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EVEN POSITIVE DEFINITE UNIMODULAR QUADRATIC FORMS OVER $\mathbb{Q}(\sqrt{3})$

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ABSTRACT. A complete list of even unimodular lattices over $\mathbb{Q}(\sqrt{3})$ is given for each dimension $n = 2, 4, 6, 8$. Siegel's mass formula is used to verify the completeness of the list. Alternate checks are given using theta series and the adjacency graph of the genus at the dyadic prime $1+\sqrt{3}$.

1. INTRODUCTION

The classification of positive definite even unimodular quadratic forms over $\mathbb{Z}$ has a long history and is consummated in Niemeier's complete enumeration of all 24-dimensional forms [10]. For dimensions greater than 24, the number of even unimodular forms grows exponentially, and complete classifications seem virtually impossible. In the case of real quadratic fields, Maass had determined in [8] all 4- and 8-dimensional even unimodular forms over the ring of integers of $\mathbb{Q}(\sqrt{5})$. Subsequently, all such forms were classified up to dimension 12 over $\mathbb{Q}(\sqrt{5})$ and up to dimension 8 over $\mathbb{Q}(\sqrt{2})$ (see [4, 14, 6, 7]). In this paper we will investigate positive definite even unimodular forms over the ring of integers of $\mathbb{Q}(\sqrt{3})$. This field is interesting, as are $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{2})$, because they are the only real quadratic fields which admit new irreducible root systems, i.e., root systems other than the classical ADE-types. For small dimensions, the root system of an even unimodular form usually determines its class. As we will see in the following, this is no longer true for the 8-dimensional forms over $\mathbb{Q}(\sqrt{3})$. In [15] Venkov gave an elegant proof of the classification of the 24-dimensional even unimodular forms over $\mathbb{Z}$ by using the theory of modular forms with spherical coefficients. In particular, he showed that the rank of any nonempty root lattice must be maximal and that all irreducible components in it have the same Coxeter number. While these two properties remain true over $\mathbb{Q}(\sqrt{5})$ and over $\mathbb{Q}(\sqrt{2})$ in the classification cited earlier, they fail to hold over $\mathbb{Q}(\sqrt{3})$ for the 8-dimensional forms. We will determine all classes of even unimodular forms up to dimension 8. Siegel's mass formula and his theorem for degree-one Hilbert-Einstein series are used to verify that our enumeration

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is complete. We will also present an adjacency graph for the 8-dimensional genus, which provides an alternative check of the completeness of the genus. For dimensions > 8, a computation of the Minkowski-Siegel mass suggests that further algebraic classification by explicit enumerations is quite infeasible. Unless otherwise indicated, all terminology and notation will follow those of [11].

2. Analytic mass formula

Let $F = \mathbb{Q}(\sqrt{3})$ and R be the ring of integers of F. Then $R = \mathbb{Z}[\varepsilon]$, where $\varepsilon = 2 + \sqrt{3}$ is the fundamental unit of F. Let L be a positive definite even unimodular R-lattice of rank n. Then $n \equiv 0 \bmod 2$ by consideration of the local dyadic structure of the lattice. Hence, there is a unique genus of such lattices in every even dimension. The Minkowski-Siegel mass of the genus of L is given by

$$M(L) = \sum_{i=1}^{h} \frac{1}{e(L_i)},$$

where $\{L_1, \dots, L_h\}$ is a complete set of representatives of the isometry classes in the genus of L, and $e(L_i)$ is the order of the orthogonal group $O(L_i)$ of L_i. By the main theorem in Siegel's analytic theory of quadratic forms, this mass is equal to an infinite product of p-adic representation densities which, when computed, yield the following formula (see [13, 6]):

$$M(L) = \frac{4^{1-n} L_F(n/2, \chi_n) \prod_{i=1}^{n/2-1} \zeta_F(2i)}{(\sqrt{12})^{-n(n-1)/2} \prod_{i=1}^{n} \pi^i \Gamma^{-2}(i/2)},$$

where $\chi_n(p) = (-1/p)^{n/2}$, $L_F(s, \chi_n) = \prod_p (1 - \chi_n(p) Np^{-s})^{-1}$, and $\zeta_F(\cdot)$ is the Dedekind zeta function. For $n \le 8$, we have the following table of masses:

n	2	4	6	8
$M(L)$	$\frac{1}{2^3 \cdot 3}$	$\frac{4}{2^6 \cdot 3^2}$	$\frac{23}{2^9 \cdot 3^4 \cdot 5}$	$\frac{23^2 \cdot 41^2}{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7}$

3. 8-dimensional forms

Let L be a positive definite even unimodular R-lattice of rank n. The *root system* of L is the set $L_2 = \{x \in L \colon Q(x) = 2\}$. The sublattice R_L of L generated by L_2 is called the *root lattice* of L. Aside from the classical root systems of ADE-types, there is a new irreducible root system over R (see [9]), namely

$$G_2 = \langle e_1 - e_2, \tfrac{1}{2}(1+\sqrt{3})e_1 + \tfrac{1}{2}(1-\sqrt{3})e_2 \rangle \cong \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix},$$

where $\{e_1, e_2\}$ is an orthonormal basis. Since $\det G_2 = 1$, the root system G_2 generates a 2-dimensional even unimodular lattice. Now if L is even unimodular of rank n, then $L \perp G_2$ is even unimodular of rank $n+2$, hence every class of n-dimensional even unimodular lattices gives rise to a unique class of $(n+2)$-dimensional even unimodular lattices. We shall construct in this section all classes of 8-dimensional even unimodular R-lattices. This will then yield complete classifications of the lower-dimensional genera. Our construction is based on Kneser's neighborhood method. For each of the lattices that we construct, we determine its root system and the order of its automorphism group with the assistance of a computer. If the root system of L generates a root lattice of maximal rank, then $e(L)$ may be computed as follows (see [3]). First we decompose the root lattice in L into irreducible components,

$$R_L = L_1 \perp L_2 \perp \cdots \perp L_t .$$

Then we let $G_2(L)$ be the factor group of $O(L)$ by the normal subgroup $S(L)$ consisting of those elements which leave invariant all the L_i. Moreover, let $G_0(L)$ be the normal subgroup of $S(L)$ consisting of those elements which, for all i, act trivially on $L_i^\#/L_i$. Here, $L_i^\#$ is the dual lattice of L_i. Finally, we let $G_1(L)$ be the factor group $S(L)/G_0(L)$. If we denote $g_k(L) = |G_k(L)|$ for $0 \le k \le 2$, then $e(L) = g_0 g_1 g_2$. On the other hand, if $\operatorname{rank} R_L < \operatorname{rank} L$, then the computation is more complicated. In any case, it is possible to find a base consisting of vectors of norm 2 or norm 4. By considering permutations of these vectors, $e(L)$ may be computed for lattices with nonmaximal root system.

We now construct the 8-dimensional even unimodular lattices.

(1) $R_L = E_8$. Let $I_8 = \langle e_1, \dots, e_8\rangle$, where $\{e_i\}$ is an orthonormal basis. Then

$$\begin{aligned} E_8 &= \{z \in I_8 \colon B(z, e_1 + \cdots + e_8) \equiv 0 \bmod 2\} + \langle (e_1 + \cdots + e_8)/2\rangle \\ &= \langle e_1 - e_2, (e_1 + \cdots + e_8)/2, e_1 + e_2, \dots, e_1 + e_7\rangle . \end{aligned}$$

E_8 is already unimodular. Its automorphism group has order $e(L) = g_0 g_1 g_2 = (2^{14} \cdot 3^5 \cdot 5^2 \cdot 7) \cdot 1 \cdot 1 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$.

(2) $R_L = D_8$. We have

$$\begin{aligned} D_8 &= \{z \in I_8 \colon B(z, e_1 + \cdots + e_8) \equiv 0 \bmod 2\} \\ &= \langle e_1 + e_2, e_1 - e_2, \dots, e_1 - e_8\rangle . \end{aligned}$$

By adjoining the vectors

$$w_1 = \frac{1-\sqrt{3}}{2}[(e_1 + e_2) + (e_1 - e_2)]$$

and

$$w_2 = \frac{1+\sqrt{3}}{2}[(e_1 - e_2) + \cdots + (e_1 - e_8)]$$

to D_8, one obtains an even unimodular lattice

$$L = \langle w_1, w_2, e_1 - e_3, e_1 - e_4, \dots, e_1 - e_8\rangle .$$

The automorphism group of L has order $e(L) = (2^7 \cdot 8!) \cdot 2 \cdot 1 = 2^{15} \cdot 3^2 \cdot 5 \cdot 7$.

(3) $R_L = E_6 \perp G_2$. Since G_2 is unimodular, it is necessary to construct a 6-dimensional unimodular lattice that has the root system E_6. Now

$$E_6 = \langle e_1 - e_2, (e_1 + \cdots + e_8)/2, e_1 + e_2, e_1 + e_3, e_1 + e_4, e_1 + e_5 \rangle .$$

By adjoining the vector

$$w = \frac{1}{\sqrt{3}} \left[\frac{e_1 + \cdots + e_8}{2} + (e_1 + e_2) + \cdots + (e_1 + e_5) \right]$$

to E_6, we have an even unimodular lattice

$$L' = \langle e_1 - e_2, (e_1 + \cdots + e_8)/2, e_1 + e_2, \ldots, e_1 + e_4, w \rangle .$$

The lattice $L = L' \perp G_2$ has root system $E_6 \perp G_2$. Its automorphism group has order $e(L) = ((2^3 \cdot 3) \cdot (2^7 \cdot 3^4 \cdot 5)) \cdot 2 \cdot 1 = 2^{11} \cdot 3^5 \cdot 5$.

(4) $R_L = D_6 \perp G_2$. Again it is necessary to construct a 6-dimensional unimodular lattice that has the root system D_6. We have $D_6 = \langle e_1 + e_2, e_1 - e_2, \ldots, e_1 - e_6 \rangle$. Two glue vectors are needed to give a unimodular lattice, namely,

$$w_1 = \frac{1 + \sqrt{3}}{2} [(e_1 + e_2) + (e_1 - e_2)]$$

and

$$w_2 = \frac{\sqrt{3}}{2}(e_1 + e_2) + \frac{1}{2}(e_1 - e_2) - \frac{1 - \sqrt{3}}{2}[(e_1 - e_3) + \cdots + (e_1 - e_6)] .$$

Let $L' = \langle w_1, w_2, e_1 - e_3, \ldots, e_1 - e_6 \rangle$. Then the lattice $L = L' \perp G_2$ has root system $D_6 \perp G_2$. Its automorphism group has order

$$e(L) = ((2^3 \cdot 3) \cdot (2^9 \cdot 3^2 \cdot 5)) \cdot 2 \cdot 1 = 2^{13} \cdot 3^3 \cdot 5 .$$

(5) $R_L = 2D_4$ (I) (decomposable). We first construct a 4-dimensional even unimodular lattice with the root system D_4. Let $D_4 = \langle e_1 + e_2, e_1 - e_2, e_1 - e_3, e_1 - e_4 \rangle$. We adjoin the vectors

$$w_1 = \frac{1 - \sqrt{3}}{2} [(e_1 + e_2) + (e_1 - e_2)]$$

and

$$w_2 = \frac{1 + \sqrt{3}}{2} [(e_1 - e_2) + (e_1 - e_3) + (e_1 - e_4)]$$

to D_4. Then $L' = \langle w_1, w_2, e_1 - e_3, e_1 - e_4 \rangle$ has root system D_4. Let $L = L_1' \perp L_2'$ be the orthogonal sum of two copies of L'. Then L has root system $2D_4$. Its automorphism group has order $e(L) = (2^6 \cdot 3)^2 \cdot (3!)^2 \cdot 2 = 2^{15} \cdot 3^3$.

(6) $R_L = 2D_4$ (II) (indecomposable). Let

$$\begin{aligned} 2D_4 &= \langle e_1 + e_2, e_1 - e_2, e_1 - e_3, e_1 - e_4 \rangle \perp \langle e_5 + e_6, e_5 - e_6, e_5 - e_7, e_5 - e_8 \rangle \\ &= \langle u_1, u_2, u_3, u_4 \rangle \perp \langle u_5, u_6, u_7, u_8 \rangle . \end{aligned}$$

If we adjoin the vectors

$$w_1 = \frac{\sqrt{3}}{2}(u_1 + u_2) + \frac{1}{2}(u_5 + u_6)$$

and

$$w_2 = \frac{\sqrt{3}}{2}(u_2 + u_3 + u_4) + \frac{1}{2}(u_6 + u_7 + u_8)$$

to $2D_4$, we obtain an even unimodular lattice with the root system $2D_4$. The lattice L is indecomposable and its automorphism group has order $e(L) = (2^6 \cdot 3)^2 \cdot 3! \cdot 2 = 2^{14} \cdot 3^3$.

(7) $R_L = 2G_2 \perp D_4$. It follows easily from the construction in (5) that there is an even unimodular lattice L with root system $2G_2 \perp D_4$. Its automorphism group has order $e(L) = ((2^3 \cdot 3)^2 \cdot (2^6 \cdot 3)) \cdot 3! \cdot 2 = 2^{14} \cdot 3^4$.

(8) $R_L = 4G_2 \cdot 4G_2$ is already unimodular. Its automorphism group has order $e(L) = (2^3 \cdot 3)^4 \cdot 1 \cdot 4! = 2^{15} \cdot 3^5$.

(9) $R_L = A_5 \perp 3A_1$. There is a basis $\{u_1, \ldots, u_8\}$ such that

$$A_5 \perp 3A_1 = \langle u_1, \ldots, u_8 \rangle \cong \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & 1 & \\ & & & 2 & 1 \\ & & & & 2 \end{pmatrix} \perp \langle 2 \rangle \perp \langle 2 \rangle \perp \langle 2 \rangle .$$

Let L be the lattice with the basis

$$\begin{aligned}\Bigg\{ & \frac{1+\sqrt{3}}{2}(\sqrt{3}u_7 - u_8),\ u_7, \\ & \frac{1+\sqrt{3}}{2\sqrt{3}}[-u_1 + 2u_2 - 3u_3 + (1-\sqrt{3})u_4 - \varepsilon^{-1}u_5 + \sqrt{3}u_6], \\ & \frac{1}{2\sqrt{3}}[-2\varepsilon^{-1}u_1 + (2-2\sqrt{3})u_2 + \sqrt{3}u_3 - 2u_4 \\ & \qquad + (4-\sqrt{3})u_5 - \sqrt{3}u_6 - 3u_7 + \sqrt{3}u_8], \\ & \frac{1}{\sqrt{3}}[(2-2\sqrt{3})u_1 - (1-2\sqrt{3})u_2 - \sqrt{3}u_3 + u_4 - \varepsilon^{-1}u_5], \\ & \frac{1}{2\sqrt{3}}[-(1-\sqrt{3})u_1 + 2u_2 + (3-\sqrt{3})u_3 - (2+4\sqrt{3})u_4 \\ & \qquad + (1+3\sqrt{3})u_5 - (3+\sqrt{3})u_6], \\ & \frac{1}{2}[(5+\sqrt{3})u_1 - (6+2\sqrt{3})u_2 + (3+\sqrt{3})u_3 - 2\varepsilon u_4 \\ & \qquad + (1+\sqrt{3})u_5 - (1+\sqrt{3})u_6], \\ & \frac{1}{2}[-(8-2\sqrt{3})u_1 + 6u_2 - (10-4\sqrt{3})u_3 + (12-4\sqrt{3})u_4 - (6-2\sqrt{3})u_5 \\ & \qquad + (1+\sqrt{3})u_6 - (3-\sqrt{3})u_7] \Bigg\}.\end{aligned}$$

Then L has the root system $A_5 \perp 3A_1$. Its automorphism group has order $e(L) = (2^3 \cdot 6!) \cdot 2 \cdot 3! = 2^9 \cdot 3^3 \cdot 5$.

(10) $R_L = 4A_1 \perp D_4$. We shall construct an even unimodular lattice with the root system $4A_1 \perp D_4$. There is a basis $\{u_1, \dots, u_8\}$ such that

$$4A_1 \perp D_4 = \langle u_1, \dots, u_8 \rangle \cong \langle 2 \rangle \perp \langle 2 \rangle \perp \langle 2 \rangle \perp \langle 2 \rangle \perp \begin{pmatrix} 2 & 0 & 1 & 1 \\ & 2 & 1 & 1 \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}.$$

Let L be the lattice with the basis

$$\left\{ \frac{1+\sqrt{3}}{2}(\sqrt{3}u_3 - u_4),\ u_3, \right.$$
$$\frac{1}{2}(\varepsilon u_1 - \varepsilon u_2 - \sqrt{3}u_3 + u_4 - u_5 + u_6),$$
$$\frac{1}{2}[-(1-\sqrt{3})u_2 - \varepsilon^{-1}u_5 - u_6 + (1-\sqrt{3})u_7 - (1-\sqrt{3})u_8],$$
$$-u_7 + u_8,\ \frac{1}{2}[-(1+\sqrt{3})u_1 + \sqrt{3}u_5 + u_6 + (3+\sqrt{3})u_7 - (1+\sqrt{3})u_8],$$
$$\frac{1}{2}[-(3+\sqrt{3})u_1 + (1+\sqrt{3})u_2 + (1+\sqrt{3})u_5 - (1-\sqrt{3})u_6],$$
$$\left. \frac{1}{2}[(1-\sqrt{3})u_1 - (2-2\sqrt{3})u_2 - (3-\sqrt{3})u_3 - 4\varepsilon^{-1}u_5 - u_6 + 4\varepsilon^{-1}u_7] \right\}.$$

Then L is our desired lattice and its automorphism group has order $e(L) = (2^4 \cdot 192) \cdot 2 \cdot 4! = 2^{14} \cdot 3^2$.

(11) $R_L = 2A_1 \perp 2A_3$. There is a basis $\{u_1, \dots, u_8\}$ such that

$$2A_1 \perp 2A_3 = \langle u_1, \dots, u_8 \rangle \cong \langle 2 \rangle \perp \langle 2 \rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 1 \\ & & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 1 \\ & & 2 \end{pmatrix}.$$

Let L be the lattice with the basis

$$\left\{ \frac{1+\sqrt{3}}{2}(\sqrt{3}u_1 - u_2),\ u_1, \right.$$
$$\frac{1}{2}(-u_3 + 2u_4 - u_5 - \varepsilon u_6 + 2\varepsilon u_7 - \varepsilon u_8),$$
$$\frac{1}{4}[-(3-\sqrt{3})u_3 - (2-2\sqrt{3})u_4 - (1+\sqrt{3})u_5 + (1-\sqrt{3})u_6$$
$$- (2-2\sqrt{3})u_7 - (1-\sqrt{3})u_8],$$
$$\frac{1}{2}(-\sqrt{3}u_1 + u_2 - u_3 + u_5 + u_6 - u_8),$$
$$\frac{1}{4}[-(3-\sqrt{3})u_1 - \varepsilon^{-1}u_3 + 2\varepsilon^{-1}u_4 - (2-3\sqrt{3})u_5 - u_6 + 2u_7 - 3u_8],$$
$$\frac{1}{2}(-\sqrt{3}u_3 - \sqrt{3}u_5 + u_6 + u_8),\ \frac{1}{2}[(1-\sqrt{3})u_3 - (3+\sqrt{3})u_5$$
$$\left. + (1-\sqrt{3})u_6 + (1+\sqrt{3})u_8] \right\}.$$

Then L has the root system $2A_1 \perp 2A_3$ and its automorphism group has order $e(L) = (2 \cdot 4!)^2 \cdot 2 \cdot 2^2 = 2^{11} \cdot 3^2$.

(12) $R_L = G_2 \perp 6A_1$. It is necessary to construct a 6-dimensional even unimodular lattice that has the root system $6A_1$. There is a basis $\{u_1, \dots, u_6\}$ such that

$$6A_1 = \langle u_1, \dots, u_6 \rangle \cong \langle 2 \rangle \perp \cdots \perp \langle 2 \rangle.$$

Let L' be the lattice with the basis

$$\left\{ \frac{1}{2}(\sqrt{3}u_1 + u_2 + \cdots + u_6),\ \frac{1+\sqrt{3}}{2}(u_1 + u_2),\right.$$
$$\frac{1-\sqrt{3}}{2}(u_2 + u_3),\ \frac{1+\sqrt{3}}{2}(u_3 + u_4),$$
$$\left. \frac{1-\sqrt{3}}{2}(u_4 + u_5),\ \frac{1+\sqrt{3}}{2}(u_5 + u_6) \right\}.$$

Then L' has the root system $6A_1$. The lattice $L = G_2 \perp L'$ is even unimodular with root system $G_2 \perp 6A_1$. Its automorphism group has order $e(L) = ((2^3 \cdot 3) \cdot 2^6) \cdot 1 \cdot (2^4 \cdot 3^2 \cdot 5) = 2^{13} \cdot 3^3 \cdot 5$.

(13) $R_L = 4A_2$. There is a basis $\{u_1, \dots, u_8\}$ such that

$$4A_2 = \langle u_1, \dots, u_8 \rangle \cong \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let L be the lattice with the basis

$$\left\{ u_3 - u_4 - \sqrt{3}u_7 + \sqrt{3}u_8,\ -u_7 + u_8,\right.$$
$$\frac{1}{3}[-2\varepsilon u_1 + \varepsilon u_2 + (1+\sqrt{3})u_3 + (1+\sqrt{3})u_4 - 2u_5 + u_6],$$
$$\frac{1}{\sqrt{3}}[2u_3 - (1+\sqrt{3})u_4 + u_5 - \varepsilon^{-1}u_6 - \varepsilon u_7 + (1+2\sqrt{3})u_8],$$
$$\frac{1}{3}(-u_1 + 2u_2 + u_3 - 2u_4 + u_5 - 2u_6 - \sqrt{3}u_7 + 2\sqrt{3}u_8),$$
$$\frac{1}{3}[-2u_1 + u_2 - (2-2\sqrt{3})u_5 + (1-\sqrt{3})u_6 - 2\varepsilon^{-1}u_7 + (5-\sqrt{3})u_8],$$
$$\frac{1}{3}[u_1 + u_2 - \varepsilon^{-1}u_3 - \varepsilon^{-1}u_4 + (3-3\sqrt{3})u_6 + (1-\sqrt{3})u_7 + (1-\sqrt{3})u_8],$$
$$\frac{1}{3}[(1-\sqrt{3})u_1 + (1-\sqrt{3})u_2 + (1-\sqrt{3})u_3 + (1-\sqrt{3})u_4 + (6-2\sqrt{3})u_5$$
$$\left. - \sqrt{3}u_6 - (10-4\sqrt{3})u_7 + (8-5\sqrt{3})u_8] \right\}.$$

Then L has the root system $4A_2$ and its automorphism group has order $e(L) = (3!)^4 \cdot 2 \cdot 12 = 2^7 \cdot 3^5$.

(14) $R_L = 8A_1$ (I). There are three inequivalent classes of even unimodular lattices with the root system $8A_1$. We construct the first one here. There is a

basis $\{u_1, \dots, u_8\}$ such that

$$8A_1 = \langle u_1, \dots, u_8\rangle \cong \langle 2\rangle \perp \cdots \perp \langle 2\rangle.$$

Let L be the lattice with the basis

$$\left\{\frac{1}{2}(-u_1 + u_2 + u_3 + \cdots + u_8),\ \frac{1+\sqrt{3}}{2}(\sqrt{3}u_1 - u_2),\right.$$
$$\frac{1}{2}(\sqrt{3}u_1 - u_2 + \sqrt{3}u_7 + u_8),\ \frac{1-\sqrt{3}}{2}(u_7 - u_8),$$
$$\frac{1}{2}(-\sqrt{3}u_5 - u_6 + u_7 + \sqrt{3}u_8),\ \frac{1-\sqrt{3}}{2}(u_5 - u_6),$$
$$\left.\frac{1}{2}(-\sqrt{3}u_3 - u_4 + u_5 + \sqrt{3}u_6),\ \frac{1+\sqrt{3}}{2}(\sqrt{3}u_3 - u_4)\right\}.$$

Then L has the root system $8A_1$. Its automorphism group has order $e(L) = 2^8 \cdot 1 \cdot (2^4 \cdot 4!) = 2^{15} \cdot 3$.

(15) $R_L = 8A_1$ (II). Let $8A_1 = \langle u_1, \dots, u_8\rangle \cong \langle 2\rangle \perp \cdots \perp \langle 2\rangle$. We construct a lattice L with the basis

$$\left\{\frac{1+\sqrt{3}}{2}(-\sqrt{3}u_7 + u_8),\ u_7,\right.$$
$$\frac{1}{2}(\sqrt{3}u_5 - u_6 - \sqrt{3}u_7 + u_8),\ u_5,\ u_4,$$
$$\frac{1}{2}(u_1 + \varepsilon u_2 + u_3 - \varepsilon u_4),$$
$$\frac{1}{2}[\varepsilon^{-1}u_1 + u_2 - (3-\sqrt{3})u_5 - (3-\sqrt{3})u_7],$$
$$\left.\frac{1}{2}[(1-\sqrt{3})u_1 - (1-\sqrt{3})u_3 - \sqrt{3}u_5 + u_6]\right\}.$$

Then L has the root system $8A_1$. Its automorphism group has order $e(L) = 2^8 \cdot 1 \cdot 2^7 = 2^{15}$.

(16) $R_L = 8A_1$ (III). Again we let $8A_1 = \langle u_1, \dots, u_8\rangle \cong \langle 2\rangle \perp \cdots \perp \langle 2\rangle$. Let L be the lattice with the basis

$$\left\{\frac{1+\sqrt{3}}{2}(\sqrt{3}u_7 - u_8),\ u_7,\right.$$
$$\frac{1+\sqrt{3}}{2}(\varepsilon^{-1}u_5 - \varepsilon^{-1}u_6),\ \frac{1+\sqrt{3}}{2}(u_4 - \sqrt{3}u_6),\ \frac{1+\sqrt{3}}{2}(\varepsilon^{-1}u_5 + \sqrt{3}\varepsilon^{-1}u_7),$$
$$\frac{1}{2}(u_1 + \varepsilon u_2 + u_3 + u_4 - \varepsilon u_5 - \sqrt{3}u_6 - \sqrt{3}u_7 + u_8),$$
$$\left.\frac{1+\sqrt{3}}{2}(\varepsilon^{-1}u_1 + u_2),\ \frac{1-\sqrt{3}}{2}(u_1 - u_3)\right\}.$$

Then L has the root system $8A_1$ and its automorphism group has order $e(L) = 2^8 \cdot 1 \cdot 8! = 2^{15} \cdot 3^2 \cdot 5 \cdot 7$.

In the following we will consider lattices the root system of which does not generate a root lattice of maximal rank. Here it is no longer possible to construct the lattices by adjoining glue vectors to the root lattice. Instead we shall construct them as neighbors of the lattices previously obtained by the glue method.

(17) $R_L = D_4$. Let $\{u_1, u_2, \dots, u_8\}$ be the basis of $L_{4A_1 \perp D_4}$ given in (10) and take L to be the neighbor of $L_{4A_1 \perp D_4}$ which contains the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3+\sqrt{3}u_4+u_8)$. Then L has the root system D_4 and its automorphism group has order $e(L) = 2^{12} \cdot 3^3$.

(18) $R_L = A_4$. Consider the basis $\{u_1, u_2, \dots, u_8\}$ of $L_{A_5 \perp 3A_1}$ given in (9). Let L be the neighbor of $L_{A_5 \perp 3A_1}$ which contains the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_4 + \sqrt{3}u_5 + u_7 + u_8)$. Then L has the root system A_4. Its automorphism group has order $e(L) = 2^7 \cdot 3^2 \cdot 5^2$.

(19) $R_L = A_3$. Again consider the basis $\{u_1, u_2, \dots, u_8\}$ of $L_{4A_1 \perp D_4}$ given in (10). Let L be the neighbor of $L_{4A_1 \perp D_4}$ which contains the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3 + u_8)$. Then L has the root system A_3 and its automorphism group has order $e(L) = 2^9 \cdot 3^3 \cdot 5$.

(20) $R_L = G_2$. It is necessary to construct a 6-dimensional even unimodular lattice with an empty root system. Let $\{u_1, u_2, \dots, u_6\}$ be the basis of the 6-dimensional lattice with root system $6A_1$ as given in (12). We take L' to be its neighbor which contains the vector $\frac{1-\sqrt{3}}{2}(u_1 + u_2)$. Then L' has an empty root system, hence $L = G_2 \perp L'$ is our desired lattice. Its automorphism group has order $e(L) = 2^{11} \cdot 3^5 \cdot 5$.

(21) $R_L = 2A_2$ (I). There are two inequivalent classes of even unimodular lattices with the root system $2A_2$. The first class arises as a neighbor of $L_{2A_1 \perp 2A_3}$. Let $\{u_1, u_2, \dots, u_8\}$ be the basis of $L_{2A_1 \perp 2A_3}$ given in (11) and let L be the neighbor of $L_{2A_1 \perp 2A_3}$ which contains the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_5+\sqrt{3}u_6+u_7)$. Then L has the root system $2A_2$ and its automorphism group has order $e(L) = 2^6 \cdot 3^4$.

(22) $R_L = 2A_2$ (II). The second class of lattices with root system $2A_2$ can be obtained as a neighbor of $L_{A_5 \perp 3A_1}$. Using the basis $\{u_1, u_2, \dots, u_8\}$ given in (9) for $L_{A_5 \perp 3A_1}$ and taking its neighbor which contains the vector $\frac{1-\sqrt{3}}{2}[\sqrt{3}u_4 + u_7 + (1 + \sqrt{3})u_8]$, we obtain a lattice L with root system $2A_2$. The automorphism group of L has order $e(L) = 2^7 \cdot 3^4$.

(23) $R_L = 4A_1$ (I). Again there are two inequivalent classes of even unimodular lattices with the root system $4A_1$. First we let $\{u_1, u_2, \dots, u_8\}$ be the basis of $L_{8A_1\,(\mathrm{I})}$ given in (14). Let L be the neighbor of $L_{8A_1\,(\mathrm{I})}$ which contains the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_1 + u_7 + u_8)$. Then L has the root system $4A_1$. Its automorphism group has order $e(L) = 2^{12}$.

(24) $R_L = 4A_1$ (II). Here we use the basis $\{u_1, u_2, \dots, u_8\}$ given in (15) for $L_{8A_1\,(\mathrm{II})}$ and take its neighbor which contains the vector

$\frac{1-\sqrt{3}}{2}(\sqrt{3}u_5+\sqrt{3}u_7+u_8)$. This yields a lattice L of root system $4A_1$. For later references, we give a basis for this lattice: $\{(1+\sqrt{3})u_1, u_2, u_3, u_1-u_4, u_5, u_4-u_6, u_6-u_7, \frac{1-\sqrt{3}}{2}(\sqrt{3}u_5+\sqrt{3}u_7+u_8)\}$. Its automorphism group has order $e(L)=2^{10}\cdot 3$.

(25) $R_L = 2A_1$. Consider the basis $\{u_1, u_2, \ldots, u_8\}$ of $L_{8A_1\,(\mathrm{II})}$ given in (15). Let L be the neighbor of $L_{8A_1\,(\mathrm{II})}$ which contains the vector $\frac{1-\sqrt{3}}{2}[\sqrt{3}u_6+(1+\sqrt{3})u_7+u_8]$. Then L has the root system $2A_1$ and its automorphism group has order $e(L)=2^{11}\cdot 3^2$.

We observed in (20) that there exists a 6-dimensional even unimodular lattice with no minimal vectors. Using a method which is analogous to that used in [4] (see also [12]) for 12-dimensional even unimodular lattices over $\mathbb{Q}(\sqrt{5})$, one can show that empty root lattices exist over $\mathbb{Q}(\sqrt{3})$ in each even dimension $n \geq 6$. More specifically, take K to be the 2-dimensional even unimodular lattice G_2 and let $\overline{K}=K/2K$ be the reduction of K mod 2. Then the quadratic map $Q\colon K \to R$ induces a nondegenerate quadratic map $\overline{Q}\colon \overline{K} \to R/4R$. $\overline{K}$ is hyperbolic, so $\overline{K} = A \oplus A'$, where A, A' are totally singular. Let $B = \{(x, \ldots, x)\colon x \in A\}$, $B' = B^{\perp} \cap (A')^n$, and $C = B \oplus B'$. Put $L = \{v \in K^n\colon \overline{v} \in C\}$, and define the quadratic map on L as $Q_L = \frac{1}{2}Q^n$, where Q^n is the quadratic map on K^n induced by Q. Then L is an even unimodular lattice of rank $2n$. Suppose $v=(v_1, \ldots, v_n) \in L$ is a minimal vector. Then $Q_L(v) = 2$, hence $Q^n(v) = 4$. We set $\overline{v}_i = x + y_i$, $x \in A$, $y_i \in A$. If no $\overline{v}_i$ vanishes, then it follows from the inequality between the arithmetic and geometric means that

$$N(4) = N\left(\sum Q(v_i)\right) \geq n^2 \sqrt[n]{N\left(\prod_{i=1}^{n} Q(v_i)\right)} \geq 4n^2.$$

This is impossible when $n \geq 3$. For such n, some $\overline{v}_i$ must vanish, which implies that $\overline{v}_i = y_i$ for all i. This means that $Q(v_i) \in 4R$. Let $Q(v_i) = 4\alpha_i$, where $\alpha_i \in R$. Since $Q^n(v) = 4$, we have $\sum_{i=1}^{n}\alpha_i = 1$, where the α_i are totally positive integers. It follows that $\alpha_j = 1$ for some j and $\alpha_i = 0$ for all $i \neq j$. On the other hand, by construction, we have $\sum \overline{v}_i = 0$, which implies that $Q^n(v) = Q(v_j) \in 8R$. This is a contradiction and hence we obtain

Proposition. *For each $n \geq 3$ there exists an even unimodular lattice over $\mathbb{Q}(\sqrt{3})$ of rank $2n$ which has an empty root system.*

We now continue our construction of 8-dimensional even unimodular lattices over $\mathbb{Q}(\sqrt{3})$. All remaining lattices will have empty root system.

(26) $R_L = \varnothing$ (I). Let $\{u_1, u_2, \ldots, u_8\}$ be the basis of $L_{8A_1\,(\mathrm{I})}$ given in (14) and let L be its neighbor containing the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3+u_7)$. Then L has empty root system and its automorphism group has order $e(L)=2^{11}\cdot 3^3\cdot 5$.

(27) $R_L = \varnothing$ (II). We use the basis $\{u_1, u_2, \ldots, u_8\}$ of $L_{8A_1\,(\mathrm{II})}$ given in (15) and take L to be its neighbor containing the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3+u_6)$. Then L has empty root system and its automorphism group has order $e(L) = 2^{11}\cdot 3^2$.

(28) $R_L = \varnothing$ (III). Consider the basis $\{u_1, u_2, \ldots, u_8\}$ of $L_{8A_1\,(\mathrm{I})}$ given in (14). Let L be the neighbor of $L_{8A_1\,(\mathrm{I})}$ containing the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_1 + \sqrt{3}u_6 + u_8)$. Then L has empty root system and its automorphism group has order $e(L) = 2^{14}\cdot 3^3$.

(29) $R_L = \varnothing$ (IV). Let $\{u_1, u_2, \ldots, u_8\}$ be the basis of $L_{4A_1\,(\mathrm{II})}$ given in (24). We take L to be the neighbor of $L_{4A_1\,(\mathrm{II})}$ containing the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3 + u_6)$. Then L has an empty root system and its automorphism group has order $e(L) = 2^7\cdot 3^5$.

(30) $R_L = \varnothing$ (V). Consider the basis $\{u_1, u_2, \ldots, u_8\}$ of $L_{8A_1\,(\mathrm{III})}$ given in (16) and take L to be its neighbor containing the vector

$$\frac{1-\sqrt{3}}{2}[\sqrt{3}u_1 + \sqrt{3}u_2 + \sqrt{3}u_4 + \sqrt{3}u_5 + \sqrt{3}u_6 + (1+\sqrt{3})u_7 + u_8].$$

Then L has an empty root system and its automorphism group has order $e(L) = 2^{14}\cdot 3^5\cdot 5^2\cdot 7$.

(31) $R_L = \varnothing$ (VI). Let $\{u_1, u_2, \ldots, u_8\}$ be the basis of $L_{4A_1\,(\mathrm{II})}$ given in (24) and let L be its neighbor containing the vector $\frac{1-\sqrt{3}}{2}(\sqrt{3}u_3+u_7)$. Then L has an empty root system and its automorphism group has order $e(L) = 2^7\cdot 3^3\cdot 5$.

This completes our construction of 8-dimensional even unimodular lattices over $\mathbb{Q}(\sqrt{3})$. We summarize our computations in Table 1.

Upon summing the reciprocals of the $e(L_i)$, we obtain

$$\sum_{i=1}^{31}\frac{1}{e(L_i)} = \frac{23^2\cdot 41^2}{2^{14}\cdot 3^5\cdot 5^2\cdot 7},$$

which is exactly the mass predicted by the mass formula. Thus we have:

Theorem 1. *There are precisely* 31 *distinct classes in the genus of positive definite even unimodular lattices of rank* 8 *over* $\mathbb{Q}(\sqrt{3})$.

As an immediate consequence of Theorem 1 and the remark made in the first paragraph of this section, we have

Theorem 2. (1) *There are precisely six distinct classes of positive definite even unimodular lattices of rank* 6 *over* $\mathbb{Q}(\sqrt{3})$, *which are distinguished by their root systems* E_6, D_6, $D_4 \perp G_2$, $3G_2$, $6A_1$, *and* $\varnothing$.

(2) *There are precisely two distinct classes of positive definite even unimodular lattices of rank* 4 *over* $\mathbb{Q}(\sqrt{3})$, *which are distinguished by their root systems* D_4 *and* $2G_2$.

(3) *There is precisely one class of positive definite even unimodular lattice of rank* 2 *over* $\mathbb{Q}(\sqrt{3})$, *namely* G_2.

TABLE 1

Even unimodular lattices over $\mathbb{Q}(\sqrt{3})$ and the orders of their automorphism groups

L_i	$g_0(L_i)$	$g_1(L_i)$	$g_2(L_i)$	$e(L_i)$
E_8	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$	1	1	$2^{14}\cdot 3^2\cdot 5\cdot 7$
D_8	$2^7\cdot 8!$	2	1	$2^{15}\cdot 3^2\cdot 5\cdot 7$
$E_6\perp G_2$	$(2^7\cdot 3^4\cdot 5)\cdot(2^3\cdot 2)$	2	1	$2^{11}\cdot 3^5\cdot 5$
$D_6\perp G_2$	$(2^9\cdot 3^2\cdot 5)\cdot(2^3\cdot 3)$	2	1	$2^{12}\cdot 3^3\cdot 5$
$2D_4$ (I)	$(2^6\cdot 3)^2$	$(3!)^2$	2	$2^{15}\cdot 3^4$
$2D_4$ (II)	$(2^6\cdot 3)^2$	$3!$	2	$2^{14}\cdot 3^3$
$2G_2\perp D_4$	$(2^3\cdot 3)^2\cdot(2^6\cdot 3)$	$3!$	2	$2^{14}\cdot 3^4$
$4G_2$	$(2^3\cdot 3)^4$	1	$4!$	$2^{15}\cdot 3^5$
$A_5\perp 3A_1$	$2^3\cdot 6!$	2	$3!$	$2^9\cdot 3^3\cdot 5$
$D_4\perp 4A_1$	$2^4\cdot 192$	2	$4!$	$2^{14}\cdot 3^2$
$2A_3\perp 2A_1$	$(2\cdot 4!)^2$	2	2^2	$2^{11}\cdot 3^2$
$G_2\perp 6A_1$	$(2^3\cdot 3)\cdot 2^6$	1	$2^4\cdot 3^2\cdot 5$	$2^{13}\cdot 3^3\cdot 5$
$4A_2$	$(3!)^4$	2	12	$2^7\cdot 3^5$
$8A_1$ (I)	2^8	1	$2^4\cdot 4!$	$2^{15}\cdot 3$
$8A_1$ (II)	2^8	1	2^7	2^{14}
$8A_1$ (III)	2^8	1	$8!$	$2^{15}\cdot 3^2\cdot 5\cdot 7$
D_4	–	–	–	$2^{12}\cdot 3^3$
A_4	–	–	–	$2^7\cdot 3^2\cdot 5^2$
A_3	–	–	–	$2^9\cdot 3^3\cdot 5$
G_2	–	–	–	$2^{11}\cdot 3^5\cdot 5$
$2A_2$ (I)	–	–	–	$2^6\cdot 3^4$
$2A_2$ (II)	–	–	–	$2^7\cdot 3^4$
$4A_1$ (I)	–	–	–	2^{12}
$4A_1$ (II)	–	–	–	$2^{10}\cdot 3$
$2A_1$	–	–	–	$2^{11}\cdot 3^2$
$\varnothing$ (I)	–	–	–	$2^{11}\cdot 3^3\cdot 5$
$\varnothing$ (II)	–	–	–	$2^{11}\cdot 3^2$
$\varnothing$ (III)	–	–	–	$2^{14}\cdot 3^3$
$\varnothing$ (IV)	–	–	–	$2^7\cdot 3^5$
$\varnothing$ (V)	–	–	–	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$
$\varnothing$ (VI)	–	–	–	$2^7\cdot 3^3\cdot 5$

Remark. In [15] Venkov showed that if $R_L\neq\varnothing$, then the root system R_L of a 24-dimensional even unimodular lattice L over $\mathbb{Z}$ possesses the following properties:

(1) $\operatorname{rank} R_L$ is maximal (i.e., $\operatorname{rank} R_L = 24$), and

(2) all irreducible components of R_L have the same Coxeter number.

These properties remain true for even unimodular lattices over $\mathbb{Q}(\sqrt{5})$ in dimensions up to 12, and over $\mathbb{Q}(\sqrt{2})$ in dimensions up to 8 (see [4, 7]), but no longer hold over $\mathbb{Q}(\sqrt{3})$ when the dimension is 8.

4. Theta series

Let $H^+ \times H^- = \{(z_1, z_2)\colon \operatorname{Im} z_1 > 0,\ \operatorname{Im} z_2 < 0\}$, i.e., $H^+ \times H^-$ is the product of the upper half plane and the lower half plane. A *Hilbert modular form* of weight k for the Hilbert modular group $SL_2(\mathbb{Z}[\varepsilon])$ is a holomorphic function f on $H^+ \times H^-$ satisfying the condition

$$f\left(\frac{az_1+b}{cz_1+d}, \frac{\overline{a}z_2+\overline{b}}{\overline{c}z_2+\overline{d}}\right) = (cz_1+d)^k(\overline{c}z_2+\overline{d})^k f(z_1, z_2)$$

for any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{Z}[\varepsilon])$. Here, $\overline{a}$ is the conjugation of a. Every Hilbert modular form f has a Fourier expansion of the form

$$f(z) = c_f(0) + \sum_{\nu \gg 0} c_f(\nu) e^{2\pi i \sigma(\nu z/2\sqrt{3})}.$$

Let L be an even unimodular lattice over $\mathbb{Q}(\sqrt{3})$ of rank n. Then the theta series of L,

$$\Theta_L(z) = \sum_{x \in L} e^{2\pi i \sigma(Q(x)z/2\sqrt{3})} = 1 + \sum_{\nu \gg 0} c_L(\nu) e^{2\pi i \sigma(\nu z/2\sqrt{3})},$$

is a Hilbert modular form of weight $\frac{n}{2}$, where $c_L(\nu) = \#\{x \in L\colon Q(x) = 2\nu\}$. If $L_1 = L, L_2, \dots, L_h$ is a complete set of representatives of the distinct classes in the genus $\operatorname{gen} L$ of L, then Siegel's theorem on the average number of representations of a number by $\operatorname{gen} L$ is given by

$$(*) \qquad \frac{1}{M(L)} \sum_{i=1}^{h} \frac{\Theta_{L_i}(z)}{e(L_i)} = G_{n/2}(z),$$

where $G_{n/2}(z) = 1 + \sum c_{n/2}(\nu) e^{2\pi i \sigma(\nu z/2\sqrt{3})}$ is the Eisenstein series of weight $\frac{n}{2}$. From [5] we have

$$c_{n/2}(\nu) = b_{n/2} \sum_{(\beta)|(\nu)} (\operatorname{sign} N\beta)^{n/2} |N\beta|^{n/2-1}$$

and

$$b_{n/2} = \frac{(2\pi)^n \sqrt{12}}{(\Gamma(\frac{n}{2}))^2 12^{n/2} \zeta_{\mathbb{Q}(\sqrt{3})}(\frac{n}{2})}.$$

For $n = 8$, we compute

$$b_4 = \frac{(2\pi)^8 \sqrt{12}}{(3!)^2 12^4 \zeta_{\mathbb{Q}(\sqrt{3})}(4)} = \frac{240}{23}.$$

Applying $(*)$ to the genus of 8-dimensional even unimodular lattices, we have

$$\frac{1}{M_8} \times \sum_{i=1}^{h} \frac{c_{L_i}(1)}{e(L_i)} = b_4 = \frac{240}{23}.$$

Using the 31 lattices in the genus and $M(L) = M_8$, we obtain

$$\begin{aligned}
\frac{1}{M_8} \times \sum_{i=1}^{31} \frac{c_{L_i}(1)}{e(L_i)} = {} & \frac{2^{14}\cdot 3^5\cdot 5^2\cdot 7}{23^2\cdot 41^2} \\
& \cdot \left(\frac{2^4\cdot 3\cdot 5}{2^{14}\cdot 3^5\cdot 5^2\cdot 7} + \frac{2^4\cdot 7}{2^{15}\cdot 3^2\cdot 5\cdot 7} + \frac{2^2\cdot 3\cdot 7}{2^{11}\cdot 3^5\cdot 5} \right. \\
& + \frac{2^3\cdot 3^2}{2^{13}\cdot 3^3\cdot 5} + \frac{2^4\cdot 3}{2^{15}\cdot 3^4} + \frac{2^4\cdot 3}{2^{14}\cdot 3^3} + \frac{2^4\cdot 3}{2^{14}\cdot 3^4} \\
& + \frac{2^4\cdot 3}{2^{15}\cdot 3^5} + \frac{2^2\cdot 3^2}{2^{9}\cdot 3^3\cdot 5} + \frac{2^5}{2^{14}\cdot 3^2} + \frac{2^2\cdot 7}{2^{11}\cdot 3^2} \\
& + \frac{2^3\cdot 3}{2^{13}\cdot 3^3\cdot 5} + \frac{2^3\cdot 3}{2^{7}\cdot 3^5} + \frac{2^4}{2^{15}\cdot 3} + \frac{2^4}{2^{15}} \\
& + \frac{2^4}{2^{15}\cdot 3^2\cdot 5\cdot 7} + \frac{2^3\cdot 3}{2^{12}\cdot 3^3} + \frac{2^2\cdot 5}{2^{7}\cdot 3^2\cdot 5} \\
& + \frac{2^2\cdot 3}{2^{9}\cdot 3^3\cdot 5} + \frac{2^2\cdot 3}{2^{11}\cdot 3^5\cdot 5} + \frac{2^2\cdot 3}{2^{6}\cdot 3^4} + \frac{2^2\cdot 3}{2^{7}\cdot 3^4} \\
& \left. + \frac{2^3}{2^{12}} + \frac{2^3}{2^{10}\cdot 3} + \frac{2^2}{2^{11}\cdot 3^2} \right) = \frac{240}{23}.
\end{aligned}$$

This calculation shows that the only lattices which admit vectors of quadratic norm 2 are exactly those previously obtained in our constructions.

5. Adjacency graph

We present in this section an adjacency graph for the 8-dimensional genus of even unimodular lattices of $\mathbb{Q}(\sqrt{3})$ at a dyadic prime. Let p be the ideal generated by $(1+\sqrt{3})$. For a lattice L in the genus, the vertices of the graph $R(L, p)$ are those lattices $M \in G = \operatorname{gen} L$ such that $M_q = L_q$ for all primes $q \neq p$. Two vertices M and M' are joined by an edge in $R(L, p)$ if $[M : M \cap M'] = [M' : M \cap M'] = Np$, where Np is the number of residue classes $\bmod p$. In this case, we say that M and M' are *neighbors* (or *adjacent*) in $R(L, p)$. If $K \in |R(L, p)|$, then $R(L, p)$ contains a representative of every class in the proper spinor genus $\operatorname{spn}^+(K)$. Let J_F^G be the subgroup of the idele group J_F consisting of those ideals (i_q) such that $i_q \in \theta(O^+(L_q))$ for all $q < \infty$, where θ is the spinor norm function. Let P_D be the subgroup of principal ideles with respect to $D = \theta(O^+(V))$. By a routine computation, we have $[J_F : P_D J_F^G] = 2$, hence there are two proper spinor genera in the genus of L. Let $g^+(L, p)$ be the number of proper spinor genera represented by $R(L, p)$. Then a result in [2] shows that

$$g^+(L, p) = 1 \quad \text{if and only if} \quad j(p) \in P_D J_F^G,$$

where $j(p) \in J_F$ is defined by

$$j(p)_q = \begin{cases} 1, & q \neq p, \\ \pi, & q = p. \end{cases}$$

Here, π is a fixed prime element in F_p. For the graph of the 8-dimensional genus of even unimodular lattices over $\mathbb{Q}(\sqrt{3})$, we have $j(p) \notin P_D J_F^G$, hence $R(L, p)$ represents both proper spinor genera in $\operatorname{gen} L$. Thus, $R(L, p)$ contains a representative of every class in $\operatorname{gen} L$. Now for each lattice in the graph, the number of neighbors is the same as the number of isotropic lines in L/pL. It is shown in [1, p. 21] that if $\dim FL = 2m$, where m is the Witt index of FL at p, then this number is given by

$$\frac{[(Np)^m - 1][(Np)^{m-1} + 1]}{Np - 1}.$$

It follows that each lattice in $R(L, p)$ has exactly 135 neighbors. We present two tables which, for each class in $\operatorname{gen} L$, give the numbers of its neighbors isometric to the various classes in $\operatorname{gen} L$. Since classes in the same proper spinor genus cannot be neighbors of one another (see [2]), it is convenient to arrange all classes of one proper spinor genus in the column and those of the opposite spinor genus in the row.

Let $N(L, K, p)$ denote the number of neighbors of L that are isometric to K. Table 2 shows $N(L, K, p)$ for L coming from a fixed proper spinor genus, say $\mathscr{S}_1$, and K coming from the opposite spinor genus $\mathscr{S}_2$. Note that each row has a sum equal to 135, which is the total number of neighbors of a fixed class. Table 3 shows $N(L, K, p)$ for L coming from $\mathscr{S}_2$ and K coming from $\mathscr{S}_1$. Note that each column now has a sum equal to 135.

Let L and K be neighbors. Then a relationship exists between $N(L, K, p)$ and $N(K, L, p)$, which is given by the following formula (see [1, p. 48]):

$$\frac{N(L, K, P)}{N(K, L, P)} = \frac{e(L)}{e(K)}.$$

This provides an alternative method for determining $e(L)$ by starting from a known lattice, say E_8, and using Tables 2 and 3, thus giving an additional check of the completeness of our list.

TABLE 2

Number of neighbors of L isometric to K, where $L \in \mathcal{S}_1$ and $K \in \mathcal{S}_2$

$L \backslash K$	D_8	$E_6 \perp G_2$	$D_4 \perp 2G_2$	$D_4 \perp 4A_1$	$2A_3 \perp 2A_1$	$8A_1$ (I)	$8A_1$ (III)	A_4	G_2	$2A_2$ (II)	$4A_1$ (II)	$2A_1$	$\varnothing$ (II)
E_8	135	0	0	0	0	0	0	0	0	0	0	0	0
$D_6 \perp G_2$	3	12	15	45	60	0	0	0	0	0	0	0	0
$2D_4$ (I)	9	0	18	18	0	81	9	0	0	0	0	0	0
$2D_4$ (II)	3	0	0	0	96	36	0	0	0	0	0	0	0
$4G_2$	0	0	54	0	0	81	0	0	0	0	0	0	0
$A_5 \perp 3A_1$	0	3	0	15	45	0	0	12	0	60	0	0	0
$G_2 \perp 6A_1$	0	0	15	45	0	0	3	0	12	0	0	60	0
$4A_2$	0	0	0	0	54	0	0	0	0	0	81	0	0
$8A_1$ (II)	0	0	2	4	32	17	0	0	0	0	32	32	16
D_4	0	0	0	3	0	0	0	96	0	0	36	0	0
A_3	0	0	0	15	0	0	0	12	3	60	0	45	0
$2A_2$ (I)	0	0	0	0	9	0	0	18	0	18	81	9	0
$4A_1$ (I)	0	0	0	1	8	2	0	0	0	32	68	8	16
$\varnothing$ (I)	0	0	0	0	0	45	0	0	0	0	0	0	90
$\varnothing$ (III)	0	0	0	0	0	36	3	0	0	0	0	96	0
$\varnothing$ (IV)	0	0	0	0	0	0	0	0	0	0	81	54	0
$\varnothing$ (V)	0	0	0	0	0	0	135	0	0	0	0	0	0
$\varnothing$ (VI)	0	0	0	0	0	0	0	0	0	0	90	0	45

TABLE 3

Number of neighbors of L isometric to K, where $L \in \mathcal{S}_2$ and $K \in \mathcal{S}_1$

$K \backslash L$	D_8	$E_6 \perp G_2$	$D_4 \perp 2G_2$	$D_4 \perp 4A_1$	$2A_3 \perp 2A_1$	$8A_1$ (I)	$8A_1$ (III)	A_4	G_2	$2A_2$ (II)	$4A_1$ (II)	$2A_1$	$\varnothing$ (II)
E_8	2	0	0	0	0	0	0	0	0	0	0	0	0
$D_6 \perp G_2$	28	27	18	6	1	0	0	0	0	0	0	0	0
$2D_4$ (I)	35	0	9	1	0	3	35	0	0	0	0	0	0
$2D_4$ (II)	70	0	0	0	4	8	0	0	0	0	0	0	0
$4G_2$	0	0	9	0	0	1	0	0	0	0	0	0	0
$A_5 \perp 3A_1$	0	108	0	32	12	0	0	5	0	9	0	0	0
$G_2 \perp 6A_1$	0	0	18	6	0	0	28	0	27	0	0	1	0
$4A_2$	0	0	0	0	32	0	0	0	0	0	8	0	0
$8A_1$ (II)	0	0	81	18	18	51	0	0	0	0	3	18	9
D_4	0	0	0	4	0	0	0	25	0	0	1	0	0
A_3	0	0	0	32	0	0	0	5	108	9	0	12	0
$2A_2$ (I)	0	0	0	0	32	0	0	100	0	36	48	32	0
$4A_1$ (I)	0	0	0	36	36	48	0	0	0	81	51	36	72
$\varnothing$ (I)	0	0	0	0	0	16	0	0	0	0	0	0	6
$\varnothing$ (III)	0	0	0	0	0	8	70	0	0	0	0	4	0
$\varnothing$ (IV)	0	0	0	0	0	0	0	0	0	8	32		0
$\varnothing$ (V)	0	0	0	0	0	0	2	0	0	0	0	0	0
$\varnothing$ (VI)	0	0	0	0	0	0	0	0	0	0	16	0	48

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