A CLASS OF SLOWLY CONVERGENT SERIES AND THEIR SUMMATION BY GAUSSIAN QUADRATURE

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ABSTRACT. Series are considered whose general term is a rational function multiplied by a fractional power. The summation of such series is reduced, via Laplace transformation techniques, to a problem of quadrature, which is then solved by Gaussian quadrature relative to Einstein and Fermi weight functions. A number of examples are worked out in detail.

1. INTRODUCTION

We consider series of the type

$$(1.0) \qquad S_0 = \sum_{k=1}^{\infty} k^{\nu-1} r(k)$$

or

$$(1.1) \qquad S_1 = \sum_{k=1}^{\infty} (-1)^{k-1} k^{\nu-1} r(k),$$

where $0 < \nu \le 1$ and $r(\cdot)$ is a rational function

$$(1.2) \qquad r(s) = \frac{p(s)}{q(s)}$$

with p, q real polynomials of degrees $\deg p \le \deg q$. Strict inequality is assumed when necessary for convergence. It is further assumed that the zeros of q all have nonpositive real parts:

$$(1.3) \qquad \text{if } q(-a) = 0 \text{ then } \operatorname{Re} a \ge 0.$$

This condition can always be achieved by a preliminary summation of a few initial terms.

The problem can be simplified by first obtaining the partial fraction decomposition of r,

$$(1.4) \quad r(s) = \sum_{\rho} \sum_{m=1}^{m_\rho} c_{\rho m} (s + a_\rho)^{-m} + \sum_{\gamma} \sum_{m=1}^{m_\gamma} [c_{\gamma m} (s + a_\gamma)^{-m} + \overline{c}_{\gamma m} (s + \overline{a}_\gamma)^{-m}],$$

where the first sum is over all real zeros $(-a_\rho)$ of q (with multiplicities m_ρ), and the second sum is over all pairs of conjugate complex zeros $(-a_\gamma, -\overline{a}_\gamma)$ (with multiplicities m_γ). The coefficients $c_{\rho m}$ in the first sum are real, those in the second complex, in general. (We have assumed in (1.4) that $\deg p <$ $\deg q$. If $\deg p = \deg q$, and we are thus dealing with S_1 in (1.1), there will be an additional constant term in (1.4). Its contribution to the series in (1.1) is expressible in terms of the Riemann zeta function; cf. (3.9) below for $m = 0$.) Once the decomposition (1.4) has been obtained (for relevant constructive methods, see, e.g., [8, §7.1]), it clearly suffices to consider

$$(1.5) \qquad r(s) = \frac{1}{(s+a)^m}, \qquad \operatorname{Re} a \geq 0, \ m \geq 1.$$

Without restriction of generality, it may be further assumed that $\operatorname{Im} a \geq 0$.

By interpreting the terms in the series (1.0) and (1.1) as Laplace transforms at integer values, it is possible to express the sum of the series as a weighted integral over $\mathbb{R}_+$ of certain special functions related to the incomplete gamma function. The weighting involves the product of a fractional power and either Einstein's function $t(e^t - 1)^{-1}$ (in the case of (1.0)), or Fermi's function $(e^t + 1)^{-1}$ (in the case of (1.1)), both having infinitely many poles on the imaginary axis of the complex plane. Properties of the required special functions are briefly developed in §2. Section 3 discusses the summation of (1.0), (1.1) via Gaussian quadrature. The case $\nu = 1$ of purely rational series is treated in §4 and complements more traditional techniques (e.g., those in [8, §7.2II]). Numerical examples for the case $\nu = \frac{1}{2}$ are given in §5, where also comparisons are made with direct summation and accelerated summation using the ε-algorithm.

2. Preliminaries

Define, for $t > 0$,

$$(2.1_{-1}) \qquad g_{-1}(t; \nu) = g_{-1}(t) = \frac{t^{-1}}{\Gamma(1-\nu)}, \qquad 0 < \nu < 1,$$

and for $t > 0$, $n = 0, 1, 2, \dots$,

$$(2.1_n) \qquad g_n(t; a, \nu) = g_n(t) = \frac{e^{-at} t^{\nu-1}}{n! \Gamma(1-\nu)} \int_0^t e^{a\tau} (t-\tau)^n \tau^{-\nu} \, d\tau,$$
$$\operatorname{Re} a \geq 0, \quad \operatorname{Im} a \geq 0, \quad a \neq 0, \quad 0 < \nu < 1.$$

Lemma 2.1. *We have*

$$(2.2) \qquad g_0(t; a, \nu) = e^{-at} \gamma^*(1 - \nu, -at),$$

where γ^* is Tricomi's form of the incomplete gamma function (cf. [3, eq. 6.5.4]). Furthermore,

$$(2.3) \qquad g_{n+1}(t) = \frac{1}{n+1} \left\{ \left(t + \frac{n+1-\nu}{a} \right) g_n(t) - \frac{t}{a} g_{n-1}(t) \right\},$$
$$n = 0, 1, 2, \dots.$$

Proof. By definition,

$$g_0(t) = \frac{e^{-at}t^{\nu-1}}{\Gamma(1-\nu)}\int_0^t e^{a\tau}\tau^{-\nu}\,d\tau,$$

which, upon the change of variables $u = -a\tau$, gives

$$g_0(t) = \frac{e^{-at}(-at)^{\nu-1}}{\Gamma(1-\nu)}\int_0^{-at} e^{-u}u^{-\nu}\,du$$

$$= \frac{e^{-at}(-at)^{-(1-\nu)}}{\Gamma(1-\nu)}\gamma(1-\nu,-at) = e^{-at}\gamma^*(1-\nu,-at)$$

(cf. [3, eqs. 6.5.2, 6.5.4]). Next,

$$g_{n+1}(t) = \frac{e^{-at}t^{\nu-1}}{(n+1)!\Gamma(1-\nu)}\int_0^t e^{a\tau}(t-\tau)^n(t-\tau)\tau^{-\nu}\,d\tau$$

$$= \frac{t}{n+1}g_n(t) - \frac{e^{-at}t^{\nu-1}}{(n+1)!\Gamma(1-\nu)}\int_0^t e^{a\tau}(t-\tau)^n\tau^{1-\nu}\,d\tau.$$

To the last integral we apply integration by parts, letting $u = \tau^{1-\nu}(t-\tau)^n$, $v' = e^{a\tau}$, hence

$$u' = (1-\nu)\tau^{-\nu}(t-\tau)^n - n\tau^{1-\nu}(t-\tau)^{n-1}$$

$$= (n+1-\nu)(t-\tau)^n\tau^{-\nu} - nt(t-\tau)^{n-1}\tau^{-\nu},$$

$$v = \frac{1}{a}e^{a\tau}.$$

This yields (2.3) for $n \geq 1$. A similar calculation gives

$$g_1(t) = \left(t + \frac{1-\nu}{a}\right)g_0(t) - \frac{1}{a\Gamma(1-\nu)},$$

which, in view of (2.1_{-1}), shows that (2.3) holds also for $n = 0$. $\square$

To the author's knowledge, no software seems to be readily available for computing the incomplete gamma function in the domain of interest here (left half plane), and one thus has to rely on standard techniques such as power series and asymptotic expansions. An effort of developing good software for Tricomi's function $\gamma^*(1-\nu,z)$, $0 < \nu < 1$, applicable for arbitrary complex z, would certainly be worthwhile. For the case $\nu = \frac{1}{2}$, however, see §5.

3. SUMMATION OF S_0 AND S_1

We employ a technique already used in [7], namely, to interpret the general term of the series as a Laplace transform and thereby converting the series into a suitably weighted integral. We first treat the series S_0 in (1.0).

Assume $r(\cdot)$ given as in (1.5), and consider first the case $a \neq 0$. Then

$$(3.1) \qquad s^{\nu-1}\cdot\frac{1}{(s+a)^m} = \mathscr{L}\left\{\frac{t^{-\nu}}{\Gamma(1-\nu)} * \frac{t^{m-1}e^{-at}}{(m-1)!}\right\}, \qquad 0 < \nu < 1,$$

where $\mathscr{L}\{\cdot\}$ is the Laplace transform and $*$ means convolution. We thus have

$$f(t) := \mathscr{L}^{-1}\left\{\frac{s^{\nu-1}}{(s+a)^m}\right\} = \frac{1}{(m-1)!\Gamma(1-\nu)}\int_0^t e^{-a(t-\tau)}(t-\tau)^{m-1}\tau^{-\nu}\,d\tau$$

$$(3.2) \quad = \frac{e^{-at}}{(m-1)!\Gamma(1-\nu)}\int_0^t e^{a\tau}(t-\tau)^{m-1}\tau^{-\nu}\,d\tau$$

$$= t^{1-\nu}g_{m-1}(t;a,\nu), \qquad m \geq 1,$$

where $g_n(t;a,\nu)$ is defined in (2.1_n). There follows

$$\sum_{k=1}^\infty \frac{k^{\nu-1}}{(k+a)^m} = \sum_{k=1}^\infty (\mathscr{L}f)(k) = \sum_{k=1}^\infty \int_0^\infty e^{-kt}f(t)\,dt$$

$$= \sum_{k=1}^\infty \int_0^\infty e^{-t}t^{1-\nu}e^{-(k-1)t}g_{m-1}(t;a,\nu)\,dt$$

$$= \int_0^\infty e^{-t}t^{1-\nu}\sum_{k=1}^\infty e^{-(k-1)t}g_{m-1}(t;a,\nu)\,dt$$

$$= \int_0^\infty t^{1-\nu}\frac{1}{e^t-1}g_{m-1}(t;a,\nu)\,dt.$$

Thus,

$$(3.3) \quad \sum_{k=1}^\infty \frac{k^{\nu-1}}{(k+a)^m} = \int_0^\infty t^{-\nu}\varepsilon(t)g_{m-1}(t;a,\nu)\,dt, \qquad m \geq 1, \; 0 < \nu < 1,$$

where $\varepsilon(\cdot)$ is "Einstein's function" (cf. [7]),

$$(3.4) \qquad\qquad\qquad \varepsilon(t) = \frac{t}{e^t-1}, \qquad 0 \leq t < \infty.$$

Since $g_{m-1}(\cdot;a,\nu)$, by Lemma 2.1, is an entire function, formula (3.3) suggests to apply Gaussian quadrature to the integral on the right, using $w(t) = t^{-\nu}\varepsilon(t)$ as a weight function on $[0,\infty]$. The required orthogonal polynomials can be computed by techniques already discussed in [7] (see, in particular, (2.3) and Example 4.4 of that paper). Gauss quadrature will converge quite rapidly, unless $\mathrm{Re}\,a$ and/or $\mathrm{Im}\,a$ is large, in which case "stratified" summation can be employed to regain rapid convergence (see Examples 5.1 and 5.4). For $\nu = \frac{1}{2}$, the first 80 recursion coefficients for the required orthogonal polynomials are given to 25 significant digits in Table 1 of the Appendix.

It is easily seen that (3.3) holds also for $a = 0$ if we define

$$(3.5) \quad g_0(t) = \frac{1}{\Gamma(2-\nu)}, \qquad g_{n+1}(t) = \frac{t}{n+2-\nu}g_n(t),$$

$$n = 0, 1, 2, \ldots \; (a = 0).$$

The series can then be expressed in terms of Riemann's zeta function,

$$(3.6) \qquad\qquad\qquad \sum_{k=1}^\infty k^{-(m+1-\nu)} = \zeta(m+1-\nu),$$

and since g_{m-1} is a monomial of degree $m-1$, the n-point Gauss formula for the integral in (3.3) gives exact answers (modulo rounding) if $n = \lfloor (m+1)/2 \rfloor$.

For the sum S_1 in (1.1), a calculation similar to the one which led to (3.3) now yields, for $a \neq 0$,

$$(3.7) \qquad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^{\nu-1}}{(k+a)^m} = \int_0^\infty t^{-\nu} \varphi(t) \cdot t g_{m-1}(t; a, \nu) \, dt,$$

$$m \geq 0, \ 0 < \nu < 1,$$

where $\varphi(\cdot)$ is the "Fermi weight function" (cf. [7]),

$$(3.8) \qquad \varphi(t) = \frac{1}{e^t + 1}, \qquad 0 \leq t < \infty.$$

The result (3.7), as noted, holds also for $m = 0$, if g_{-1} is defined as in (2.1_{-1}). If $a = 0$, then (3.7) holds with g_{m-1}, $m \geq 0$, defined in (2.1_{-1}) and (3.5), and represents the series

$$(3.9) \qquad \sum_{k=1}^{\infty} (-1)^{k-1} k^{-(m+1-\nu)} = (1 - 2^{-m+\nu}) \zeta(m+1-\nu).$$

Again, Gauss quadrature for the integral in (3.7), in this case, is exact if we take $n = \lfloor (m+2)/2 \rfloor$ points.

The first 80 recursion coefficients for the orthogonal polynomials with respect to the weight function $t^{-\nu} \varphi(t)$, $\nu = \frac{1}{2}$, are listed in Table 2 of the Appendix.

Remarks. 1. The fractional power $k^{\nu-1}$ in (1.0), (1.1) can easily be generalized to $(k+b)^{\nu-1}$, $\operatorname{Re} b \geq 0$, since this only introduces a factor e^{-bt} in (3.1) and gives $f(t) = t^{1-\nu} e^{-bt} g_{m-1}(t; a-b, \nu)$ in place of (3.2), hence

$$(3.10) \qquad \sum_{k=1}^{\infty} \frac{(k+b)^{\nu-1}}{(k+a)^m} = \int_0^\infty t^{-\nu} \varepsilon(t) e^{-bt} g_{m-1}(t; a-b, \nu) \, dt,$$

and the similarly generalized version of (3.7),

$$(3.11) \qquad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(k+b)^{\nu-1}}{(k+a)^m} = \int_0^\infty t^{-\nu} \varphi(t) \cdot t e^{-bt} g_{m-1}(t; a-b, \nu) \, dt.$$

2. As an alternative to Gauss quadrature with weight function $t^{-\nu} \varepsilon(t)$, one could write the integral in (3.3) as

$$(3.12) \qquad \int_0^\infty t^{-\nu} \varepsilon(t) g_{m-1}(t; a, \nu) \, dt = \int_0^\infty t^{-\nu} e^{-t} \cdot \frac{t}{1 - e^{-t}} g_{m-1}(t; a, \nu) \, dt$$

and apply Gauss-Laguerre quadrature to the integral on the right. The poles that were previously incorporated in the weight function $t^{-\nu} \varepsilon(t)$ then, however, become part of the integrand, which may adversely affect the speed of convergence of the quadrature scheme. This was indeed found to be the case when a is relatively small, but for larger values of a, in particular when used

in conjunction with stratified summation (cf. Examples 5.1 and 5.4), the Gauss-Laguerre method is competitive and, in Example 5.1, even more efficient.

The same alternative approach is possible for the integral in (3.7), if written as

$$(3.13) \quad \int_0^\infty t^{-\nu} \varphi(t) \cdot t g_{m-1}(t; a, \nu)\, dt = \int_0^\infty t^{-\nu} e^{-t} \cdot \frac{t}{1 + e^{-t}} g_{m-1}(t; a, \nu)\, dt,$$

although this time the poles affect convergence more severely, since they are half as close to the real axis than before. Still, for sufficiently large values of a, Gauss-Laguerre quadrature here, too, gains the upper hand.

Finally, if many different values of ν were contemplated, then Gauss-Laguerre would also be preferable, since the recursion coefficients for the respective orthogonal polynomials—the generalized Laguerre polynomials—are then explicitly known and need not be tabulated. In the case of the weight function $w(t) = t^{-\nu} e^{-t}$ on $(0, \infty)$, the corresponding (monic) orthogonal polynomials $\pi_k(\cdot) = \pi_k(\cdot; w)$ indeed satisfy

$$(3.14) \quad \begin{aligned} \pi_{k+1}(t) &= (t - \alpha_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \qquad k = 0, 1, 2, \ldots, \\ \pi_{-1}(t) &= 0, \qquad \pi_0(t) = 1, \end{aligned}$$

with the coefficients $\alpha_k = \alpha_k(w)$, $\beta_k = \beta_k(w)$ $[\beta_0(w) = \int_0^\infty w(t)\, dt]$ having the particularly simple form

$$(3.15) \quad \begin{aligned} \alpha_k(w) &= 2k + 1 - \nu, \qquad k \geq 0; \\ \beta_0(w) &= \Gamma(1 - \nu), \qquad \beta_k(w) = k(k - \nu), \quad k \geq 1 \end{aligned} \qquad (w = t^{-\nu} e^{-t}).$$

4. Series of Purely Rational Terms

So far, we assumed that $0 < \nu < 1$ in (1.0) and (1.1). The same techniques, however, are applicable when $\nu = 1$, i.e., for series

$$(4.1) \quad S_0' = \sum_{k=1}^\infty r(k), \qquad S_1' = \sum_{k=1}^\infty (-1)^{k-1} r(k).$$

One finds, when

$$(4.2) \quad r(s) = \frac{1}{(s + a)^m}, \qquad \operatorname{Re} a \geq 0,$$

for $m \geq 2$ that

$$(4.3) \quad \sum_{k=1}^\infty \frac{1}{(k + a)^m} = \frac{1}{(m - 1)!} \int_0^\infty \varepsilon(t) t^{m-2} e^{-at}\, dt,$$

and for $m \geq 1$ that

$$(4.4) \quad \sum_{k=1}^\infty \frac{(-1)^{k-1}}{(k + a)^m} = \frac{1}{(m - 1)!} \int_0^\infty \varphi(t) t^{m-1} e^{-at}\, dt,$$

with ε and φ as given in (3.4) and (3.8), respectively.

For more general rational functions

$$(4.5) \qquad r(s) = \sum_i \sum_{m=1}^{m_i} c_{im}(s + a_i)^{-m},$$

we get from (4.3)

$$(4.6) \qquad \sum_{k=1}^{\infty} r(k) = \int_0^{\infty} \varepsilon(t)g(t)\,dt,$$

where

$$(4.7) \qquad g(t) = \sum_i \sum_{m=1}^{m_i} \frac{c_{im}}{(m-1)!} t^{m-2} e^{-a_i t}.$$

Evidently, g is again an entire function, provided

$$(4.8) \qquad \sum_i c_{i1} = 0,$$

which is required for the series in (4.6) to converge. Similarly,

$$(4.9) \qquad \sum_{k=1}^{\infty} (-1)^{k-1} r(k) = \int_0^{\infty} \varphi(t)h(t)\,dt,$$

where

$$(4.10) \qquad h(t) = \sum_i \sum_{m=1}^{m_i} \frac{c_{im}}{(m-1)!} t^{m-1} e^{-a_i t},$$

an entire function for any choice of the coefficients c_{im}.

The first 40 recursion coefficients for the polynomials orthogonal with respect to ε and φ can be found to 25 significant digits in [7, Appendices A1 and A2].

The method described in this section provides an alternative to other summation/integration methods, such as those discussed in [8, §7.2II]. An advantage of the present method is that it leads to Gaussian quadrature of *entire* functions, a possible complication, that the interval of integration is infinite.

5. Examples

In all of our examples we take $\nu = \frac{1}{2}$. In this case, the function g_0 in (2.2) is given by (cf. [3, eq. 6.5.18])

$$(5.1) \qquad g_0\left(t; a, \frac{1}{2}\right) = e^{-at} \gamma^*\left(\frac{1}{2}, -at\right) = \frac{2}{\sqrt{\pi}} \frac{F(\sqrt{at})}{\sqrt{at}},$$

where F is Dawson's integral,

$$(5.2) \qquad F(z) = e^{-z^2} \int_0^z e^{t^2}\,dt.$$

For real z, this can be evaluated with an accuracy of up to about 20 significant digits, using the rational Chebyshev approximations given in [2].

All computations reported below were done in double precision on the Cyber 205 computer (the equivalent of about 29 decimal places).

Example 5.1. $S_0 = \sum_{k=1}^{\infty} k^{-1/2}/(k+a)^m$.

This series with $a = m = 1$ was communicated to the author by Professor P. J. Davis, who encountered it in his study of spirals [4].

We computed S_0 from (3.3) (with $\nu = \frac{1}{2}$), (5.1), and (2.1$_{-1}$), (2.3), for $a = .5, 1., 2., 4., 8.$ and $m = 1(1)5$. The integral in (3.3) was evaluated by n-point Gaussian quadrature rules, which were generated with the help of the recursion coefficients in Table 1 of the Appendix and well-known eigenvalue techniques (see, e.g., [6, §1.3(iv)]). In Table 5.1 we show only the results for $m = 1$; those for $m > 1$ are similar, but exhibit somewhat slower convergence. It is evident that, as a increases, convergence of the Gauss quadrature formula slows down considerably. The reason for this is the behavior of the function g_0 on the right of (5.1), which for increasing a approaches a discontinuous function (see Figure 5.1).

TABLE 5.1

n-point quadrature approximations to the integral in (3.3) *with*
$\nu = \frac{1}{2}$, $m = 1$, $a = .5, 1., 2., 4., 8.$

n	$a = .5$	$a = 1.$	$a = 2.$
5	2.1344163	1.8599	1.537
10	2.1344166429861	1.860025078	1.53967
15	2.1344166429862372611	1.86002507922117	1.539680509
20	2.1344166429862372611	1.860025079221190306	1.539680512350
25		1.8600250792211903071	1.53968051235329
30		1.8600250792211903072	1.539680512353302010
35			1.5396805123533020128
40			1.5396805123533020128

n	$a = 4.$	$a = 8.$
5	1.19	.8
10	1.217	.91
15	1.21826	.930
20	1.218273	.9312
25	1.218274011	.93135
30	1.21827401461	.931371
35	1.218274014668	.9313727
40	1.2182740146698	.93137291

To achieve better accuracy, when a is large, we proceed as follows. With $a_0 = \lfloor a \rfloor$ denoting the largest integer $\leq a$, and $a = a_0 + a_1$, where $a_0 \geq 1$, $0 \leq a_1 < 1$, the summation over all $k \geq 1$ may be "stratified" by letting $k = \lambda + \kappa a_0$ and summing over all $\kappa \geq 0$ for $\lambda = 1, 2, \ldots, a_0$. Thus,

$$
S_0 = \sum_{k=1}^{\infty} \frac{k^{-1/2}}{(k + a_0 + a_1)^m} = \sum_{\lambda=1}^{a_0} \sum_{\kappa=0}^{\infty} \frac{(\lambda + \kappa a_0)^{-1/2}}{(\lambda + \kappa a_0 + a_0 + a_1)^m}
$$

$$
(5.3) \quad = a_0^{-(m+1/2)} \sum_{\lambda=1}^{a_0} \sum_{\kappa=0}^{\infty} \frac{(\kappa + \lambda/a_0)^{-1/2}}{(\kappa + 1 + (\lambda + a_1)/a_0)^m}
$$

$$
= a_0^{-(m+1/2)} \sum_{\lambda=1}^{a_0} \left\{ \sum_{\kappa=1}^{\infty} \frac{(\kappa + \lambda/a_0)^{-1/2}}{(\kappa + 1 + (\lambda + a_1)/a_0)^m} + \frac{(\lambda/a_0)^{-1/2}}{(1 + (\lambda + a_1)/a_0)^m} \right\}.
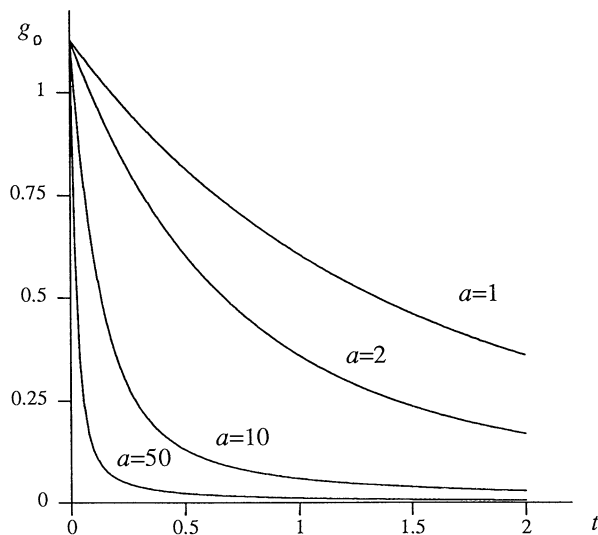$$

FIGURE 5.1

The function g_0 in (5.1)

To the inner sum we now apply (3.10) to obtain

(5.4)
$$\sum_{\kappa=1}^{\infty} \frac{(\kappa + \lambda/a_0)^{-1/2}}{(\kappa + 1 + (\lambda + a_1)/a_0)^m}$$
$$= \int_0^{\infty} t^{-1/2} \varepsilon(t) e^{-(\lambda/a_0)t} g_{m-1}\left(t; 1 + \frac{a_1}{a_0}, \frac{1}{2}\right) dt.$$

The "effective" parameter in g_{m-1} is now $1 + a_1/a_0$, a number close to 1, and the coefficient λ/a_0 in the exponential is bounded by 1. Gauss quadrature applied to the integral in (5.4) should therefore converge quite rapidly, indeed, more so the larger a_0! This is borne out by the results displayed for $m = 1$ and $a = a_0 = 8., 16., 32.$ in Table 5.2.

The correct number of significant decimal digits produced by direct summation of the series, using 1000 terms, is shown in Table 5.3. The numbers in parentheses are the correct digits obtained by applying the ε-algorithm with the same number of terms. For the entries marked by an asterisk, we used (5.3),

TABLE 5.2

Approximations to S_0 in (5.3) for $m = 1$, $a_1 = 0$, $a_0 = 8.,$
16., 32., using n-point quadrature in (5.4)

n	$a = 8.$	$a = 16.$	$a = 32.$
5	.93098	.6946	.5097
10	.9313726	.6949315	.5099264
15	.931372933	.6949317145	.5099265169
20	.9313729340028	.6949317146409	.5099265170271
25	.9313729340031036	.6949317146410454	.5099265170272112
30	.9313729340031038714	.6949317146410455900	.5099265170272113479
35	.9313729340031038716 9	.6949317146410455901 6	.5099265170272113488 04
40	.9313729340031038716 9	.6949317146410455901 6	.5099265170272113488 04

TABLE 5.3

Number of correct significant decimal digits in direct (and accelerated) summation of S_0 using 1000 terms

m	$a = .5$	$a = 1.$	$a = 2.$	$a = 4.$	$a = 8.$
1	2 (2)	1 (2)	2 (2)	2 (2)	1 (1)
2	4 (5)	4 (5)	4 (5)	4 (5)	3 (4)
3	7 (9)	7 (9)	6 (8)	6 (8)	5 (7)*
4	10 (13)	9 (12)	9 (12)	8 (11)*	8 (10)*
5	14 (17)	13 (15)	11 (14)	11 (13)*	10 (12)*

(5.4) to verify the number of correct digits. As is evident from Table 5.3, the ε-algorithm is only marginally effective on this particular series.

Example 5.2. $S_1 = \sum_{k=1}^{\infty}(-1)^{k-1}k^{-1/2}/(k+a)^m$.

We now apply Gauss quadrature (obtained from the recursion coefficients in Table 2 of the Appendix) relative to the weight function $t^{-1/2}\varphi(t)$ to the integral in (3.7) (with $\nu = \frac{1}{2}$), using the same values of a and m as in Example 5.1. The results are similar to those in Table 5.1 of Example 5.1, except that convergence is slightly slower. Stratified summation similar to (5.3), (5.4), on the other hand, gives

$$(5.5) \qquad S_1 = a_0^{-(m+1/2)} \sum_{\lambda=1}^{a_0} (-1)^\lambda \left\{ s_\lambda - \frac{(\lambda/a_0)^{-1/2}}{(1 + (\lambda + a_1)/a_0)^m} \right\},$$

where

$$(5.6) \qquad s_\lambda = \begin{cases} -\displaystyle\sum_{\kappa=1}^{\infty} \frac{(\kappa + \lambda/a_0)^{-1/2}}{(\kappa + 1 + (\lambda + a_1)/a_0)^m} & \text{if } a_0 \text{ is even}, \\ \displaystyle\sum_{\kappa=1}^{\infty} \frac{(-1)^{\kappa-1}(\kappa + \lambda/a_0)^{-1/2}}{(\kappa + 1 + (\lambda + a_1)/a_0)^m} & \text{if } a_0 \text{ is odd}, \end{cases}$$

that is, by (3.10) and (3.11),

$$(5.7) \quad s_\lambda = \begin{cases} -\displaystyle\int_0^{\infty} t^{-1/2}\varepsilon(t)e^{-(\lambda/a_0)t} g_{m-1}(t; 1 + a_1/a_0, 1/2)\, dt, & a_0 \text{ even}, \\ \displaystyle\int_0^{\infty} t^{-1/2}\varphi(t) \cdot te^{-(\lambda/a_0)t} g_{m-1}(t; 1 + a_1/a_0, 1/2)\, dt, & a_0 \text{ odd}. \end{cases}$$

The use of (5.5), with n-point quadrature applied to (5.7), yields results converging at the same speed as those in Table 5.2.

Direct summation using 1000 terms yields 2–3 more correct digits than in Table 5.3, but the ε-algorithm is now surprisingly effective, giving full accuracy (20 decimals) with only 23–27 terms!

Example 5.3. $S_0(b) = \sum_{k=1}^{\infty} (k+b)^{-1/2}/(k+b+1)$, $0 \leq b < 1$.

This is another series of interest in P. J. Davis's work on spirals [4]. It can be readily evaluated with the help of Remark 1 in §3, taking $m = 1$, $a = b + 1$ in (3.10), and noting (5.1); one finds

$$(5.8) \quad \begin{aligned} S_0(b) &= \int_0^\infty t^{-1/2} \varepsilon(t) e^{-bt} g_0\left(t; 1, \frac{1}{2}\right) dt \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-bt} \frac{F(\sqrt{t})}{\sqrt{t}} \cdot t^{-1/2} \varepsilon(t) \, dt. \end{aligned}$$

Thus, $S_0(b)$ is the Laplace transform of the function $(F(\sqrt{t})/\sqrt{t})t^{-1/2}\varepsilon(t)$, which has a square root singularity at the origin and poles at integer multiples of $2\pi i$. The last integral in (5.8) is easily computed by Gaussian quadrature relative to the weight function $t^{-1/2}\varepsilon(t)$. No more than 24 quadrature points are needed to get 14 correct significant decimal digits in the range $0 < b < 1$.

Example 5.4. $S_0 = \sum_{k=1}^{\infty} k^{-1/2}/(k+i\alpha)$, $\alpha > 0$.

Here, $a = i\alpha$, $m = 1$, hence

$$(5.9) \qquad S_0 = \int_0^\infty t^{-1/2} \varepsilon(t) g_0\left(t; i\alpha, \frac{1}{2}\right) dt.$$

Use of (5.1) and a simple change of variables gives

$$g_0\left(t; i\alpha, \frac{1}{2}\right) = \frac{ie^{-z^2}}{z} \operatorname{erf}(-iz), \qquad z = \sqrt{i\alpha t},$$

where erf is the error function. Letting $-iz = \frac{1}{2}\sqrt{\pi}(1-i)x$, i.e., $x = \sqrt{2\alpha t/\pi}$, one finds (cf. [5, eq. 7.3.22])

$$(5.10) \qquad g_0(t; i\alpha, \tfrac{1}{2}) = \sqrt{2/(\alpha t)} e^{-i\alpha^2 t^2} [C(\sqrt{2\alpha t/\pi}) + iS(\sqrt{2\alpha t/\pi})].$$

Here, $C(x)$, $S(x)$ denote the Fresnel integrals [5, eqs. 7.3.1, 7.3.2]. They can be computed with an accuracy of up to 18 significant digits from the rational Chebyshev approximations provided (on microfiche cards) in [1]. Gaussian quadrature of (5.9), with g_0 given by (5.10), yields the results shown in Table 5.4. For each n, the first entry is the real part, the second the imaginary part of the Gauss approximation to S_0. Convergence is seen to deteriorate rapidly with increasing α, which is to be expected in view of the highly oscillatory behavior of g_0 in (5.10) when α is large. The device of stratified summation, used successfully in Examples 5.1 and 5.2, however, can also be applied here,

TABLE 5.4

n-point quadrature approximations to the integral in (3.3) with
$\nu = \frac{1}{2}$, $m = 1$, $a = i\alpha$, $\alpha = .5, 1., 2., 4., 8.$

n	$\alpha = .5$	$\alpha = 1.$
10	2.382181322854	2.006153
	−.564259325223	−.7964883
20	2.38218132285517164	2.0061526552273
	−.56425932522086830	−.79648812356982
40	2.38218132285517164	2.00615265522741423
	−.56425932522086830	−.79648812356984801
80		2.00615265522741424
		−.79648812356984802

n	$\alpha = 2.$	$\alpha = 4.$	$\alpha = 8.$
10	1.52	1.14	1.0
	−.846	−.91	−.3
20	1.51822	1.10	.9
	−.84397	−.72	−.7
40	1.518231590364	1.0978	.73
	−.843981047692	−.745	−.54
80	1.51823159036615356	1.0976938	.77
	−.84398104769701668	−.7460348	−.599

and gives, with $\alpha_0 = \lfloor \alpha \rfloor$, $\alpha = \alpha_0 + \alpha_1$, $\alpha_0 \geq 1$, $0 \leq \alpha_1 < 1$,

$$
(5.11) \quad S_0 = \alpha_0^{-3/2} \sum_{\lambda=1}^{\alpha_0} \left\{ \int_0^\infty t^{-1/2}\varepsilon(t)e^{-(\lambda/\alpha_0)t} g_0\left(t; i\left(1 + \frac{\alpha_1}{\alpha_0}\right), \frac{1}{2}\right) dt \right.
$$
$$
\left. + \frac{(\alpha_0/\lambda)^{3/2}}{1 + i(\alpha_0 + \alpha_1)/\lambda} \right\}.
$$

Using (5.11) instead of (5.9) yields the results shown in Table 5.5.

Direct summation using 1000 terms gives 1–2 correct decimal digits in the real part, and 4–5 in the imaginary part. The epsilon algorithm produces no more than one additional correct digit.

Example 5.5. $S_1 = \sum_{k=1}^\infty (-1)^{k-1} k^{-1/2}/(k + i\alpha)$, $\alpha > 0$.

Applying Gauss quadrature to

$$
(5.12) \quad S_1 = \int_0^\infty t^{-1/2}\varphi(t) \cdot t g_0(t; i\alpha, \tfrac{1}{2}) dt,
$$

with g_0 as in (5.10), or to formulae analogous to those in (5.5)–(5.7), yields results similar in quality to those in Table 5.4 (but converging at a slightly slower rate) and to those in Table 5.5. As in Example 5.2, here too, the ε-algorithm produces full accuracy (20 decimals) with as few as 25–28 terms.

TABLE 5.5

Approximations to S_0 for $\alpha_1 = 0$, $\alpha_0 = 8.$, $16.$, $32.$, using n-point quadrature for the integral in (5.11)

n	$\alpha = 8.$	$\alpha = 16.$	$\alpha = 32.$
5	.78217	.55456	.39250
	−.6028	−.46405	−.34703
10	.78214786	.554548189	.392496065
	−.60290377	−.464094441	−.347063784
15	.782147849839	.554548181558	.392496059680
	−.602903762412	−.464094436689	−.347063781179
20	.7821478498420740	.5545481815605363	.3924960596818862
	−.6029037624091237	−.46409443668759260	−.3470637811774942
25	.78214784984207490	.55454818156053686	.39249605968188663
	−.60290376240912468	−.46409443668759260	−.34706378117749456
30	.78214784984207491	.55454818156053686	.39249605968188663
	−.60290376240912469	−.46409443668759260	−.34706378117749456

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APPENDIX

Coefficients α_k, β_k in the recurrence relation (3.14) for the (monic) polynomials $\pi_k(\cdot; w_1)$ and $\pi_k(\cdot; w_2)$ orthogonal on $[0, \infty]$ with respect to the weight functions $w_1(t) = t^{-1/2}\varepsilon(t)$ and $w_2(t) = t^{-1/2}\varphi(t)$, where ε and φ are the Einstein and Fermi functions, respectively.

TABLE 1

Recursion coefficients for the polynomials $\{\pi_k(\cdot; w_1)\}$

k	alpha(k)	beta(k)
0	0.7702686701927817973619158D+00	0.2315157373394117000425819D+01
1	0.3187598556761524679366414D+01	0.1024084687983407303423387D+01
2	0.5263746923045607509713590D+01	0.4414290426470050351659216D+01
3	0.7301718513979321228472401D+01	0.9811302569554005490176024D+01
4	0.9325649742557246403991717D+01	0.1722019846597339069655335D+02
5	0.1134253380338705910700859D+02	0.2663816927777788994670644D+02
6	0.1335527862000350912776166D+02	0.3806308145171306188997379D+02
7	0.1536534463823498763879841D+02	0.5149347226123903980540211D+02
8	0.1737355807916766756284331D+02	0.6692831418597948517773595D+02
9	0.1938042671340346598781799D+02	0.8436685949264783306411561D+02
10	0.2138628230757703840201835D+02	0.1038085462169082643494963D+03
11	0.2339135212161886387283204D+02	0.1252529402397974594712127D+03
12	0.2539579780077243186696702D+02	0.1486996983438284336506672D+03
13	0.2739973790812632438336057D+02	0.1741485438093782211156081D+03
14	0.2940326166032188956953619D+02	0.2015992497717325190612671D+03
15	0.3140643765975902698313727D+02	0.2310516275456567160834720D+03
16	0.3340931964524983754293190D+02	0.2625055182290468743398986D+03
17	0.3541195039481598759588694D+02	0.2959607865315635323564955D+03
18	0.3741436444410603887748966D+02	0.3314173161504727690082355D+03

TABLE 1 (*continued*)

k	alpha(k)	beta(k)
19	0.39416590023263585452261533D+02	0.368875006246128392348 9100D+03
20	0.41418650464761856741811497D+02	0.408333768714383691640 4452D+03
21	0.43420565244991268776951773D+02	0.449793526046735232033 4015D+03
22	0.45422350767174583074598772D+02	0.493254209630791249026 9402D+03
23	0.47424020958286094251402127D+02	0.538715758385355637850 4431D+03
24	0.49425587730056977442132632D+02	0.586178117653102867958 4024D+03
25	0.51427061339201650743365927D+02	0.635641238293898737723 5091D+03
26	0.53428450671916934060193812D+02	0.687105075936104137096 7609D+03
27	0.55429763470782432459859497D+02	0.740569590353507437407 0377D+03
28	0.57431006517359307228941782D+02	0.796034744943072204068 2676D+03
29	0.59432185780362916516760592D+02	0.853500506284272427120 6002D+03
30	0.61433306536828190182706840D+02	0.912966843764972260308 6484D+03
31	0.63434373471900528796986440D+02	0.974433729261976155387 5038D+03
32	0.65435390761571238355914527D+02	0.103790113686679875151 00730D+04
33	0.67436362141699625774572290D+02	0.110336904264907444334 03161D+04
34	0.69437290965930137820205927D+02	0.117083742445148283899 4380D+04
35	0.71438180254556600517079700D+02	0.124030626171120925450 2793D+04
36	0.73439032735960077220210817D+02	0.131177553530386321478 5656D+04
37	0.75439850881918665662853807D+02	0.138524522740649775269 0127D+04
38	0.77440636937832458610121710D+02	0.146071532137694943812 1714D+04
39	0.79441392948707162373152675D+02	0.153818580164718472949 1338D+04
40	0.81442120781582404323611067D+02	0.161765665362871620509 1856D+04
41	0.83442822144965808849797744D+02	0.169912783628460750692 831D+04
42	0.85443498605734155233472288D+02	0.178259941877366496143 6173D+04
43	0.87444151603882807452353537D+02	0.186807130694473184256 5880D+04
44	0.89444782465439899808580567D+02	0.195554351671495596710 0250D+04
45	0.91445392413809230825583847D+02	0.204501603729631834934 3018D+04
46	0.93445982579762955930272487D+02	0.213648858584906127960 66675D+04
47	0.95446554010270029651380267D+02	0.222996197064527668694 9499D+04
48	0.97447107677631740756087072D+02	0.232543536461338444603 9056D+04
49	0.99447644479857079327406547D+02	0.242290903171733416219 0448D+04
50	0.10144816525999212006277577D+03	0.252238296637158191367 86609D+04
51	0.10344867079849836496548267D+03	0.262385715277372840136 9017D+04
52	0.10544916182476443092111057D+03	0.272733159143466494221 1447D+04
53	0.10744963902022114574067757D+03	0.283280672725958087121 38972D+04
54	0.10945010302232161211502807D+03	0.294028118948488450562 5506D+04
55	0.11149756534428124814500172D+03	0.304975653456390232510 21868D+04
56	0.11345099379752861586368017D+03	0.316123170488532992205 5971D+04
57	0.11545142165619198045292147D+03	0.327470729132299266275 1263D+04
58	0.11745183849818112574823177D+03	0.339018308930678633464 7040D+04
59	0.11945224478836990902782437D+03	0.350765909343183993429 0477D+04
60	0.12145264096462098035075577D+03	0.362713529851955153020 8766D+04
61	0.12345302743977098129155527D+03	0.374861169960454681697 1876D+04
62	0.12545340460344026333460347D+03	0.387208829192258832071 0826D+04
63	0.12745377282368516849534217D+03	0.399756507089935192025 6672D+04
64	0.12945413244850880957019367D+03	0.412504203213999584511 6965D+04
65	0.13145448380724445251977887D+03	0.425451917141945482446 7366D+04
66	0.13345482721182400692152747D+03	0.438599684467339817394 194D+04
67	0.13545516295794273463653277D+03	0.451947396798980086287 2556D+04
68	0.13745549132613006528000637D+03	0.465495161760106664220 5789D+04
69	0.13945581258273533553232217D+03	0.479242942987667742772 9711D+04
70	0.14145612698083632750973707D+03	0.493190740131630926933 5963D+04
71	0.14345643476107765202062297D+03	0.507338552854338940525 4518D+04
72	0.14545673615244529077211677D+03	0.521686380829905702801 4556D+04
73	0.14745703137298296479311847D+03	0.536234223743649773618 5135D+04
74	0.14945732063045542362519097D+03	0.550982081291562379711 8366D+04
75	0.15145760412296324184459297D+03	0.565929953179807477016 4711D+04
76	0.15345788203951325814744977D+03	0.581077839124251522404 0107D+04
77	0.15545815456054839058441507D+03	0.596425738850020825278 0461D+04
78	0.15745842185844020353565277D+03	0.611973652091084527535 6023D+04
79	0.15945868409794728243623573D+03	0.627721578589861421505 6571D+04

TABLE 2
Recursion coefficients for the polynomials $\{\pi_k(\cdot\,;w_2)\}$

k	alpha(k)	beta(k)
0	0.6324588697185093623661046D+00	0.1072154929940191339530897D+01
1	0.2618492484147360028201222D+01	0.6752170963175943015639712D+00
2	0.4579564963641043610171703D+01	0.3167583560258398238821178D+01
3	0.6564030779299476746087191D+01	0.7696159230518900819569277D+01
4	0.8555069923101791491989842D+01	0.1422284065830645961871458D+02
5	0.1054905551208604898429030D+02	0.2274710395677869363743930D+02
6	0.1254466067583060408401432D+02	0.3326936386778514311350517D+02
7	0.1454126841626332776533809D+02	0.4579000339917602081974728D+02
8	0.1653854758410777936978112D+02	0.6030931435273240849088733D+02
9	0.1853630238902031385655892D+02	0.7682751599549647344840230D+02
10	0.2053440866145723808122330D+02	0.9534477556959725017684018D+02
11	0.2253278334043201314293342D+02	0.1158612232118805656707516D+03
12	0.2453136848117392119444544D+02	0.1383769621807920822429557D+03
13	0.2653012226419184859211550D+02	0.1628920758679360137987405D+03
14	0.2852901364926778224060180D+02	0.1894066326819196264761766D+03
15	0.3052801904751258293005461D+02	0.2179206895182090910153207D+03
16	0.3252712016881291968149760D+02	0.2484342942741886876099256D+03
17	0.3452630258357236264912560D+02	0.2809474877069469776593879D+03
18	0.3652555473451833018142980D+02	0.3154603048296664459865959D+03
19	0.3852486724106963523121097D+02	0.3519727759780056334528845D+03
20	0.4052423239911886032854784D+02	0.3904849276362956159680060D+03
21	0.4252364381449926929769257D+02	0.4309967830860900405116312D+03
22	0.4452309612986561756291508D+02	0.4735083629213687739354546D+03
23	0.4652258481809677698689649D+02	0.5180196854622827391158301D+03
24	0.4852210602388269609628421D+02	0.5645307670907364429349043D+03
25	0.5052416544075394216094150D+02	0.6130416225250636069044409D+03
26	0.5252123321454810195841400D+02	0.6635522650467400693493852D+03
27	0.5452083386684847465695719D+02	0.7160627066889587821461353D+03
28	0.5652045623368841980826095D+02	0.7705729583946056944673543D+03
29	0.5852009841604980400440341D+02	0.8270830301494799173258353D+03
30	0.6051975873956415655717186D+02	0.8855929310953301741134029D+03
31	0.6251943572146066221850836D+02	0.9461026696263162082092677D+03
32	0.6451912804326958639256424D+02	0.1008612253471766812206150D+04
33	0.6651883452813302280453478D+02	0.1073121689767537171298761D+04
34	0.6851855412183125570209516D+02	0.1139630985117825228888483D+04
35	0.7051828578682640946987237D+02	0.1208140145648959135593239D+04
36	0.7251802893877224443308380D+02	0.1278649177056392968932633D+04
37	0.7451778253505194512639873D+02	0.1351158084645929041756438D+04
38	0.7651754596499318109446470D+02	0.1425666873370009681906469D+04
39	0.7851731859147789236506330D+02	0.1502175547859779860352491D+04
40	0.8051709983717/8497355304D+02	0.1580684112453507229192462D+04
41	0.8251688916100884308951775D+02	0.1661192571221852446032812D+04
42	0.8451668608731183487411340D+02	0.1743700927990405810243678D+04
43	0.8651649016653273904289527D+02	0.1828209186359842875030183D+04
44	0.8851630098839871314378399D+02	0.1914717349723999207645930D+04
45	0.9051611817484278269255930D+02	0.2003225421286120784146841D+04
46	0.9251594137682471448048561D+02	0.2093733440407350998874209D+04
47	0.9451577027152721523962350D+02	0.2186241300950756537132996D+04
48	0.9651560455987618875274102D+02	0.2280749114631716813598169D+04
49	0.9851544396434169830412752D+02	0.2377256847690383261846901D+04
50	0.1005152882269828385903095D+03	0.2475764502570766919286996D+04
51	0.1025151371077051766150250D+03	0.2576272081595900380206470D+04
52	0.1045149903827039775922085D+03	0.2678779586976054963886758D+04
53	0.1065148478430702518687758D+03	0.2783287020816254278645068D+04
54	0.1085147092935398729290198D+03	0.2889794385123156407511188D+04
55	0.1105145745513687301106930D+03	0.2998301681811368343859689D+04
56	0.1125144434453191782850635D+03	0.3108808912709248866969463D+04
57	0.1145143158147450000930684D+03	0.3221316079564249594294866D+04
58	0.1165141915087637613485474D+03	0.3335823184047838333965474D+04
59	0.1185140703855068636699258D+03	0.3452330227760043965955907D+04

TABLE 2 *(continued)*

k	alpha(k)	beta(k)
60	0.12051395231143881879755563D+03	0.3570837212233657801133153D+04
61	0.1225138371607383182344134D+03	0.3691344138938122617672131D+04
62	0.1245137248147345762237603D+03	0.3813851009283137281105733D+04
63	0.1265136151613932056286678D+03	0.3938357824622001955589468D+04
64	0.1285135080948465633552084D+03	0.4064864586254726357105646D+04
65	0.1305134035149640899308410D+03	0.4193371295430921239345142D+04
66	0.1325133013269586796188234D+03	0.4323877953352491301430689D+04
67	0.1345132014410255638900482D+03	0.4456384561176145930510639D+04
68	0.1365131037720105813681102D+03	0.4590891120015742613214127D+04
69	0.1385130082391050492885239D+03	0.4727397630944476443598310D+04
70	0.1405129147655647516722025D+03	0.4865904094996927900448475D+04
71	0.1425128232784509234276992D+03	0.5006410513170979945361452D+04
72	0.1445127337083905422629163D+03	0.5148916886429614489121902D+04
73	0.1465126459893562457077958D+03	0.5293423215702597373710512D+04
74	0.1485125600584607722422590D+03	0.5439929501888060208879617D+04
75	0.1505124758557679865105743D+03	0.5588435745853986675137462D+04
76	0.1525123933241170914867563D+03	0.5738941948439610250053694D+04
77	0.1545123124089595574811505D+03	0.5891448110456729724026014D+04
78	0.1565122330582076109876996D+03	0.6045954232690948338002006D+04
79	0.1585121552220933272468049D+03	0.6202460315902841892951491D+04

BIBLIOGRAPHY

1. W. J. Cody, *Chebyshev approximations for the Fresnel integrals*, Math. Comp. **22** (1968), 450–453. Loose microfiche suppl. A1–B4.

2. W. J. Cody, K. A. Paciorek, and H. C. Thacher, Jr., *Chebyshev approximations for Dawson's integral*, Math. Comp. **24** (1970), 171–178.

3. P. J. Davis, *Gamma function and related functions*, Handbook of Mathematical Functions, Chapter 6 (M. Abramowitz and I. A. Stegun, eds.), NBS Appl. Math. Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

4. ____, *Spirals: From Theodorus of Cyrene to Meta-Chaos*, The Hedrick Lectures, Math. Assoc. Amer., August, 1990.

5. W. Gautschi, *Error function and Fresnel integrals*, Handbook of Mathematical Functions, Chapter 7 (M. Abramowitz and I. A. Stegun, eds.), NBS Appl. Math. Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

6. ____, *Computational aspects of orthogonal polynomials*, Orthogonal Polynomials—Theory and Practice (P. Nevai, ed.), NATO ASI Series, Series C: Mathematical and Physical Sciences, vol. 294, Kluwer, Dordrecht, 1990, pp. 181–216.

7. W. Gautschi and G. V. Milovanović, *Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series*, Math. Comp. **44** (1985), 177–190.

8. P. Henrici, *Applied and computational complex analysis*, vol. 1, Wiley, New York, 1984.

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