A TABLE OF TOTALLY REAL QUINTIC NUMBER FIELDS

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ABSTRACT. We give a table of the 1077 totally real number fields of degree five having a discriminant less than 2 000 000. There are two nonisomorphic fields of discriminant 1 810 969 and two nonisomorphic fields of discriminant 1 891 377. All the other number fields in the table are characterized by their discriminant. Among these fields, three are cyclic and four have a Galois closure whose Galois group is the dihedral group D_5. The Galois closure for all the other fields in the table has a Galois group isomorphic to S_5.

1. INTRODUCTION

A systematic study of 5th-degree number fields was done by H. Cohn in [4]; in that paper he gave a list of small discriminants for each of the three possible signatures, and he made the conjecture that these tables are complete in the ranges covered by the given values. In [3] P. Cartier and Y. Roy proved that the different polynomials of [4] having the same discriminant generate isomorphic number fields.

Some years later, J. Hunter [5], using methods of the geometry of numbers, determined the minimal discriminants for the three signatures of the number fields of degree five. They are: 1 649 for the signature $(1, 2)$, $-4 511$ for the signature $(3, 1)$, and 14 641 for the totally real quintics. He also enlarged the conjectural tables constructed by H. Cohn.

More recently, D. G. Rish [10] gave complete tables for quintics having one or three real places. Unfortunately, the author does not give proofs of the inequalities used, and these inequalities seem to have been obtained in a manner analogous to that used by J. Liang and H. Zassenhaus in [7] that are known to be incorrect (cf. [9]).

For the totally real quintics K. Takeuchi [11] found all the number fields of discriminants smaller than 150 000, and his list confirms the conjectural results in [4, 5] for that signature. It should be pointed out, however, that K. Takeuchi's table of totally real number fields of degree five contains two errors to which J. Martinet has drawn my attention: the polynomials $X^5+2X^4-6X^3-4X^2+5X-1$ of discriminant 107 653 and $X^5 - 8X^3 - 2x^2 + 3X + 1$ of discriminant 146 205

do not generate quintic number fields because they are reducible: the first one is divisible by X^2+3X-1 and the second by X^2+3X+1. Our computations show that the table of K. Takeuchi is correct if we eliminate these two discriminants.

The table given in the Supplement section contains the complete list of the 1077 totally real number fields of degree five of discriminant smaller than 2 000 000. For the computation of the table we used the techniques of the geometry of numbers, and we will give a brief description of them in the next section. These methods have already been used by different authors in the construction of extensive tables of number fields [1, 2].

In §3 we describe the simplifications used to reduce as much as possible the number of polynomials to be considered. We also explain what tests are used for the determination of the signature of the fields, the irreducibility of the polynomials, and the exact value of the discriminant of the fields.

In §4 we show how we found the possible existence of isomorphisms among the number fields generated by the roots of the different polynomials having the same discriminant. Thus, we prove the existence of two nonisomorphic fields of discriminant 1 810 969 and two nonisomorphic fields of discriminant 1 891 377. All the other fields in the table are characterized by their discriminant.

Finally, in §5, we study the Galois group of the Galois closure of each number field in the table. We then prove that in the range of the table there exist exactly three abelian number fields and four number fields for which the Galois group of the Galois closure is the dihedral group D_5. All the other number fields in the table have a Galois closure having a Galois group isomorphic to the symmetric group S_5.

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2. THE METHOD

Let K be a totally real number field of degree 5 having a discriminant $d_K < 2 000 000$, and let $\mathbf{Z}_K$ denote its ring of integers. We know that K is always a primitive extension of $\mathbf{Q}$ generated by a root θ of an irreducible polynomial $P(X) = X^5 + a_1 X^4 + a_2 X^3 + a_3 X^2 + a_4 X + a_5 \in \mathbf{Z}[X]$ having only real roots.

We will also use Newton's relations

$$(1) \qquad\qquad S_k + k \cdot a_k + \sum_{i=1}^{k-1} a_i \cdot S_{k-i} = 0,$$

valid for all $k \in \mathbf{N}$ if we define $a_k = 0$ for $k > 5$ and where the S_k denote the kth power sums,

$$S_k = S_k(\theta) = \mathrm{Tr}_{K/\mathbf{Q}}(\theta^k).$$

If we fix an order for the roots of the polynomial $P(X) = \prod_{i=1}^{5}(X - \theta_i)$, we can obtain an embedding of K in $\mathbf{R}^5$ by using the map

$$\tau \begin{cases} K \to \mathbf{R}^5, \\ \theta \mapsto (\theta_1, \ldots, \theta_5), \end{cases}$$

and under this map $\mathbf{Z}_K$ becomes a lattice M of $\mathbf{R}^5$ of discriminant d_K.
The map

$$q \begin{cases} \mathbf{Z}_K \to \mathbf{R}, \\ \theta \mapsto \sum_{i=1}^{5} \theta_i^2 \end{cases}$$

is a positive definite quadratic form. The restriction of q to the sublattice $\mathbf{M}'$, the image by τ of the ring $\mathbf{Z}[\theta]$, is still a positive definite quadratic form whose matrix relative to the basis $\{1, \theta, \ldots, \theta^4\}$ is

$$\begin{pmatrix} 5 & S_1 & S_2 & S_3 & S_4 \\ S_1 & S_2 & S_3 & S_4 & S_5 \\ S_2 & S_3 & S_4 & S_5 & S_6 \\ S_3 & S_4 & S_5 & S_6 & S_7 \\ S_4 & S_5 & S_6 & S_7 & S_8 \end{pmatrix}$$

and has determinant $d(\theta) = f^2 \cdot d_K$, where f is the index of the ring $\mathbf{Z}[\theta]$ in $\mathbf{Z}_K$.

Our computations are based upon the following fundamental result:

Theorem 1 (Hunter [5], Pohst [9], Martinet [8]). *There exists an algebraic integer* $\theta \in K$, $\theta \notin \mathbf{Q}$, *such that*

$$(2) \qquad\qquad 0 \leq \mathrm{Tr}_{K/\mathbf{Q}}(\theta) \leq 2,$$

$$(3) \qquad\qquad 6 \leq \mathrm{Tr}_{K/\mathbf{Q}}(\theta^2) \leq \frac{(\mathrm{Tr}_{K/\mathbf{Q}}(\theta))^2}{5} + 35.566.$$

For each value of S_2 in the range defined by inequalities (3) we have:

Lemma 1. *The polynomial* $P(X)$ *for which* θ *is a root satisfies*

$$(4) \qquad\qquad |a_5| = |P(\theta)| \leq (S_2/5)^{2.5},$$

$$(5) \qquad\qquad |P(1)| \leq \left(\frac{S_2 - 2S_1 + 5}{5}\right)^{2.5},$$

$$(6) \qquad\qquad |P(-1)| \leq \left(\frac{S_2 + 2S_1 + 5}{5}\right)^{2.5}.$$

This follows from the inequalities between arithmetic and geometric means.

Conditions (2)–(6) suffice to construct all the polynomials where a root generates each one of the fields K under consideration; however, in order to minimize the number of polynomials to be considered, we will try to find, in a

different way, more inequalities for the coefficients of $P(X)$ and for the symmetric functions S_k for $k = 3, 4$. Let us fix the value of a_5; if we write $N = |a_5|$ ($\neq 0$) and $T = S_2$ we have:

Theorem 2 (Pohst [9]). *Let* y_t *denote the smallest positive root of the equation*

$$t \cdot \left(\frac{N}{x^{5-t}}\right)^{2/t} + (5-t) \cdot x^2 = T, \qquad t = 1, 2, 3, 4,$$

and

$$T_m = \max_{1 \leq t < 4} \left\{ t \cdot \left(\frac{N}{y_t^{5-t}}\right)^{m/t} + (5-t) y_t^m \right\} \quad \text{for } m = -1, 3, 4.$$

Then we have the inequalities

(7) $-|a_5| T_{-1} \leq a_4 \leq |a_5| T_{-1}$,

(8) $-T_3 \leq S_3 \leq T_3$,

(9) $0 < S_4 \leq T_4$.

Moreover, the coefficients of $P(X)$ satisfy Newton's inequalities because the roots of this polynomial are real:

(10) $2a_1 a_3 \leq a_2^2$,

(11) $2a_2 a_4 \leq a_3^2$,

(12) $5a_3 a_5 \leq 2a_4^2$.

We should also bear in mind the fact that the second derivative of $P(X)$ has three real roots and, consequently, a positive discriminant. Following [5], we obtain the inequality

(13) $|25a_3 - 15a_1 a_2 + 4a_1^3| \leq \sqrt{2(2a_1^2 - 5a_2)^3}$.

The lower bound in inequality (9) can be improved if we consider the lattice $\mathbf{M}'$ and the restriction of the quadratic form q to the submodules containing points of $\mathbf{M}'$, where at least one of their coordinates in the basis $\{1, \theta, \theta^2, \theta^3, \theta^4\}$ is zero. These restrictions are also positive definite quadratic forms and therefore have a positive determinant. A first rough estimate is obtained by the inequality

(14) $\begin{vmatrix} S_2 & S_3 \\ S_3 & S_4 \end{vmatrix} > 0$,

and a stronger lower bound is obtained by

(15) $\begin{vmatrix} 5 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} > 0$.

3. THE COMPUTATIONS

We begin by fixing the (integer) value of S_2 in the range (3), and we consider the values of S_1 in the interval (2) having the same parity as that of S_2 because, according to the relations (1), the residue class $(\bmod k)$ of S_k is entirely determined by the residue class $(\bmod k)$ of $S_1, \ldots, S_{k-1}$ for $k = 2, 3, 4, 5$.

We first determine the coefficients a_1 and a_2 of $P(X)$, and then we compute $D' = 1.25(S_2 - a_1^2/5)^4$. According to Theorem 1 we know that all the number fields K of discriminant $d_K < D'$ have already been found for values of S_2 smaller than those we are now considering. The computation of D' allows us to restrict our research to the values of the discriminant d_K in the range $D' < d_K < 2\,000\,000$, which becomes narrower when the value of S_2 is crossed. The search for the inessential factors of the discriminant of the polynomial is then made easier.

Next we consider the intervals obtained for a_5, $P(1)$ and $P(-1)$, using inequalities (4)–(6). In the special case where $a_1 = 0$ we can take $a_5 > 0$; the value of a_5 being fixed, we use Theorem 2 to determine T_{-1}, T_3, and T_4.

We then choose the values for S_3 in $|S_3| \le T_3'$ with $T_3' = \min(T_3, \sqrt{S_2 T_4})$ because, according to inequality (14), we must have $S_3^2 < S_2 S_4$. We can now compute the value of a_3, which must satisfy inequalities (10) and (13).

Inequality (15) gives us a lower bound T_4' for S_4, and we take S_4 in the interval $[T_4', T_4]$ which is sometimes empty. This allows us to compute the last coefficient a_4 which is still unknown. We then use inequalities (7), (11), (12), (5), and (6) in that order. For the polynomials whose coefficients and power sums satisfy inequalities (1)–(15), we determine the number of real roots, using Sturm's Theorem, and when all the roots are real, we compute them with an accuracy of less than 10^{-11}.

To test whether or not $P(X)$ is reducible, we first consider the polynomial obtained from $P(X)$ by reduction $(\bmod 2)$ of their coefficients, and we compare it with the complete list of irreducible polynomials of degree five in $\mathbf{F}_2[X]$. When the reduced polynomial appears in this list, we know that $P(X)$ is irreducible in $\mathbf{Z}[X]$ and the test is finished. Otherwise, we use the roots to verify if there exist divisors of degree 1 or 2 of $P(X)$. Let us suppose that $P(X)$ is irreducible; then we compute an approximate value of its discriminant, using the formula $d_P = \prod_{i=1}^5 P'(\theta_i)$, and its exact value in the finite fields $\mathbf{F}_{100\,003}$ and $\mathbf{F}_{100\,019}$. The Chinese Remainder Theorem suffices to determine the exact value of d_K.

Next we try to decompose d_P in the form $d_P = f^2 \cdot d_K$ with $D' < d_K < 2\,000\,000$, and when this is possible, we consider the different prime divisors of f to decide whether or not they are inessential divisors of the discriminant. This can be done using Dedekind's Criterium (cf. [9]). If p is a prime which divides f and appears in the decomposition in primes of d_P at a high power, we construct a basis of integers of K to determine the exact power of p dividing d_K.

4. ISOMORPHIC FIELDS

In spite of the fact that the use of the lower bound D' reduces the number of polynomials generating number fields of the same discriminant, we usually obtain several polynomials for each discriminant. To decide whether or not two such polynomials correspond to isomorphic fields, we proceed in the following way: we fix a polynomial, among those having the same discriminant, for which the index f is as small as possible (this is the polynomial which appears in the table), and we fix an order for its roots $\theta_1, \ldots, \theta_5$. For each one of the other polynomials having the same value of d_K (designated in the sequel by $P'(X)$ with roots $\theta'_1, \ldots, \theta'_5$), we successively consider the products

$$s_1(\sigma) = \sum_{i=1}^{5} \theta_i \theta'_{\sigma(i)},$$

where σ describes all the permutations in five elements.

Considering the permutation σ, when we find a value of $s_1(\sigma)$ near an integer, we compute the products

$$s_k = s_k(\sigma) = \sum_{i=1}^{5} \theta_i^k \theta'_{\sigma(i)} \quad \text{for } k = 2, 3, 4,$$

and if s_2, s_3, and s_4 are also near integers, we try to solve the system

$$(16) \qquad \begin{pmatrix} 5 & S_1 & S_2 & S_3 & S_4 \\ S_1 & S_2 & S_3 & S_4 & S_5 \\ S_2 & S_3 & S_4 & S_5 & S_6 \\ S_3 & S_4 & S_5 & S_6 & S_7 \\ S_4 & S_5 & S_6 & S_7 & S_8 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix},$$

where $s_0 = S_1(\theta')$.

If $f \cdot x_i$ is near an integer for $0 \le i \le 4$, we verify that the polynomial with roots equal to

$$\theta'_i = x_0 + x_1 \theta_i + x_2 \theta_i^2 + x_3 \theta_i^3 + x_4 \theta_i^4 \quad \text{for } i = 1, \ldots, 5$$

is exactly $P'(X)$.

Thus, we obtain the following theorem.

Theorem 3. *Within the range covered by the table there exist two nonisomorphic fields of discriminant* $1\,810\,969$ *and two nonisomorphic fields of discriminant* $1\,891\,377$. *All the other fields in the table are characterized by their discriminant.*

Proof. In only two cases did the system (16) lack a solution $(x_0, x_1, x_2, x_3, x_4)$ with $f \cdot x_i$ integer for $i = 0, \ldots, 4$: this was the case for the polynomials $P_1(X) = X^5 - 4X^4 - 7X^3 + 7X^2 + 6X - 5$ with $P_1(\theta) = 0$ and $P'_1(X) = X^5 - 2X^4 - 8X^3 + 9X^2 + 16X - 7$ with $P'_1(\theta') = 0$ of discriminant $1\,810\,969$ as well as for the polynomials $P_2(X) = X^5 - X^4 - 7X^3 + 6X^2 + 9X - 6$ with

$P_2(\mu) = 0$ and $P_2'(X) = X^5 - 2X^4 - 9X^3 + 8X^2 + 26X + 9$ with $P_2'(\mu') = 0$ of discriminant $1\,891\,377 = 3^3 \cdot 70051$.

To prove that the fields $K = \mathbf{Q}(\theta)$ and $K' = \mathbf{Q}(\theta')$ are nonisomorphic, we consider the decomposition of the ideal (3) in these two fields; we have:
(3) remains prime in K,
$(3) = \wp_1 \cdot \wp_1' \cdot \wp_3$ in $\tilde{K}$, where $\wp_1 = (3, \theta' + 1)$, $\wp_1' = (3, \theta' - 1)$, and $\wp_3 = (3, \theta'^3 + \theta'^2 + 2\theta' + 1)$.
The fields $K_1 = \mathbf{Q}(\mu)$ and $K_1' = \mathbf{Q}(\mu')$ are not isomorphic either because we have
$(3) = \wp_1^3 \cdot \wp_2$ in K_1, where $\wp_1 = (3, \mu)$ and $\wp_2 = (3, \mu^2 + 2\mu + 2)$, and
$(3) = \wp_1 \cdot \wp_1' \cdot \wp_1''^3$ in K_1', where $\wp_1 = (3, \mu')$, $\wp_1'' = (3, \mu' + 1)$, and $\wp_1'' = (3, \mu' - 1)$. $\square$

5. GALOIS GROUPS

Theorem 4. *The fields of discriminant* $14\,641 = 11^4$, $390\,625 = 5^8$, *and* $923\,521 = 31^4$ *are cyclic.*

These fields are well known.

Remarks. 1. The field of discriminant $14\,641$ is generated by a root θ of the polynomial $X^5 - X^4 - 4X^3 + 3X^2 + 3X - 1$. It can easily be verified that the other roots are $-\theta^2 + 2$, $\theta^3 - 3\theta$, $-\theta^4 + 4\theta^2 - 2$, and $\theta^4 - \theta^3 - 3\theta^2 + 2\theta + 1$.

2. The field of discriminant $390\,625$ is generated by a root θ of the polynomial $X^5 - 10X^3 + 5X^2 + 10X + 1$ and the other roots are

$$(-2\theta^4 + \theta^3 + 23\theta^2 - 18\theta - 25)/7, \qquad (4\theta^4 - 2\theta^3 - 39\theta^2 + 36\theta + 22)/7,$$

$$(\theta^4 + 3\theta^3 - 8\theta^2 - 19\theta + 9)/7, \qquad (-3\theta^4 - 2\theta^3 + 24\theta^2 - 6\theta - 6)/7.$$

3. The field of discriminant $923\,521$ is generated by a root θ of the polynomial $X^5 - X^4 - 12X^3 + 21X^2 + X - 5$ and the other roots are

$$(2\theta^4 - \theta^3 - 22\theta^2 + 31\theta)/5, \qquad (2\theta^4 + 4\theta^3 - 17\theta^2 - 14\theta + 10)/5,$$

$$(-3\theta^4 - \theta^3 + 33\theta^2 - 24\theta - 15)/5, \qquad (-\theta^4 - 2\theta^3 + 6\theta^2 + 2\theta + 10)/5.$$

Theorem 5. *The Galois group of the Galois closure of the fields of discriminant* $160\,801 = 401^2$, $667\,489 = 19^2 \cdot 43^2$, $1\,194\,649 = 1\,093^2$, *and* $1\,940\,449 = 7^2 \cdot 199^2$ *is the dihedral group* D_5.

It can easily be verified that the quadratic fields $\mathbf{Q}(\sqrt{401})$, $\mathbf{Q}(\sqrt{817})$, $\mathbf{Q}(\sqrt{1\,093})$, and $\mathbf{Q}(\sqrt{1\,393})$ have class number equal to 5.

Theorem 6. *The Galois group of the Galois closure of the totally real quintic number fields of discriminant less than* $2\,000\,000$ *not indicated in Theorems 4 and 5 above is the symmetric group* S_5.

Proof. We know from [8] that the minimal discriminant for the totally real quintics whose Galois closure has a Galois group isomorphic to the affine group Aff_5 is $2\,382\,032$ and out of the range of the table. The discriminant of the number fields having a Galois closure with Galois group isomorphic to A_5 is a square, but there are only seven perfect squares among the discriminants in the table and their group has been established in Theorems 4 and 5. This proves our theorem. $\square$

Remark. The minimal discriminant for the totally real quintics whose Galois group of its Galois closure is the group A_5 seems to be the field of discriminant $3\,104\,644 = 2^2 \cdot 881^2$ generated by a root of the polynomial $X^5 - X^4 - 11X^3 + X^2 + 12X - 4$.

BIBLIOGRAPHY

1. A.-M. Bergé, J. Martinet, and M. Olivier, *The computation of sextic fields with a quadratic subfield*, Math. Comp. **54** (1990), 869–884.

2. J. Buchmann and D. Ford, *On the computation of totally real quartic fields of small discriminant*, Math. Comp. **52** (1989), 161–174.

3. P. Cartier and Y. Roy, *On the enumeration of quintic fields with small discriminant*, J. Reine Angew. Math. **268/269** (1974), 213–215.

4. H. Cohn, *A numerical study of quintics of small discriminant*, Comm. Pure Appl. Math. **8** (1955), 377–386.

5. J. Hunter, *The minimum discriminant of quintic fields*, Glasgow Math. Assoc. **3** (1957), 57–67.

6. S. H. Kwon and J. Martinet, *Sur les corps résolubles de degré premier*, J. Reine Angew. Math. **375/376** (1987), 12–23.

7. J. Liang and H. Zassenhauss, *The minimum discriminant of sixth degree totally complex algebraic number fields*, J. Number Theory **9** (1977), 16–35.

8. J. Martinet, *Méthodes géométriques dans la recherche des petits discriminants*, Séminaire de Théorie des Nombres de Paris 1983/1984, Birkhäuser, Basel, 1985, pp. 147–179.

9. M. Pohst, *On the computation of number fields of small discriminant including the minimum discriminants of sixth degree fields*, J. Number Theory **14** (1982), 99–117.

10. D. G. Rish, *On algebraic number fields of degree five*, Vestnik Moscov. Univ. Ser. I. Mat. Mekh. **37** (1982), 76–80.

11. K. Takeuchi, *Totally real algebraic number fields of degree 5 and 6 with small discriminant*, Saitama Math. J. **2** (1984), 21–31.

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