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ON CERTAIN SLOWLY CONVERGENT SERIES OCCURRING IN PLATE CONTACT PROBLEMS

WALTER GAUTSCHI

ABSTRACT. A simple computational procedure is developed for accurately summing series of the form $\sum_{k=0}^{\infty}(2k+1)^{-p}z^{2k+1}$, where z is complex with $|z| \le 1$ and $p = 2$ or 3, as well as series of the type

$$\sum_{k=0}^{\infty}(2k+1)^{-p}\cosh(2k+1)x/\cosh(2k+1)b$$

and

$$\sum_{k=0}^{\infty}(2k+1)^{-p}\sinh(2k+1)x/\cosh(2k+1)b,$$

where $0 \le x \le b$, $p = 2$ or 3. The procedures are particularly useful in cases where the series converge slowly. Numerical experiments illustrate the effectiveness of the procedures.

1. INTRODUCTION

Our concern, in §§2–4, is with series of the type

$$R_p(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^p} \tag{1.1$_p$}$$

or the type

$$S_p(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)^p}, \tag{1.2$_p$}$$

where

$$z \in \mathbb{C}, \quad |z| \le 1, \quad \text{and} \quad p = 2 \text{ or } 3. \tag{1.3}$$

Of particular interest to us is the numerical evaluation of these series in cases of slow convergence, i.e., when $|z|$ is close or equal to 1. It clearly suffices to concentrate on the first of the two series, R_p, since

$$S_p(z) = iR_p(-iz). \tag{1.4}$$

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Furthermore, $R_p(-z) = -R_p(z)$ and $R_p(\overline{z}) = \overline{R_p(z)}$, so that attention can be restricted to the first quadrant of the complex plane.

Series of the type (1.1_p), with

$$(1.5) \qquad z = A, \quad 0 < A \le 1, \qquad \text{and} \qquad z = e^{i\alpha}, \quad \alpha \in \mathbb{R},$$

occur in the mathematical treatment of unilateral plate contact problems, and their numerical evaluation, in this context, has recently been discussed by K. M. Dempsey, D. Liu, and J. P. Dempsey [1]. The method proposed by these authors consists in applying Plana's summation formula, which in turn requires the numerical evaluation of several definite integrals—for example by Romberg integration.

Here we develop a technique which appears to be considerably simpler. All it requires is the application (in the backward direction) of a three-term recurrence relation, once a set of numerical constants has been precomputed. Results of high accuracy are easily achieved, even for $|z|$ near or equal to 1.

Some of the series (1.1_p), (1.2_p) with $p = 2$ or $p = 3$ can be summed explicitly as Fourier series when z is given by the second expression in (1.5). We thus have

$$(1.6_2) \quad \sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^2} = \pi(\pi - 2|\alpha|)/8, \qquad -\pi \le \alpha \le \pi \quad [8,\ (17.2.16)],$$

$$(1.6_3) \quad \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^3} = \pi\alpha(\pi - |\alpha|)/8, \qquad -\pi \le \alpha \le \pi \quad [8,\ (14.2.21)],$$

and analogous formulae for

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{-2} \sin(2k+1)\alpha, \qquad \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-3} \cos(2k+1)\alpha,$$

which can be obtained from (1.6) by applying (1.4). When $z = 1$, the sum of (1.1_p) is expressible in terms of the Riemann zeta function,

$$(1.7) \qquad R_p(1) = (1 - 2^{-p})\zeta(p),$$

whereas $S_2(1)$ is known as Catalan's constant, and $S_3(1) = \pi^3/32$. All these explicit formulae will be useful for testing purposes.

In §5 we combine our techniques of §2 with series expansion to deal with the more difficult series

$$(1.8_p) \qquad T_p(x, b) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^p} \frac{\cosh(2k+1)x}{\cosh(2k+1)b}$$

and

$$(1.9_p) \qquad U_p(x, b) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^p} \frac{\sinh(2k+1)x}{\cosh(2k+1)b},$$

where

$$0 \le x \le b, \quad b > 0, \quad \text{and} \quad p = 2, 3. \tag{1.10}$$

Both are also of interest in plate contact problems [1]. Here again, we are able to sum these series effectively and to high precision, the major (as yet unresolved) difficulty occurring when b is very small.

2. SUMMATION OF R_p AND S_p, $p = 2$ AND $p = 3$

We begin with an idea used previously in [7, 6], namely to express part of each term of the series (not the whole term, as in [7, 6]) as a Laplace transform with integer argument. Specifically,

$$\frac{1}{(k+1/2)^p} = (\mathscr{L} f)(k), \qquad \mathscr{L} = \text{Laplace transform}, \tag{2.1}$$

where

$$f(t) = \frac{1}{(p-1)!} t^{p-1} e^{-t/2}. \tag{2.2}$$

Then

$$\begin{aligned} R_p(z) &= \frac{z}{2^p} \sum_{k=0}^{\infty} \frac{z^{2k}}{(k+1/2)^p} = \frac{z}{2^p} \sum_{k=0}^{\infty} z^{2k} \int_0^\infty e^{-kt} \cdot \frac{t^{p-1} e^{-t/2}}{(p-1)!} dt \\ &= \frac{z}{2^p (p-1)!} \int_0^\infty \sum_{k=0}^{\infty} (z^2 e^{-t})^k \cdot t^{p-1} e^{-t/2} dt \\ &= \frac{z}{2^p (p-1)!} \int_0^\infty \frac{1}{1 - z^2 e^{-t}} t^{p-1} e^{-t/2} dt, \end{aligned}$$

that is,

$$R_p(z) = \frac{z}{2^p (p-1)!} \int_0^\infty \frac{t^{p-1} e^{t/2}}{e^t - z^2} dt. \tag{2.3}$$

We distinguish two cases.

Case 1: $z = 1$. In this case, (2.3) takes the form

$$R_p(1) = \frac{1}{2^p (p-1)!} \int_0^\infty \frac{t}{e^t - 1} \cdot t^{p-2} e^{t/2} dt \tag{2.4}$$

and can be evaluated by Gaussian quadrature relative to the weight function (cf. [7])

$$\varepsilon(t) = \frac{t}{e^t - 1} \quad \text{on } [0, \infty] \qquad (\text{"Einstein function"}). \tag{2.5}$$

However, there is no real need for this, since by (1.7) the sum is expressible in terms of the well-tabulated Riemann zeta function [9]. In particular, $R_2(1) = \pi^2/8$.

The more difficult case is

Case 2: $z \neq 1$. Here we could proceed similarly as in (2.4) and write

$$R_p(z) = \frac{z}{2^p(p-1)!} \int_0^\infty \varepsilon(t) \cdot \frac{e^t - 1}{e^t - z^2} \cdot t^{p-2} e^{t/2} \, dt. \tag{2.6}$$

Unfortunately, the second factor in the integrand is quite ill-behaved when $|z|$ is close to 1, exhibiting a steep boundary layer near $t = 0$. (Consider, e.g., $z^2 = 1 - \eta$, $0 < \eta \ll 1$.) Gaussian quadrature, therefore, will no longer be effective.

Instead, we make the change of variable $e^{-t} = \tau$ in (2.3) (and then replace τ again by t) to obtain

$$R_p(z) = \frac{1}{2^p(p-1)!z} \int_0^1 \frac{[\ln(1/t)]^{p-1}}{\sqrt{t}} \frac{dt}{z^{-2} - t}. \tag{2.7}$$

This expresses $R_p(z)$ as a Stieltjes transform of the weight function

$$w_p(t) = \frac{[\ln(1/t)]^{p-1}}{\sqrt{t}} \quad \text{on } [0, 1]. \tag{2.8}$$

Our assumptions on z are such that the point z^{-2} at which the transform is evaluated lies *outside* of the interval $[0, 1]$,

$$z^{-2} \in \mathbb{C} \backslash [0, 1]. \tag{2.9}$$

The integral in (2.7), therefore, can be evaluated by backward recursion, as is discussed in [3, §5].

Indeed, if

$$y_{n+1} = (z^{-2} - \alpha_n) y_n - \beta_n y_{n-1}, \qquad n = 0, 1, 2, \ldots, \tag{2.10}$$

is the recurrence relation for the orthogonal polynomials $\{\pi_n(z^{-2}; w_p)\}$ relative to the weight function (2.8), thus,

$$\alpha_n = \alpha_n(w_p), \quad \beta_n = \beta_n(w_p) \qquad \left[\beta_0(w_p) = \int_0^1 w_p(t)\, dt\right], \tag{2.11}$$

and if we define the sequence $\{r_{n-1}^{[\nu]}(z)\}_{n=0}^{\nu+1}$ for any integer $\nu > 0$ by

$$r_\nu^{[\nu]}(z) = 0, \qquad r_{n-1}^{[\nu]}(z) = \frac{\beta_n}{z^{-2} - \alpha_n - r_n^{[\nu]}(z)}, \quad n = \nu, \nu-1, \ldots, 1, 0, \tag{2.12}$$

then (cf. [3, equation (5.2) for $N = 0$])

$$\int_0^1 \frac{w_p(t)\, dt}{z^{-2} - t} = \lim_{\nu \to \infty} r_{-1}^{[\nu]}(z). \tag{2.13}$$

Thus, by (2.7),

$$R_p(z) = \frac{r_{-1}^{[\infty]}(z)}{2^p(p-1)!z}. \tag{2.14}$$

Convergence in (2.13) is faster the further away z^{-2} is from the interval $[0, 1]$.

The evaluation of $r_{-1}^{[\infty]}(z)$ is quite cheap, once the coefficients α_n, β_n in (2.11) have been precomputed for sufficiently many n. One simply lets ν increase through a sequence $\{\nu_i\}$ of integers $0 < \nu_1 < \nu_2 < \cdots$ and stops at the smallest i, say $i = i_{\min}$, for which $|r_{-1}^{[\nu_i]}(z) - r_{-1}^{[\nu_{i-1}]}(z)| \le \varepsilon |r_{-1}^{[\nu_i]}(z)|$, where ε is a preset error tolerance. One then accepts $r_{-1}^{[\nu_i]}(z)$ with $i = i_{\min}$ as the desired approximation of $r_{-1}^{[\infty]}(z)$ in (2.14). For the two choices of z in (1.5), practical guidelines for determining an acceptable value of ν (i.e., one for which $r_{-1}^{[\nu]}(z)$ sufficiently approximates $r_{-1}^{[\infty]}(z)$) will be given in §4.

The coefficients α_n, β_n can be computed by known methods, as will be further discussed in §3. The first 100 coefficients are tabulated in Tables 1 and 2 of the Appendix for $p = 2$ and $p = 3$ to an accuracy of 25 and 20 significant decimal digits, respectively.

The procedure (2.12)–(2.14), in view of (1.4), is readily adapted to the series S_p in (1.2_p). Indeed, letting $s_n^{[\nu]}(z) = -r_n^{[\nu]}(-iz)$, one finds

$$S_p(z) = \frac{s_{-1}^{[\infty]}(z)}{2^p (p-1)! z}, \tag{2.15}$$

where

$$s_\nu^{[\nu]}(z) = 0, \quad s_{n-1}^{[\nu]}(z) = \frac{\beta_n}{z^{-2} + \alpha_n - s_n^{[\nu]}(z)}, \quad n = \nu, \nu-1, \dots, 1, 0. \tag{2.16}$$

Since $S_p(z)$ is effectively the Stieltjes transform of $w_p(\cdot)$ evaluated at $-z^{-2}$, the process (2.15), (2.16) converges more rapidly (as $\nu \to \infty$) the further away $-z^{-2}$ is from the interval $[0, 1]$. In particular, it converges rapidly for $z = 1$, yielding a fast way of computing Catalan's constant when $p = 2$. Indeed, taking $\nu = 16$ in (2.16) produces $s_{-1}^{[\infty]}(1)$, hence $S_2(1)$, accurately to 25 decimal digits!

3. Generating the coefficients $\alpha_n(w_p)$, $\beta_n(w_p)$ for $p = 2$ and $p = 3$

Consider the weight function

$$w_p(t; \alpha) = t^\alpha [\ln(1/t)]^{p-1}, \qquad 0 < t \le 1, \ \alpha > -1, \ p \ge 2, \tag{3.1}$$

and let

$$m_n(\alpha; p) = \int_0^1 P_n^*(t) w_p(t; \alpha)\, dt, \qquad n = 0, 1, 2, \dots, \tag{3.2}$$

denote the "modified moments" of $w_p(\cdot; \alpha)$ relative to the shifted Legendre polynomials $P_n^*(t) = P_n(2t-1)$. In the case $p = 2$ these modified moments are explicitly known (cf. [2]):

$$m_n(\alpha; 2) = \frac{1}{\alpha+1}\left\{\frac{1}{\alpha+1} + 2\sum_{k=1}^{n}\frac{k}{(k+\alpha+1)(k-\alpha-1)}\right\} \cdot \prod_{k=1}^{n}\frac{\alpha+1-k}{\alpha+1+k}. \tag{3.3}$$

It is also well known how the modified moments of a weight function w can be used to generate the recursion coefficients $\alpha_n(w)$, $\beta_n(w)$ of the respective orthogonal polynomials $\{\pi_k(\cdot\,; w)\}$ by means of the so-called modified Chebyshev algorithm [4, §2.4]. This algorithm indeed works particularly well in the case of the weight function (3.1) with $p = 2$, $\alpha = -\frac{1}{2}$, i.e., for $w(t) = w_2(t)$ (cf. (2.8)), as was demonstrated in [5, Example 5.3]. This, then, is the way we computed the quantities $\alpha_n(w_p)$, $\beta_n(w_p)$ for $p = 2$. Compensating for a loss of about four decimal digits, when n runs from 0 to 99, we tabulate the results in Table 1 of the Appendix to only 25 decimals (having done the computation in 29-decimal arithmetic).

In order to get the same quantities for $p = 3$, it suffices to observe that

$$\frac{\partial w_p}{\partial \alpha}(t; \alpha) = -w_{p+1}(t; \alpha),$$

and therefore

$$m_n(\alpha; p+1) = -\int_0^1 P_n^*(t)\frac{\partial w_p}{\partial \alpha}(t; \alpha)\, dt = -\frac{\partial m_n}{\partial \alpha}(\alpha; p).$$

Thus, the required modified moments $m_n(\alpha; 3)$ can be obtained by differentiating both sides of (3.3) with respect to α (after multiplication by -1). The result is

$$m_n(\alpha; 3) = \frac{2}{(\alpha+1)^3}\left\{1 + 2(\alpha+1)\sum\nolimits_1 + 2(\alpha+1)^2\sum\nolimits_1^2 - 2(\alpha+1)^3\sum\nolimits_2\right\}\Pi, \tag{3.4}$$

where

$$\begin{aligned} \sum\nolimits_1 &= \sum_{k=1}^{n}\frac{k}{(k+\alpha+1)(k-\alpha-1)}, \\ \sum\nolimits_2 &= \sum_{k=1}^{n}\frac{k}{(k+\alpha+1)^2(k-\alpha-1)^2}, \\ \Pi &= \prod_{k=1}^{n}\frac{\alpha+1-k}{\alpha+1+k}. \end{aligned} \tag{3.5}$$

Putting $\alpha = -\frac{1}{2}$ in (3.4), and using the resulting quantities as input to the modified Chebyshev algorithm, produces the coefficients $\alpha_n(w_3)$, $\beta_n(w_3)$. The procedure is somewhat less stable than in the case $p = 2$, suffering a loss of about eight to nine decimal digits when applied up to $n = 99$. For this reason we tabulate $\alpha_n(w_3)$, $\beta_n(w_3)$ in Table 2 of the Appendix to only 20 decimals.

4. Implementation and numerical examples

It would clearly be desirable in our procedure (2.12) to know a priori what value to choose for the starting index ν, given any z in the first quadrant of $\mathbb{C}$ and given the required accuracy. The recursion in (2.12) then would need to be run through only once, and the iterative procedure suggested in §2 could be dispensed with.

To deal with this problem, we consider only the two cases of practical interest stated in (1.5). More precisely, we address the following related problem: Given ν and the desired relative accuracy ε, determine the set of values A in $[0, 1]$, resp. α in $[0, \pi/2]$, for which $r_{-1}^{[\nu]}$ in (2.13) approximates $r_{-1}^{[\infty]}$ within a relative error of ε.

As to the values of A, we note that the speed of convergence in (2.13) decreases as A increases in $[0, 1]$. The desired set of A-values must thus have the form $0 \le A \le A(\nu, \varepsilon) \le 1$, and the problem is to determine $A(\nu, \varepsilon)$. We solve this empirically by a bisection procedure: Start with two numbers A_0^-, A_0^+ such that $A_0^- \le A(\nu, \varepsilon) \le A_0^+$, for example, $A_0^- = 0$, $A_0^+ = 1$. Having already obtained A_{k-1}^- and A_{k-1}^+ with $A_{k-1}^- < A_{k-1}^+$, test the midpoint $M = \frac{1}{2}(A_{k-1}^- + A_{k-1}^+)$ to see whether at M the procedure (2.12) yields an approximation $r_{-1}^{[\nu]}$ with relative error larger or smaller than ε. In the former case we set $A_k^- = A_{k-1}^-$, $A_k^+ = M$, in the latter case $A_k^- = M$, $A_k^+ = A_{k-1}^+$. We quit this iteration as soon as, say, $A_k^+ - A_k^- \le \frac{1}{2}10^{-6}$ and take $\frac{1}{2}(A_k^- + A_k^+)$ to approximate $A(\nu, \varepsilon)$. In order to determine the relative errors of $r_{-1}^{[\nu]}$, as required in this procedure, we approximate $r_{-1}^{[\infty]}$ by $r_{-1}^{[99]}$ and, at the same time, check to see that $r_{-1}^{[99]}$ and $r_{-1}^{[98]}$ agree to within a relative accuracy $\varepsilon/100$. If they do, it is safe to assume that $r_{-1}^{[99]}$ can reliably substitute $r_{-1}^{[\infty]}$ in determining whether $r_{-1}^{[\nu]}$ has relative error $> \varepsilon$ or $< \varepsilon$. If they do not, we print a cautionary message, and take A_k^- as a (conservative) estimate from below of $A(\nu, \varepsilon)$.

The results of this procedure are summarized in Table 4.1 for both $p = 2$ and $p = 3$. An asterisk indicates a conservative lower estimate of $A(\nu, \varepsilon)$ for reasons explained above.

We can see from Table 4.1, for example, that if we are interested in 12-digit accuracy and only in positive values of A satisfying $A \le .99$, then we can safely use $\nu = 50$ in (2.12) when $p = 2$, and $\nu = 40$ when $p = 3$. On the other hand, the choice $\nu = 10$ for the same range of A-values, always gives at least four correct decimal digits.

Interestingly, the procedure (2.12), (2.13) seems to converge even for $A = 1$, albeit slowly, but there is no theoretical justification for it (to our knowledge).

TABLE 4.1

Values of $A(\nu, \varepsilon)$, $\varepsilon = \frac{1}{2}10^{-acc}$, *such that* $r_{-1}^{[\nu]}$ *approximates* $r_{-1}^{[\infty]}$ *to acc digits for all* A *with* $0 \le A \le A(\nu, \varepsilon)$

ν	*acc*	$p=2$	$p=3$	ν	*acc*	$p=2$	$p=3$	ν	*acc*	$p=2$	$p=3$
10	4	.9902	1.0000	40	4	.9999	1.0000	70	4	1.0000	1.0000
	8	.9313	.9592		8	.9962	.9993		8	.9989	1.0000
	12	.8422	.8732		12	.9889	.9936		12	.9961*	.9980*
	16	.7384	.7688		16	.9786	.9842		16	.9922*	.9955
	20	.6325	.6603		20	.9653	.9717		20	.9887	.9914
20	4	.9985	1.0000	50	4	1.0000	1.0000	80	4	1.0000	1.0000
	8	.9827	.9931		8	.9977	.9998		8	.9990*	1.0000
	12	.9551	.9685		12	.9931	.9963		12	.9961*	.9980*
	16	.9178	.9330		16	.9864	.9903		16	.9922*	.9961*
	20	.8729	.8891		20	.9777	.9823		20	.9914	.9922*
30	4	.9996	1.0000	60	4	1.0000	1.0000	90	4	1.0000	1.0000
	8	.9927	.9980		8	.9985	1.0000		8	.9990*	1.0000
	12	.9800	.9873		12	.9953	.9976		12	.9961*	.9980*
	16	.9620	.9708		16	.9906	.9936		16	.9922*	.9961*
	20	.9395	.9492		20	.9845	.9880		20	.9922*	.9922*

For the second choice $z = e^{i\alpha}$, $0 \le \alpha \le \pi/2$, in (1.5), it was observed empirically that the speed of convergence in (2.13) decreases—slowly at first, and then faster—as α decreases from $\pi/2$ to 0. Therefore, a similar procedure as above for A-values can be applied to determine the number $\omega(\nu, \varepsilon)$ with the property that for all α satisfying $0 \le \omega(\nu, \varepsilon)\pi/2 \le \alpha \le \pi/2$, the procedure (2.12) produces $r_{-1}^{[\nu]}$ with (at least approximately) $|(r_{-1}^{[\nu]} - r_{-1}^{[\infty]})/r_{-1}^{[\infty]}| \le \varepsilon$. The results are displayed in Table 4.2.

TABLE 4.2

Values of $\omega(\nu, \varepsilon)$, $\varepsilon = \frac{1}{2}10^{-acc}$, *such that* $r_{-1}^{[\nu]}$ *approximates* $r_{-1}^{[\infty]}$ *to acc digits for all* α *with* $\omega(\nu, \varepsilon)\pi/2 \le \alpha \le \pi/2$

ν	*acc*	$p=2$	$p=3$	ν	*acc*	$p=2$	$p=3$	ν	*acc*	$p=2$	$p=3$
10	4	.0159	0.0000	40	4	.0002	0.0000	70	4	0.0000	0.0000
	8	.1066	.0690		8	.0054	.0013		8	.0020*	0.0000
	12	.2864	.2313		12	.0152	.0096		12	.0046	.0025
	16	.6789	.5576		16	.0294	.0226		16	.0092	.0078*
	20	1.0000*	1.0000*		20	.0482	.0404		20	.0156*	.0120
20	4	.0026	0.0000	50	4	.0001	0.0000	80	4	0.0000	0.0000
	8	.0247	.0118		8	.0033	.0005		8	.0020*	0.0000
	12	.0649	.0481		12	.0095	.0056		12	.0039*	.0020*
	16	.1251	.1045		16	.0185	.0138		16	.0078*	.0078*
	20	.2095	.1844		20	.0304	.0249		20	.0156*	.0089*
30	4	.0007	0.0000	60	4	0.0000	0.0000	90	4	0.0000	0.0000
	8	.0103	.0036		8	.0022	.0002		8	.0020*	0.0000
	12	.0278	.0190		12	.0064	.0036		12	.0039*	.0020*
	16	.0535	.0428		16	.0126	.0091		16	.0078*	.0078*
	20	.0878	.0754		20	.0208	.0168		20	.0156*	.0078*

TABLE 4.3
Results for Example 1

A	$p=2$	$p=3$	$R_2(A)$	$R_3(A)$
.8	21	16	.8772880939214647253008518	.82248858052014232615
.9	30	23	1.025938951111110172771877	.93414857586540185586
.95	43	31	1.114099577929052481501213	.99191543992242877550
.99	95	65	1.202075664776857538062901	1.0395722318736413458
.999	–	–	1.2293981974	1.0505677498304
1.000	–	–	1.2336	1.051799789

Example 1. $R_p(A)$ for $A = .8, .9, .95, .99, .999, 1.000$, and $p = 2, 3$.

We applied the procedure (2.12) with $\nu = 1, 2, 3, \ldots$, terminating it for the first value of ν, $\nu = \nu_{\min}$, for which $|(r_{-1}^{[\nu]} - r_{-1}^{[\nu-1]})/r_{-1}^{[\nu]}| \le \varepsilon$, where $\varepsilon = \frac{1}{2}10^{-25}$ for $p = 2$, and $\varepsilon = \frac{1}{2}10^{-20}$ for $p = 3$. Table 4.3 shows the values of $\nu_{\min}$ along with 25-, resp. 20-digit results for $R_p(A)$, $p = 2, 3$.

For $A \ge .999$, full accuracy could not be achieved with $\nu \le 99$, only the partially accurate results shown in Table 4.3.

Example 2. $R_p(e^{i\alpha})$ for $\alpha = \omega\pi/2$, $\omega = .2, .1, .05, .01, .001, 0.000$, and $p = 2, 3$.

The same experiment as in Example 1 was run in this case, with the results being shown in Table 4.4. The first entry under each heading $R_p(e^{i\omega\pi/2})$ represents the real part, the second the imaginary part. The results for $\operatorname{Re} R_2$, $\operatorname{Im} R_3$ were checked against formulas (1.6_2) and (1.6_3), respectively, and revealed agreement to all digits shown.

TABLE 4.4
Results for Example 2

ω	$p=2$	$p=3$	$R_2(e^{i\omega\pi/2)})$	$R_3(e^{i\omega\pi/2})$
.2	27	21	.9869604401089358618834491	.96915102126251836837
			.4474022700859631972532577	.34882061265337297697
.1	37	28	1.110330495122552844618880	1.0268555576593748316
			.2783029792855803918158969	.18409976778928018229
.05	51	38	1.172015522629361335986596	1.0444944153967221625
			.1663915239689736941195221	.09447224926028851460
.01	–	76	1.2213635446348081290808	1.0514082919738793229
			.04592009281744058404956	.01928202831056145067
.001	–	–	1.232466849	1.051794454929
			.006400460	.001936923346
0.000	–	–	1.2337	1.051799789
			0.	0.

5. Summation of T_p and U_p, $p = 2$ and $p = 3$

We first take up the series (1.8_p). We expand the ratio of hyperbolic cosines as follows:

$$\frac{\cosh(2k+1)x}{\cosh(2k+1)b} = \sum_{n=0}^{\infty}(-1)^n\{e^{-(2k+1)[(2n+1)b-x]} + e^{-(2k+1)[(2n+1)b+x]}\}.$$

Then, upon using again the Laplace transform technique (2.1), (2.2), and interchanging the summations over k and n, one obtains after an elementary calculation

$$T_p(x, b) = \frac{1}{2^p(p-1)!}\sum_{n=0}^{\infty}(-1)^n e^{(2n+1)b}[\varphi_n(-x) + \varphi_n(x)], \tag{5.1}$$

where

$$\varphi_n(s) = e^s\int_0^1 \frac{w_p(t)\,dt}{e^{2[(2n+1)b+s]} - t}, \qquad -b \le s \le b. \tag{5.2}$$

The integral in (5.2) again is a Stieltjes transform of the weight function (2.8), this time evaluated at $u = \exp(2[(2n+1)b+s])$. Clearly, $u > 1$, unless $n = 0$ and $s = -b$, in which case, by (2.7) and (1.7),

$$\varphi_0(-b) = e^{-b}\int_0^1 \frac{w_p(t)}{1-t}\,dt = (2^p - 1)(p-1)!\zeta(p)e^{-b}. \tag{5.3}$$

The integral in (5.2), hence both $\varphi_n(x)$ and $\varphi_n(-x)$ in (5.1) (the latter if $n > 0$ or $x < b$), can be computed, as before, by the recursive procedure (2.12), (2.13) (where z^{-2} is to be replaced by u). For large n, this procedure converges almost instantaneously.

The series in (5.1), on the other hand, converges geometrically, with ratio $\exp(-2b)$. This is easily seen by noting that its general term (including the factor in front of the series) behaves like $2(-1)^n\cosh x \cdot e^{-2nb}$ as $n \to \infty$. Thus, convergence is quite satisfactory, unless b is small, the speed of convergence being independent of x. Table 5.1 shows the number of terms, N, required

Table 5.1
Number of terms required in the series of (5.1) *to achieve an accuracy of acc significant decimal digits*

b	acc	$p=2$	$p=3$	b	acc	$p=2$	$p=3$	b	acc	$p=2$	$p=3$
.05	4	104	105	.20	4	26	26	.80	4	7	7
	8	196	198		8	49	49		8	13	13
	12	288	290		12	72	72		12	18	18
	16	380	382		16	95	96		16	24	24
	20	473	474		20	118	119		20	30	30
.10	4	52	53	.40	4	13	13	1.60	4	4	4
	8	98	99		8	25	25		8	6	6
	12	144	145		12	36	36		12	9	9
	16	190	191		16	48	48		16	12	12
	20	236	237		20	59	59		20	15	15

in (5.1) to achieve various accuracies. As mentioned, N does not depend on x. It can be seen that the convergence characteristics of the series are virtually the same for $p = 2$ and $p = 3$. (When x is very close to b, the backward recursion (2.12) with $\nu \leq 99$ for evaluating $\varphi_0(-x)$ in (5.1) may provide only limited accuracy; cf. Example 1.)

For the series (1.9_p) one finds similarly

$$U_p(x, b) = \frac{1}{2^p(p-1)!} \sum_{n=0}^{\infty} (-1)^n e^{(2n+1)b} [\varphi_n(-x) - \varphi_n(x)], \tag{5.4}$$

with $\varphi_n(\cdot)$ defined in (5.2); the convergence behavior, when $x > 0$, is similar to the one shown in Table 5.1 for the series (5.1).

Series of the types (1.8_p), (1.9_p), which include alternating sign factors, can be treated similarly.

Appendix

Recursion coefficients α_n, β_n for the (monic) polynomials $\{\pi_k(\cdot\,; w_2)\}$ and $\{\pi_k(\cdot\,; w_3)\}$ orthogonal on $[0, 1]$ with respect to the weight functions $w_2(t) = t^{-1/2}\ln(1/t)$ and $w_3(t) = t^{-1/2}[\ln(1/t)]^2$.

Table 1

Recursion coefficients for the polynomials $\{\pi_k(\cdot\,; w_2)\}$

n	alpha(n)	beta(n)
0	0.1111111111111111111111111D+00	0.4000000000000000000000000D+01
1	0.4661483641075477810171688D+00	0.2765432098765432098765432D-01
2	0.4880690581976426561739654D+00	0.5534292684170711183265476D-01
3	0.4938743419208057331274822D+00	0.5940526298488865183067045D-01
4	0.4962639578613459263700277D+00	0.6077714606674732893827287D-01
5	0.4974805136345470499404327D+00	0.6140371143126410746951299D-01
6	0.4981846424539394712819088D+00	0.6174167659202270796881379D-01
7	0.4986290336259843770529448D+00	0.6194453627914717711328688D-01
8	0.4989276082849235415546195D+00	0.6207576580933626144340940D-01
9	0.4991379564664850047980802D+00	0.6216550244588411861001340D-01
10	0.4992917697449965925976574D+00	0.6222955193630939853437094D-01
11	0.4994076708859089483520375D+00	0.6227685326078544899112281D-01
12	0.4994971916094638566242202D+00	0.6231277082877488477563886D-01
13	0.4995677851751364266156599D+00	0.6234068199929453251339864D-01
14	0.4996244439462620957645566D+00	0.6236279899520571298076283D-01
15	0.4996706145979840808548842D+00	0.6238061988995864213325718D-01
16	0.4997087392464561491728915D+00	0.6239518834725249753338138D-01
17	0.4997405876531736626190872D+00	0.6240724948356001710689111D-01
18	0.4997674679010946606417245D+00	0.6241734675384434899468191D-01
19	0.4997903638440694047466612D+00	0.6242588405027882485913615D-01
20	0.4998100270509691381859106D+00	0.6243316658846858305615025D-01
21	0.4998270396901901811053116D+00	0.6243942847487754986053115D-01
22	0.4998418584011269547061628D+00	0.6244485169209210663968073D-01
23	0.4998548454525784424706020D+00	0.6244957942452422353437694D-01
24	0.4998662912324218943801592D+00	0.6245372557342242600457226D-01
25	0.4998764307204424397405948D+00	0.6245738165737186124658778D-01
26	0.4998854557169246669449019D+00	0.6246062188808478979166957D-01
27	0.4998935240328226886591572D+00	0.6246350695269707453479568D-01
28	0.4999007664750365107918077D+00	0.6246608686595204798542240D-01
29	0.4999072922115382430628671D+00	0.6246840314472866468672816D-01
30	0.4999131929321822564939469D+00	0.6247049048282836967879567D-01
31	0.4999185461046623069180692D+00	0.6247237805306714282029063D-01
32	0.4999234175438090965764987D+00	0.6247409052851061161429977D-01
33	0.4999278634549460876866086D+00	0.6247564888999735836355421D-01
34	0.4999319320708932838633869D+00	0.6247707106956441823595053D-01

TABLE 1 (*continued*)

n	alpha(n)	beta(n)
35	0.4999356649724540623274387D+00	0.6247837246679940094973984D-01
36	0.4999390981604717422591136D+00	0.6247956636600570708844248D-01
37	0.4999422629314923704886477D+00	0.6248066427536794284683246D-01
38	0.4999451865971175754297572D+00	0.6248167620434672378022245D-01
39	0.4999478930781540114073128D+00	0.6248261089183034006005443D-01
40	0.4999504033978688392463883D+00	0.6248347599478382249321288D-01
41	0.4999527360934751182615016D+00	0.6248427824502116118751316D-01
42	0.4999549075609863005157917D+00	0.6248502358010969536816872D-01
43	0.4999569323454961738619833D+00	0.6248571725317097460386049D-01
44	0.4999588233865400022085640D+00	0.6248636392537765834044482D-01
45	0.4999605922263117426170318D+00	0.6248696774419352905015697D-01
46	0.4999622491870299058307070D+00	0.6248753240981319504469108D-01
47	0.4999638035225698808616725D+00	0.6248806123179200596514722D-01
48	0.4999652635485445800661969D+00	0.6248855717748684742433619D-01
49	0.4999666367542657434951760D+00	0.6248902291363342497098043D-01
50	0.4999679298994151058697458D+00	0.6248946084214908332126385D-01
51	0.4999691490977670252895535D+00	0.6248987313105962943007473D-01
52	0.4999702998899082096191761D+00	0.6249026174129439284484899D-01
53	0.4999713873065772549159492D+00	0.6249062844996838433907696D-01
54	0.4999724159239822749292821D+00	0.6249097487066807275398102D-01
55	0.4999733899122375046656649D+00	0.6249130247117342087934172D-01
56	0.4999743130778803590509507D+00	0.6249161258897980601024296D-01
57	0.4999751889012818397363903D+00	0.6249190644492645302489131D-01
58	0.4999760205696396844641164D+00	0.6249218515519076536316924D-01
59	0.4999768110061406606236641D+00	0.6249244974186864670497904D-01
60	0.4999775628957922328764517D+00	0.6249270114232811644774820D-01
61	0.4999782787083515100262005D+00	0.6249294021749607086844032D-01
62	0.4999789607187184893133689D+00	0.6249316775921498850954361D-01
63	0.4999796110251092079375052D+00	0.6249338449678696015999789D-01
64	0.4999802315652808832931335D+00	0.6249359110280602015518800D-01
65	0.4999808241310441662847611D+00	0.6249378819836585975999631D-01
66	0.4999813903812661709776500D+00	0.6249397635771819984114800D-01
67	0.4999819318535410926892670D+00	0.6249415611244704749850512D-01
68	0.4999824499746822529848129D+00	0.6249432795521547829357330D-01
69	0.4999829460701697066273291D+00	0.6249449234313423932856533D-01
70	0.4999834213726706073831202D+00	0.6249464970079516550903631D-01
71	0.4999838770297349356284462D+00	0.6249480042300698114169578D-01
72	0.4999843141107565888762890D+00	0.6249494487726638748190090D-01
73	0.4999847336132789314898947D+00	0.6249508340599330177430746D-01
74	0.4999851364687144438985805D+00	0.6249521632855620635385 42D-01
75	0.4999855235475398955558700D+00	0.6249534394310585118905247D-01
76	0.4999858956640213131307627D+00	0.6249546652824932049195604D-01
77	0.4999862535805167765288282D+00	0.6249558434456138111659900D-01
78	0.4999865980113996234652464D+00	0.6249569763596903056671454D-01
79	0.4999869296266398705890535D+00	0.6249580663101061396102570D-01
80	0.4999872490550774736510815D+00	0.6249591154398574866660389D-01
81	0.4999875568874173723953749D+00	0.6249601257600626690066200D-01
82	0.4999878536789730305318126D+00	0.6249610991595779263883089D-01
83	0.4999881399521823296847290D+00	0.6249620374138053097068322D-01
84	0.4999884161989171590890288D+00	0.6249629421927693288786306D-01
85	0.4999886828826058173914092D+00	0.6249638150685309052629656D-01
86	0.4999889404401853724423155D+00	0.6249646575220000346266933D-01
87	0.4999891892838993776656731D+00	0.6249654709492022401174982D-01
88	0.4999894298029547919706246D+00	0.6249662566670482840875654D-01
89	0.4999896623650505703804859D+00	0.6249670159186516247938764D-01
90	0.4999898873177891638918131D+00	0.6249677498782336725853075D-01
91	0.4999901049899810715268292D+00	0.6249684596556529537947903D-01
92	0.4999903156928516094091928D+00	0.6249691463005907714323314D-01
93	0.4999905197211581872778772D+00	0.6249698108064228095881835D-01
94	0.4999907173542256001838438D+00	0.6249704541138033192705538D-01
95	0.4999909088569061417086197D+00	0.6249710771139860088117153D-01
96	0.4999910944804707157151021D+00	0.6249716806519035083547536D-01
97	0.4999912744634365583230273D+00	0.6249722655290252557984423D-01
98	0.4999914490323366734035746D+00	0.6249728325060118350439253D-01
99	0.4999916184024356271670790D+00	0.624973382[illegible]51821636937157D-01

TABLE 2
Recursion coefficients for the polynomials $\{\pi_k(\cdot; w_3)\}$

n	alpha(n)	beta(n)
0	0.37037037037037037037D-01	0.16000000000000000000D+02
1	0.35811288669783060917D+00	0.66282578875171467764D-02
2	0.44293596764346020311D+00	0.41154551017361395415D-01
3	0.46925333322630284096D+00	0.51741782003936091009D-01
4	0.48077976287000283670D+00	0.56064845754865753829D-01
5	0.48684740638240343111D+00	0.58228578157740470460D-01
6	0.49043291987640869651D+00	0.59461384764548385798D-01
7	0.49272768237210454609D+00	0.60229080854911299451D-01
8	0.49428489903347684509D+00	0.60739066803493998597D-01
9	0.49539008804668943172D+00	0.61094906789182697675D-01
10	0.49620279412350964003D+00	0.61352955641236678499D-01
11	0.49681787613674196493D+00	0.61545998375888748886D-01
12	0.49729461823511554334D+00	0.61694155907229941430D-01
13	0.49767162298766581034D+00	0.61810329736885846614D-01
14	0.49797490489096250254D+00	0.61903100105428254212D-01
15	0.49822251490262493942D+00	0.61978352734467430133D-01
16	0.49842729720309510051D+00	0.62040233674523126765D-01
17	0.49859859328823678290D+00	0.62091731596138376834D-01
18	0.49874332901963438146D+00	0.62135044810738308068D-01
19	0.49886672742402863250D+00	0.62171819377072260690D-01
20	0.49897278759666604684D+00	0.62203307551568758377D-01
21	0.49906461349768728317D+00	0.62230475638549453317D-01
22	0.49914464410949686541D+00	0.62254078896209909882D-01
23	0.49921481738564071689D+00	0.62274714515063386935D-01
24	0.49927668889965310293D+00	0.62292859708354606101D-01
25	0.49933151895604900098D+00	0.62308899510226347107D-01
26	0.49938033739401414223D+00	0.62323147341070148965D-01
27	0.49942399238220700438D+00	0.62335860413249654858D-01
28	0.49946318757058857679D+00	0.62347251405111280344D-01
29	0.49949851066980882148D+00	0.62357497401578225918D-01
30	0.49953045564672500675D+00	0.62366746808956619876D-01
31	0.49955944011544773239D+00	0.62375124751988228320D-01
32	0.49958581907688664129D+00	0.62382737322232212745D-01
33	0.49960989585754508391D+00	0.62389674948882016821D-01
34	0.49963193088162418308D+00	0.62396015093213554691D-01
35	0.49965214875344591167D+00	0.62401824417420930673D-01
36	0.49967074401221995461D+00	0.62407160541823668615D-01
37	0.49968788583618920278D+00	0.62412073477358958151D-01
38	0.49970372190980540385D+00	0.62416606800160811616D-01
39	0.49971838161991717404D+00	0.62420798619957045131D-01
40	0.49973197871081541506D+00	0.62424682382629097021D-01
41	0.49974461350037973769D+00	0.62428287538611475284D-01
42	0.49975637473833709910D+00	0.62431640102160245772D-01
43	0.49976734117120030262D+00	0.62434763121387056841D-01
44	0.49977758286563735392D+00	0.62437677074965693541D-01
45	0.49978716233197190813D+00	0.62440400208298018187D-01
46	0.49979613548158744430D+00	0.62442948819471456645D-01
47	0.49980455244572020271D+00	0.62445337503398093115D-01
48	0.49981245827811267119D+00	0.62447579360980611775D-01
49	0.49981989355998203428D+00	0.62449686178915178922D-01
50	0.49982689492252312177D+00	0.62451668584748955026D-01
51	0.49983349549954855405D+00	0.62453536181008802223D-01
52	0.49983972532074252462D+00	0.62455297661568124452D-01
53	0.49984561165426967645D+00	0.62456960912889718816D-01
54	0.49985117930605904348D+00	0.62458533102349863062D-01
55	0.49985645088191383563D+00	0.62460020755493639085D-01
56	0.49986144701763254412D+00	0.62461429823778722643D-01
57	0.49986618658152699341D+00	0.62462765744122685831D-01
58	0.49987068685305789539D+00	0.62464033491367810709D-01
59	0.49987496368075360253D+00	0.62465237624609943064D-01
60	0.49987903162211334570D+00	0.62466382328197957392D-01
61	0.49988290406780632029D+00	0.62467471448093066886D-01
62	0.49988659335214961618D+00	0.62468508524178538288D-01
63	0.49989011085157064972D+00	0.62469496819027146677D-01
64	0.49989346707252485814D+00	0.62470439343563310277D-01

TABLE 2 (*continued*)

n	alpha(n)	beta(n)
65	0.49989667173013992394D+00	0.62471338879997137574D-01
66	0.49989973381868792364D+00	0.62472198002356840682D-01
67	0.49990266167484177651D+00	0.62473019094902672552D-01
68	0.49990546303454826112D+00	0.62473804368668536520D-01
69	0.49990814508424340115D+00	0.62474555876345704732D-01
70	0.49991071450704447675D+00	0.62475275525695845215D-01
71	0.49991317752447402614D+00	0.62475965091657110617D-01
72	0.49991553993420306482D+00	0.62476626227286812920D-01
73	0.49991780714424177721D+00	0.62477260473666718445D-01
74	0.49991998420395478897D+00	0.62477869268881843823D-01
75	0.49992207583223368018D+00	0.62478453956170477586D-01
76	0.49992408644312069750D+00	0.62479015791331707690D-01
77	0.49992602016914386125D+00	0.62479555949466760887D-01
78	0.49992788088259415486D+00	0.62480075531121750293D-01
79	0.49992967221494964853D+00	0.62480575567891808625D-01
80	0.49993139757462874885D+00	0.62481057027539907959D-01
81	0.49993306016323485719D+00	0.62481520818677805912D-01
82	0.49993466299043719936D+00	0.62481967795051404622D-01
83	0.49993620888761714558D+00	0.62482398759468269883D-01
84	0.49993770052039570566D+00	0.62482814467401053793D-01
85	0.49993914040014582863D+00	0.62483215630297026601D-01
86	0.49994053089458246073D+00	0.62483602918620793101D-01
87	0.49994187423751384843D+00	0.62483976964654494941D-01
88	0.49994317253782916515D+00	0.62484338365077338409D-01
89	0.49994442778779006308D+00	0.62484687683344099460D-01
90	0.49994564187068709249D+00	0.62485025451880310790D-01
91	0.49994681656791599301D+00	0.62485352174110100732D-01
92	0.49994795356552355925D+00	0.62485668326331105650D-01
93	0.49994905446026804320D+00	0.62485974359449494323D-01
94	0.49995012076523481262D+00	0.62486270700586905479D-01
95	0.49995115391504418198D+00	0.62486557754569991412D-01
96	0.49995215527068492031D+00	0.62486835905312266817D-01
97	0.49995312612400387501D+00	0.62487105517097069732D-01
98	0.49995406770187939389D+00	0.62487366935769639540D-01
99	0.49995498117010374556D+00	0.62487620489845595326D-01

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DEPARTMENT OF COMPUTER SCIENCES, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

E-mail address: wxg@cs.purdue.edu