof of (3.10) and (3.11) is established in a way similar to that for (3.8). \square

Suppose that $f(z)$ is a meromorphic function which has M simple poles at the points z_m ($m = 1, 2, \dots, M$) outside of ε_ρ with residues $\text{Res } f(z_m)$.

Then, performing the contour integral of (3.3) gives

$$(3.14) \quad V_k^N(f) = -\frac{2}{\pi} \sum_{m=1}^M \operatorname{Res} f(z_m) \tilde{U}_k(z_m) / \omega_{N+1}(z_m), \quad k \geq 0.$$

Put $z = (w + w^{-1})/2$; then the Chebyshev polynomial can be expressed as

$$(3.15) \quad T_n(z) = (w^n + w^{-n})/2, \quad w = z + \sqrt{z^2 - 1},$$

$|w| > 1$ for $z \notin [-1, 1]$.

From (1.3), (3.4), (3.14) and (3.15) it is seen that $|V_k^N(f)| = O(r^{-k-N})$, where $r = \min_{1 \leq m \leq M} |z_m + \sqrt{z_m^2 - 1}| > 1$. Thus, from (3.8) we may estimate the error for $Q_N(f; c)$ as follows:

$$(3.16) \quad |Q(f; c) - Q_N(f; c)| \lesssim 4 |V_0^N(f)|(r+1)/(r-1).$$

Now, we wish to estimate $|V_0^N(f)|$ in terms of the available coefficients a_k^N of the truncated Chebyshev series $p_N(t)$ (1.2). Elliott [6] gives the expression

$$(3.17) \quad a_k^N = \frac{2}{\pi i} \oint_{\epsilon_p} \frac{T_{N-k}(z) f(z)}{\omega_{N+1}(z)} dz, \quad 0 \leq k \leq N.$$

Performing the contour integral in (3.17) and comparing with (3.14) gives the estimates

$$(3.18) \quad |V_0^N| \sim |a_N^N| r / (r^2 - 1)$$

and $|a_k^N| \sim r |a_{k+1}^N|$, unless the poles z_m of $f(z)$ are close to the segment $[-1, 1]$ on the real axis. Finally, from (3.16) and (3.18) we could obtain an estimate of the truncation error $E_N(f; c)$ for $Q_N(f; c)$ as follows:

$$(3.19) \quad E_N(f; c) = 8 (|a_N^N|/2) r / (r - 1)^2,$$

where we note that $a_N^N/2$ is the coefficient of the last term in the truncated Chebyshev series (1.2). The constant r may be estimated from the asymptotic behavior of $\{a_k^N\}$ in a way similar to that in the stopping criterion described in [12].

If $|a_k^N|$ decreases slowly as k increases, that is, $r \rightarrow 1+$, we prefer a rather cautious error estimation similar to that given in the stopping criterion of [12] in place of (3.19). See also [16].

Next, we turn to estimate the error (3.10) of $Q_{5N/4}(f; c)$ in terms of the available coefficients b_k^N of $p_{5N/4}(t)$ (2.2).

Lemma 3.4. *Let $f(z)$ be single-valued and analytic inside and on ϵ_p . Further, define*

$$(3.20) \quad J_k^N(\sigma) = \frac{-1}{\pi i} \oint_{\epsilon_p} \frac{T_{N/\sigma-k}(z) f(z) dz}{\omega_{N+1}(z) \{T_{N/\sigma}(z) - \cos 2\pi\beta_\sigma\}},$$

$1 \leq k \leq N/\sigma, \sigma = 2, 4,$

4. NUMERICAL EXAMPLES

We now show numerical results obtained with the present automatic quadrature scheme for the following test problems:

- (4.1) $\int_{-1}^1 \frac{\exp\{a(t-1)\}}{t-c} dt, \quad a = 4, 8, 16,$
- (4.2) $\int_{-1}^1 \frac{(t^2 + a^2)^{-1}}{t-c} dt, \quad a = 1, 1/4, 1/8,$
- (4.3) $\int_0^1 \frac{\cos 2\pi at}{t-c} dt, \quad a = 8, 16, 32,$
- (4.4) $\int_{-1}^1 \frac{1-a^2}{1-2at+a^2} \cdot \frac{1}{t-c} dt, \quad a = 0.8, 0.9, 0.95,$
- (4.5) $\int_0^1 \frac{\sqrt{1-t^2}}{t-c} dt.$

TABLE 1

Comparison of the performance of the present method with QAWC in QUADPACK [17] for $\int_{-1}^1 e^{a(t-1)}/(t-c) dt, a = 4, 8, 16$. N denotes the number of abscissae required to satisfy the tolerance ϵ_a . The present method computes all the integrals for a set of the values of c by using $N - 1$ abscissae once and for all, and by using the number of the corresponding values of c .

		$\epsilon_a = 10^{-6}$				$\epsilon_a = 10^{-10}$			
		present method		QUADPACK		present method		QUADPACK	
a	c	N	error	N	error	N	error	N	error
4	0.2	↑ (+1)	1×10^{-10}	25	2×10^{-15}	↑ (+1)	9×10^{-15}	105	3×10^{-13}
	0.5	17+1	2×10^{-11}	25	4×10^{-15}	21+1	1×10^{-16}	105	4×10^{-14}
	0.95	↓ (+1)	6×10^{-11}	25	1×10^{-14}	↓ (+1)	2×10^{-14}	105	2×10^{-13}
8	0.2	↑ (+1)	5×10^{-10}	105	7×10^{-13}	↑ (+1)	9×10^{-13}	145	7×10^{-13}
	0.5	21+1	3×10^{-10}	105	2×10^{-15}	25+1	3×10^{-13}	185	4×10^{-13}
	0.95	↓ (+1)	8×10^{-11}	65	4×10^{-15}	↓ (+1)	1×10^{-12}	145	2×10^{-13}
16	0.2	↑ (+1)	7×10^{-13}	105	7×10^{-13}	↑ (+1)	7×10^{-13}	185	7×10^{-13}
	0.5	33+1	6×10^{-13}	145	8×10^{-13}	33+1	6×10^{-13}	225	8×10^{-13}
	0.95	↓ (+1)	2×10^{-14}	105	6×10^{-16}	↓ (+1)	2×10^{-14}	185	1×10^{-13}

TABLE 2
 Comparison of the performance of the present method with QAWC
 in QUADPACK [17] for $f_{-1}^1 (t^2+a^2)^{-1}/(t-c) dt$, $a = 1, 1/4, 1/8$.

		$\varepsilon_a = 10^{-6}$				$\varepsilon_a = 10^{-10}$			
		present method		QUADPACK		present method		QUADPACK	
a	c	N	error	N	error	N	error	N	error
1	0.2	$\uparrow (+1)$	1×10^{-8}	65	4×10^{-13}	$\uparrow (+1)$	6×10^{-13}	145	4×10^{-13}
	0.5	21+1	1×10^{-8}	65	5×10^{-13}	33+1	5×10^{-13}	145	5×10^{-13}
	0.95	$\downarrow (+1)$	4×10^{-9}	65	5×10^{-13}	$\downarrow (+1)$	1×10^{-13}	105	7×10^{-13}
1/4	0.2	$\uparrow (+1)$	3×10^{-7}	225	2×10^{-11}	$\uparrow (+1)$	7×10^{-13}	365	3×10^{-12}
	0.5	81+1	3×10^{-8}	215	9×10^{-12}	129+1	5×10^{-13}	325	3×10^{-12}
	0.95	$\downarrow (+1)$	9×10^{-10}	165	1×10^{-11}	$\downarrow (+1)$	1×10^{-13}	235	2×10^{-12}
1/8	0.2	$\uparrow (+1)$	4×10^{-7}	335	6×10^{-12}	$\uparrow (+1)$	1×10^{-12}	505	1×10^{-11}
	0.5	161+1	3×10^{-7}	225	3×10^{-11}	257+1	2×10^{-13}	445	1×10^{-12}
	0.95	$\downarrow (+1)$	1×10^{-7}	255	5×10^{-12}	$\downarrow (+1)$	5×10^{-13}	325	1×10^{-11}

Tables 1 – 5 compare the results of the present scheme with those of QAWC in the subroutine package QUADPACK [17] for each problem (4.1)–(4.5). We show the number of function evaluations N required to satisfy the requested absolute accuracy ε_a for each integral and the actual errors.

It should be noted that the present scheme can efficiently give all the approximations to the integrals (1.1) for a set of c -values by using the common number of function evaluations once and for all, except for each function value $f(c)$ at c , for smooth functions $f(t)$. Consequently, in each Table 1–5, the present method requires only $N + \text{extra } 2 (= N + 2)$ function evaluations to compute the three integrals for the three values of c . For example, in Table 1, $20 \{= N + 2 = (17 + 1) + 2\}$ function evaluations are sufficient for the three integrals with the parameter $a = 4$ to satisfy the tolerance $\varepsilon_a = 10^{-6}$.

The computation was carried out in double-precision arithmetic (about 16 significant digits).

TABLE 3
Comparison of the performance of the present method with QAWC in QUADPACK [17] for $f_0^1 \cos 2\pi at/(t-c) dt$, $a = 8, 16, 32$.

		$\epsilon_a = 10^{-6}$				$\epsilon_a = 10^{-10}$			
		present method		QUADPACK		present method		QUADPACK	
a	c	N	error	N	error	N	error	N	error
8	0.6	$\uparrow^{(+1)}$	2×10^{-10}	325	2×10^{-11}	$\uparrow^{(+1)}$	1×10^{-13}	495	4×10^{-12}
	0.8	49+1	1×10^{-10}	355	1×10^{-11}	65+1	1×10^{-13}	425	5×10^{-12}
	0.95	$\downarrow_{(+1)}$	3×10^{-11}	305	3×10^{-12}	$\downarrow_{(+1)}$	2×10^{-13}	505	4×10^{-12}
16	0.6	$\uparrow^{(+1)}$	8×10^{-11}	555	2×10^{-11}	$\uparrow^{(+1)}$	6×10^{-13}	875	2×10^{-13}
	0.8	81+1	1×10^{-10}	635	1×10^{-11}	97+1	5×10^{-14}	785	1×10^{-11}
	0.95	$\downarrow_{(+1)}$	5×10^{-12}	595	2×10^{-11}	$\downarrow_{(+1)}$	2×10^{-14}	975	2×10^{-12}
32	0.6	$\uparrow^{(+1)}$	2×10^{-14}	1055	4×10^{-12}	$\uparrow^{(+1)}$	2×10^{-14}	1615	3×10^{-12}
	0.8	161+1	9×10^{-14}	1205	5×10^{-12}	161+1	9×10^{-14}	1405	7×10^{-12}
	0.95	$\downarrow_{(+1)}$	7×10^{-14}	1125	3×10^{-11}	$\downarrow_{(+1)}$	7×10^{-14}	1595	8×10^{-13}

TABLE 4
Comparison of the performance of the present method with QAWC in QUADPACK [17] for $f_{-1}^1(1-a^2)(1-2at+a^2)^{-1}/(t-c) dt$, $a = 0.8, 0.9, 0.95$.

		$\epsilon_a = 10^{-6}$				$\epsilon_a = 10^{-10}$			
		present method		QUADPACK		present method		QUADPACK	
a	c	N	error	N	error	N	error	N	error
0.8	0.15	$\uparrow^{(+1)}$	4×10^{-9}	195	7×10^{-13}	$\uparrow^{(+1)}$	3×10^{-12}	305	3×10^{-13}
	0.45	97+1	4×10^{-9}	205	1×10^{-12}	129+1	4×10^{-12}	305	4×10^{-13}
	0.95	$\downarrow_{(+1)}$	4×10^{-8}	305	2×10^{-12}	$\downarrow_{(+1)}$	1×10^{-11}	385	2×10^{-12}
0.9	0.15	$\uparrow^{(+1)}$	4×10^{-8}	255	5×10^{-13}	$\uparrow^{(+1)}$	1×10^{-11}	365	3×10^{-13}
	0.45	193+1	2×10^{-8}	265	9×10^{-13}	257+1	2×10^{-11}	375	4×10^{-13}
	0.95	$\downarrow_{(+1)}$	9×10^{-8}	335	2×10^{-12}	$\downarrow_{(+1)}$	1×10^{-10}	445	5×10^{-13}
0.95	0.15	$\uparrow^{(+1)}$	6×10^{-9}	315	3×10^{-13}	$\uparrow^{(+1)}$	6×10^{-15}	425	2×10^{-13}
	0.45	385+1	1×10^{-7}	325	6×10^{-13}	641+1	2×10^{-13}	435	3×10^{-13}
	0.95	$\downarrow_{(+1)}$	5×10^{-7}	395	2×10^{-12}	$\downarrow_{(+1)}$	8×10^{-13}	505	6×10^{-13}

TABLE 5

Comparison of the performance of the present method with QAWC in QUADPACK [17] for $\int_0^1 \sqrt{1-t^2} / (t-c) dt$. The number in the parentheses indicates failure to achieve the required accuracy.

c	$\epsilon_a = 10^{-3}$				$\epsilon_a = 10^{-5}$			
	present method		QUADPACK		present method		QUADPACK	
	N	error	N	error	N	error	N	error
0.6	↑ (+1)	4×10^{-4}	(65)	2×10^{-3}	↑ (+1)	6×10^{-7}	315	3×10^{-9}
0.9	97+1	2×10^{-4}	285	3×10^{-7}	1025+1	6×10^{-6}	405	4×10^{-9}
0.95	↓ (+1)	1×10^{-4}	295	5×10^{-7}	↓ (+1)	7×10^{-6}	445	3×10^{-9}

APPENDIX A

Here, we prove (3.7). By using the relation

$$(A.1) \quad 2 T_n(t) T_m(t) = T_{n+m}(t) + T_{|n-m|}(t), \quad n, m \geq 0,$$

and the definition of $\omega_{N+1}(t)$ (1.3) in (3.6), it follows that

$$(A.2) \quad \begin{aligned} 2 \Omega_k^N(c) &= \int_{-1}^1 \frac{\omega_{N+k+1}(t) - \omega_{N+k+1}(c)}{t-c} dt \\ &\pm \int_{-1}^1 \frac{\omega_{|N-k|+1}(t) - \omega_{|N-k|+1}(c)}{t-c} dt, \quad k \geq 0. \end{aligned}$$

In the above, the plus sign is taken if $N-k \geq 1$ and the minus sign if $k-N \geq 1$. Further, the second term in the right-hand side should be ignored when $k = N$.

Elliott [6] gives the identity involving the Chebyshev polynomial of the second kind $U_k(t)$:

$$(A.3) \quad T_{k+1}(t) - T_{k+1}(c) = 2(t-c) \sum_{n=0}^k U_{k-n}(c) T_n(t), \quad k \geq 0.$$

Using the identities $U_k(t) - U_{k-2}(t) = 2 T_k(t)$ ($k \geq 1$), where we define $U_{-1}(t) = 0$, and (A.3) in (A.2) gives

$$(A.4) \quad \begin{aligned} \Omega_k^N(c) &= 2 \sum_{n=0}^{N+k} T_{N+k-n}(c) \int_{-1}^1 T_n(t) dt \\ &\pm 2 \sum_{n=0}^{|N-k|} T_{|N-k|-n}(c) \int_{-1}^1 T_n(t) dt. \end{aligned}$$

Thus, $\Omega_k^N(c)$ is bounded by

$$(A.5) \quad |\Omega_k^N(c)| \leq 2 \sum_{n=0}^{N+k} \left| \int_{-1}^1 T_n(t) dt \right| + 2 \sum_{n=0}^{|N-k|} \left| \int_{-1}^1 T_n(t) dt \right|.$$

If one notes in (A.5) that the integral $\int_{-1}^1 T_n(t) dt$ equals $2/(1-n^2)$ if n is even, and vanishes otherwise, it is easy to verify (3.7).

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