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THE 5-MODULAR REPRESENTATIONS OF THE TITS SIMPLE GROUP IN THE PRINCIPAL BLOCK

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ABSTRACT. In this paper we show how to construct the 5-modular absolutely irreducible representations of the Tits simple group in the principal block, which is the only block of positive defect. Starting with the smallest nontrivial ones, all the others except one pair are obtained as constituents of tensor products of dimension at most 729. The last two we get from a permutation representation of degree 1600. We give an exact description of the construction of the first one of degree 26 by extending its restrictions to several subgroups, a method first used in the existence proof of the Janko group J_4. Using the explicit matrices obtained from the above constructions, we work out the Green correspondents and sources of all the representations and state their socle series.

0. INTRODUCTION

One aim of modular representation theory is the construction of the p-modular absolutely irreducible representations of finite groups G, where p is a prime dividing the group order $|G|$. Having done this, invariants of these representations like vertices, Green correspondents, and sources may be computed to get more information about them. This paper deals with the 5-modular representations of the Tits simple group ${}^2F_4(2)'$ in the principal block. This is in some sense the only interesting block for this problem in ${}^2F_4(2)'$, since all the other blocks are blocks of defect 0. The following tables show some small permutation representations and tensor products of ${}^2F_4(2)'$ written as sums of their 5-modular constituents. To calculate these tables, we have used the decomposition matrix of ${}^2F_4(2)' \bmod 5$, which can be found in [4].

We have indicated in bold type how we construct the representations in this paper. Thus, one has to handle a tensor product of dimension $27 * 78 = 2106$ to get the smallest representation of degree 26 out of permutation modules and tensor products. But on the other hand, once having constructed 26, all the other representations, except the pair $(460, 460')$, can be constructed by looking at tensor products of maximal dimension 729, and the last pair can be found in a permutation representation of degree 1600. So this will be the way we construct the 5-modular absolutely irreducible representations of ${}^2F_4(2)'$ in the principal

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TABLE 1
Some small permutation modules

$$1600 = 4*1 + 3*(27+27^*) + 78 + 2*(109+109') + \mathbf{460} + \mathbf{460'}$$

$$1755 = 2*1 + 27 + 27^* + 2*78 + 109 + 109' + \text{projectives}$$

$$2304 = 4*1 + 4*(27+27^*) + 78 + 2*(109+109') + 460*460' + \text{projectives}$$

TABLE 2
Some small tensor products

$$26 \otimes 26 = 1 + \mathbf{27} + \mathbf{27^*} + \mathbf{78} + \mathbf{109} + \mathbf{109'} + \text{projectives} = 26^* \otimes 26^*$$

$$26 \otimes 26^* = 1 + \text{projectives}$$

$$26 \otimes 27 = 109 + \mathbf{593} = 26^* \otimes 27^*$$

$$26 \otimes 27^* = 109' + \mathbf{593'} = 26^* \otimes 27$$

$$27 \otimes 27 = 1 + 27 + 2*27^* + 78 + 109 + 109' + \mathbf{351}$$

$$27^* \otimes 27^* = 1 + 2*27 + 27^* + 78 + 109 + 109' + \mathbf{351^*}$$

$$27 \otimes 27^* = 1 + 78 + \text{projectives}$$

$$27 \otimes 78 = 26 + 26^* + 2*27 + 27^* + 351 + 2*351^* + 460 + 460'$$

block. We start with the smallest nontrivial representation of degree 26 in §1, constructing it from its restrictions to several subgroups of ${}^2F_4(2)'$. To do this, we use the same method as Benson, Conway, Norton, Parker, and Thackray in the existence proof of the Janko group J_4 [1], and we will give a detailed description of this way of constructing modular representations. Another description of this method can be found in a recent paper by Parker and Wilson [7], where they describe the construction of the 111-dimensional representation of the Lyons group Ly over GF(5). In §2 we try to get our explicit matrices for ${}^2F_4(2)'$ compatible with a description of ${}^2F_4(2)'$ as a permutation group on 1600 letters. On the one hand, we need this for later calculations to work out the restrictions of representations to some subgroups; on the other hand, we need a permutation representation of ${}^2F_4(2)'$ of degree 1600 to construct the last pair of representations we are interested in; all the others will follow from the one of degree 26 by building tensor products. The constructions of all these representations will be described in §3. Section 4 deals with the computation of the Green correspondents and sources of all the 5-modular absolutely irreducible representations of ${}^2F_4(2)'$ in the principal block. We will use the computer algebra system CAYLEY [2] and programs by Schneider [9], first to construct the Green correspondents and sources, and second to work out their socle series. The reader is referred to the book of Landrock [5] and to the paper by Schneider [9] for the necessary background from representation theory and a detailed description of the algorithms of §4. As usual, modules will be denoted by their dimensions, and for a given module M we write M^* for its dual and M' for its image under the outer automorphism of ${}^2F_4(2)'$. If there are several modules of the same dimension, we may also use subscripts to distinguish between them.

1. The 26-dimensional representations

In this section we want to construct the representation of degree 26, and we will do this with the method used in the construction of J_4 [1]. The idea is first to restrict the desired representation to some subgroups of the given group, and then to try to extend these restrictions to a representation of the whole group. In comparison to the existence proof of J_4, we will use this method for semisimple modules, where some aspects of the construction are easier; unfortunately, we will end up with a large number of cases to check. Everything is based on the following

Theorem 1.1. *Let G be a finite group, K a finite field, and $U_1 = \langle x_1, \dots, x_t \rangle$, $U_2 = \langle y_1, \dots, y_m \rangle$ two subgroups of G such that $G = \langle U_1, U_2 \rangle$. Define $V = U_1 \cap U_2$ and let $D_1 \colon U_1 \to \mathrm{GL}_n(K)$, $D_2 \colon U_2 \to \mathrm{GL}_n(K)$ be representations of U_1, U_2, respectively, such that $D_1 \downarrow_V = D_2 \downarrow_V$. Furthermore, let $T_1, \dots, T_k$ be a full set of representatives for the double cosets*

$$C_{\mathrm{GL}_n(K)}(D_2) \backslash C_{\mathrm{GL}_n(K)}(D_2 \downarrow_V) / C_{\mathrm{GL}_n(K)}(D_1).$$

If D_1 and D_2 can be extended to a representation D of G, i.e., if there exists a representation $D \colon G \to \mathrm{GL}_n(K)$ such that $D \downarrow_{U_1} \simeq D_1$ and $D \downarrow_{U_2} \simeq D_2$, then there exists $T \in \{T_1, \dots, T_k\}$ with $D \simeq \langle D_1, D_2^T \rangle$, i.e., the matrices $D_1(x_1), \dots, D_1(x_t)$, $T^{-1} D_2(y_1) T, \dots, T^{-1} D_2(y_m) T$ define the extension D for the group G.

Proof. Suppose that D_1 and D_2 can be extended to a representation D of G. By conjugation with a suitable matrix we can assume that $D \downarrow_{U_1} = D_1$. Since $D \downarrow_{U_2} \simeq D_2$, there exists a matrix $T \in \mathrm{GL}_n(K)$ such that $T^{-1} D_2 T = D \downarrow_{U_2}$. It follows that

$$D_2 \downarrow_V = D_1 \downarrow_V = D \downarrow_V = T^{-1} D_2 T \downarrow_V,$$

so $T \in C_{\mathrm{GL}_n(K)}(D_2 \downarrow_V)$. Now choose an arbitrary $S \in C_{\mathrm{GL}_n(K)}(D_2)$. Then ST has the same properties as T, so it is enough to check a right transversal of $C_{\mathrm{GL}_n(K)}(D_2 \downarrow_V)$. On the other hand, it is clear that for every $S \in C_{\mathrm{GL}_n(K)}(D_1)$ the matrix TS will define a representation equivalent to D, since

$$\langle D_1, D_2^{TS} \rangle = \langle D_1^S, D_2^{TS} \rangle = S^{-1} \langle D_1, D_2^T \rangle S = D^S \simeq D,$$

so it is in fact enough to check representatives for the double cosets. □

Remarks 1.2. (1) The condition $D_1 \downarrow_V = D_2 \downarrow_V$ is assumed here for simplicity, but whenever $D_1 \downarrow_V$ and $D_2 \downarrow_V$ are isomorphic representations, then there exists a matrix M such that $D_1 \downarrow_V = M^{-1} D_2 M \downarrow_V$. This M, however, may be hard to find in special cases, but when the restrictions are semisimple modules, then everything works fine and the representatives $T_1, \dots, T_k$ can be constructed without any difficulties, as we will see later.

(2) By checking all representatives $T_1, \dots, T_k$, one may prove that the representations D_1 and D_2 cannot be extended to a representation D for a given group G.

Before applying the theorem to construct the 26-dimensional representations of ${}^2F_4(2)'$, let us first recall some facts about ${}^2F_4(2)'$, which can be found in [10].

Lemma 1.3. *The group* ${}^2F_4(2)'$ *has two conjugacy classes of maximal subgroups isomorphic to* $L_3(3){:}2$. *Let* U_1 *be one such subgroup, and let* V *be the Sylow-3-normalizer of* U_1. *Then there exists an outer automorphism* σ *of* ${}^2F_4(2)'$ *with the following properties*:

(a) $U_2 = \sigma(U_1)$ *is not conjugate to* U_1 *in* ${}^2F_4(2)'$.

(b) σ *is an outer automorphism of* $V \simeq 3^{1+2}{:}D_8$.

Since V *is maximal in* U_1, *we have* $V = U_1 \cap U_2$.

The notation of Lemma 1.3 indicates how we want to use Theorem 1.1 for the construction of the 26-dimensional representations of ${}^2F_4(2)'$. There is good reason for this choice of U_1, U_2, and V, because the restriction of 26 to $L_3(3){:}2$ stays irreducible and the order of $L_3(3){:}2$ is prime to 5; therefore, the further restriction of 26 to V is semisimple, which makes it easier to construct its centralizer.

$L_3(3)$ is known as the automorphism group of the projective plane of order 3, and adding the duality automorphism which interchanges points and lines, we get $L_3(3){:}2$ as a permutation group on 26 letters (13 points and 13 lines). The permutations

$$\begin{aligned} a &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) \\ &\quad (14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26), \\ b &= (1, 3, 13)(2, 7, 6)(4, 8, 11)(14, 20, 23)(15, 17, 18)(16, 25, 24), \\ c &= (1, 25)(2, 24)(3, 23)(4, 22)(5, 21)(6, 20)(7, 19) \\ &\quad (8, 18)(9, 17)(10, 16)(11, 15)(12, 14)(13, 26) \end{aligned}$$

generate $L_3(3){:}2$ with $\langle a, b\rangle = L_3(3)$. A Sylow-3-normalizer of $L_3(3){:}2$ is, e.g., given by the elements

$$x = b^{caba^7} b^{caba^7 c^{a^9}} b^2 c^{a^9}, \qquad y = ((ab)^4)a^{11}.$$

This gives $U_1 = \langle a, b, c\rangle$, $V = \langle x, y\rangle$, and $U_2 = \sigma(U_1) = \langle \sigma(a), \sigma(b), \sigma(c)\rangle$, and since σ is an automorphism of V, we know that $\sigma(x), \sigma(y) \in \langle x, y\rangle$. To apply the theorem, we have to know $\sigma(x)$, $\sigma(y)$ as words in a, b, c. Obviously, $\sigma(x)$, $\sigma(y)$ satisfy the same relations as x, y, namely

$$x^6 = y^2 = x^3yx^{-1}yx^{-3}yx^{-1}y = (x^2y)^2x^{-1}y(xy)^2x^{-1}y = 1,$$

but the pair $(\sigma(x), \sigma(y))$ is not conjugate to the pair (x, y) in V, because σ is an outer automorphism of V. Up to conjugacy there is only one such pair of elements in V, e.g.,

$$x' = b^{caba^7c^{a^9}b^{caba^7}}y, \quad y' = (c^{a^9})y.$$

1.1. **The restrictions to U_1, U_2, and V.** By the above choice for U_1, U_2, and V we only have to construct the restriction of 26 to $L_3(3){:}2$ for the generators a, b, and c, and we get the restrictions to U_1, U_2, and V immediately because we can choose the same matrices for U_1 and U_2 and we know generators for V as a subgroup both of U_1 and U_2. But the restrictions of 26 and 26^* to $L_3(3){:}2$ are irreducible, isomorphic modules, and we use the CAYLEY system for the first time in this paper to construct one of them in the following way.

The centralizer of c in $L_3(3){:}2$ has index 234, and this gives a permutation representation of $L_3(3){:}2$ of this degree. Looking at the Atlas [3], we see that our restriction is a composition factor of this module, and using the CAYLEY version of the Meat-Axe [6], we get matrices A, B, and C for our elements a, b, and c to define D_1, the restriction of 26 (and 26^*) to $U_1 = L_3(3){:}2$. But we can use the same matrices also to define D_2, the restriction of 26 to $U_2 = \langle\sigma(a), \sigma(b), \sigma(c)\rangle$. Furthermore, with the above description of the generating pairs (x, y) and (x', y'), we are able to compute matrices X, Y and X', Y', to get $D_1\downarrow_V$ and $D_2\downarrow_V$, respectively. Unfortunately, we have $D_1\downarrow_V \simeq D_2\downarrow_V$, but not equality. So we have to find a transformation matrix M as in Remark 1.2(1), which means we have to find a matrix M such that

$$X = M^{-1}X'M \quad \text{and} \quad Y = M^{-1}Y'M.$$

This can be done using the idea of standard base [6]. Of course, we have to transform all generating matrices for D_2 with M, which gives $D_2 \simeq D_2^M = \langle A^M, B^M, C^M\rangle$. For simplicity, and according to Theorem 1.1, we write D_2 again for this representation of U_2.

1.2. **The double coset representatives.** As remarked earlier, the restriction of 26 to V is semisimple; in fact, it decomposes as $D_2\downarrow_V = 4_1 \oplus 4_2 \oplus 6_1 \oplus 6_2 \oplus 6_3$, a direct sum of five pairwise nonisomorphic, irreducible representations. By transforming $D_2\downarrow_V$ into the corresponding block structure it is an easy exercise to write down its centralizer. At this point we have to worry about the field K of Theorem 1.1. We are working in characteristic 5, and for the construction of D_1 and D_2, GF(5) was sufficient. But looking at the character table of ${}^2F_4(2)'$, we see that we have to use GF(25) to construct 26 for the whole group. Therefore, we have to choose our double cosets in the centralizer of $D_2\downarrow_V$ over GF(25), an abelian group of order 24^5. Since D_1 and D_2 are irreducible, their centralizers are just equal to the center of $GL_{26}(25)$, and the total number of double coset representatives we have to check is $24^4 = 331776$. It should be remarked again, that because of the semisimplicity of $D_2 \downarrow_V$ and its block structure we can write down all these representatives in a very easy way.

1.3. **The construction of the representations.** Since both 26 and 26* restrict to the same module for $L_3(3){:}2$, we have to find both of them with the help of our 24^4 double coset representatives. So we are looking for two transformation matrices T_1, T_2 such that D_1 and $D_2^{T_i}$, $i = 1, 2$, give two representations for ${}^2F_4(2)'$. It is hard to prove that some matrices represent a given group, but it may be much easier to show that they do not make sense as a representation. So we will check all double coset representatives T and for nearly all of them we have to prove that D_1 and D_2^T do not give a representation for ${}^2F_4(2)'$. This can be done in the following way:

(i) For any double coset representative T we choose an element P of $\langle D_1, D_2^T\rangle$, e.g., a product of two matrices, one from D_1 and one from D_2^T.

(ii) If, for this representative T, $\langle D_1, D_2^T\rangle$ represents ${}^2F_4(2)'$, then the order of P is bounded by the orders of the elements of ${}^2F_4(2)'$; therefore, order(P) has to be less than or equal to 16.

(iii) Choosing a random vector in $GF(25)^{26}$ and multiplying it with P again and again, we have an easy first check for the order of P.

Although this test seems to be a weak one, it is strong enough to eliminate all but two of our 331776 double coset representatives, and we can construct the representations 26 and 26* for ${}^2F_4(2)'$. These calculations have been done during the author's stay at the Scientific Centre of IBM Germany at Heidelberg.

More precisely, we have two sets of matrices $\{A, B, C, D\}$ and $\{A, B, C, D'\}$, which give the two representations, where $\{A, B, C\}$ generate the above restriction of 26 to $L_3(3){:}2$. The matrices A, B, C, and D are given in the Appendix. Knowing only the matrices, we have no chance to find a Sylow-5-subgroup or a Sylow-5-normalizer of ${}^2F_4(2)'$ in this representation, which is necessary for the construction of Green correspondents and sources. So what we need is

(a) ${}^2F_4(2)'$ as a permutation group,

(b) permutations a', b', c', d' for the matrices A, B, C, D.

In the next section we will construct such a permutation representation for ${}^2F_4(2)'$, and we will switch to other generators and matrices for easier calculations in Sylow-5-subgroups and Sylow-5-normalizers.

2. ${}^2F_4(2)'$ AS A PERMUTATION GROUP ON 1600 LETTERS

To construct ${}^2F_4(2)'$ as a permutation group, we start with the generators and relations of Parrott [8]. Using his notation, the following is an easy exercise in CAYLEY.

Lemma 2.1. *The subgroup* $H = \langle s_8, s_6r_5s_8(r_1r_8)^2s_8r_1r_7^5s_1\rangle$ *has index* 1600 *in* ${}^2F_4(2)'$, *hence* $H \simeq L_3(3){:}2$.

From the above lemma we get two permutations for $L_3(3){:}2$ as a permutation group on 1600 letters, say x_1 and x_2. On the other hand, from the construction in §1, we know permutations a, b, c on 26 letters which also generate $L_3(3){:}2$. Since we know matrices A, B, C for them, we want to find them again in $\langle x_1, x_2\rangle$, i.e., we want permutations in $\langle x_1, x_2\rangle$ for our matrices A, B, C. To get such permutations, we look at the relations for a, b, c, which are

$$\begin{aligned} a^{13} &= b^3 = c^2 = (ac)^2 = (ab^{-1})^3 = a^2cbabc(a^{-1}b^{-1})^2 \\ &= a^2(bc)^2ba^2b^{-1}a^{-1}b = (ab)^2(cb^{-1})^2a^{-1}b^{-1}a^{-1}b \\ &= ab^{-1}c^{-1}bc^{-1}b^{-1}c^{-1}a^{-1}b^{-1}ac^{-1}b = 1. \end{aligned}$$

We use the CAYLEY system first to get these relations, and second to look for permutations in $\langle x_1, x_2\rangle$ satisfying these relations. Up to conjugacy, there is only one such set of permutations $\{a', b', c'\}$ in $\langle x_1, x_2\rangle$, and so we have three of the four permutations on 1600 letters we are looking for. (It does not matter which $L_3(3){:}2$ in ${}^2F_4(2)'$ we take, because there is only one class of subgroups of type $L_3(3){:}2$ in ${}^2F_4(2)'$ under automorphisms.)

Having permutations for A, B, and C to generate $L_3(3){:}2$, we only have to find one more extra element to get ${}^2F_4(2)'$. To do this, we first construct a matrix $Z \in \langle A, B, C, D\rangle$ with $\text{order}(Z) = 2$, $\text{trace}(Z) = -1$, because there are only 1755 such elements in ${}^2F_4(2)'$. In our permutation representation of ${}^2F_4(2)'$ above on 1600 letters we only have to find this class of involutions again. (This is done by finding one such involution, e.g., the permutation corresponding to the generator r_1 in Parrott's notation, and conjugating it with a transversal of its centralizer in ${}^2F_4(2)'$.) To find the right permutation for the matrix Z, we check some easy relations of it with A, B, and C for all the conjugacy class members. Finally, we will end with a fourth permutation z', and the following holds:

Fact. *We have four permutations* a', b', c', *and* z' *on* 1600 *letters which generate* ${}^2F_4(2)'$, *and we have corresponding matrices* A, B, C, *and* Z *which define a* 26*-dimensional representation of* ${}^2F_4(2)'$ *over* GF(25).

As remarked at the end of §1, we now switch to new generators for ${}^2F_4(2)'$ and the corresponding matrices. This is necessary, because we are interested in a Sylow-5-subgroup and a Sylow-5-normalizer of ${}^2F_4(2)'$ and we want to restrict our representations of ${}^2F_4(2)'$ to these subgroups. If we choose the right generators, the restrictions will make no difficulties. So we use CAYLEY again to find generators for a Sylow-5-normalizer of ${}^2F_4(2)'$, and we work out some products to get permutations u and v with the following properties:

(a) ${}^2F_4(2)' = \langle u, v, a'u\rangle = \langle v, a'u\rangle$.

(b) $N = \langle u, v \rangle$ is a Sylow-5-normalizer of ${}^2F_4(2)'$.

(c) $S = \langle s = (u^2v)^2,\ t = (uv^2)^2 \rangle$ is the Sylow-5-subgroup of N, hence a Sylow-5-subgroup of ${}^2F_4(2)'$.

Using the concept of base and strong generating set and programs by Schneider [9], we can compute the matrices for u and v in our given representation $\langle A, B, C, Z \rangle$ for ${}^2F_4(2)'$. So we are able to write down the module 26 for our new set of generators of ${}^2F_4(2)'$.

For the rest of the paper we will view ${}^2F_4(2)'$ as this permutation group on 1600 letters, generated by the three permutations u, v, and $w = a'u$. In this presentation, it is now easy to restrict a given module for ${}^2F_4(2)'$ to a Sylow-5-normalizer; just forget about the matrix for w and look only at the matrices for u and v. With the above information, it is also merely a problem of matrix multiplication to restrict such a module further down to a Sylow-5-subgroup, because we know how to write generators for such a subgroup as words in u and v. So we can start to work out the other representations of ${}^2F_4(2)'$ in the principal block in §3 and to concentrate on their Green correspondents and sources in §4.

3. Other representations of ${}^2F_4(2)'$

Once the modules 26 and 26^* have been constructed, it is more or less a question of computer time to get the other 5-modular irreducible representations of ${}^2F_4(2)'$ in the principal block. First, we work out some tensor products using the CAYLEY system again. As ordinary characters, the product $26 \otimes 26$ decomposes as

$$26 \otimes 26 = \chi_8 + \chi_9,$$

where we use the Atlas-notation for the ordinary characters of ${}^2F_4(2)'$. Here, χ_8 is a projective character modulo 5, so we concentrate on χ_9. With the help of the Meat-Axe we get the modules 27, 27^*, 78, 109, and $109'$ as composition factors, as can be seen in Table 2. So we have all the 5-modular irreducible representations of ${}^2F_4(2)'$ up to dimension 109. It should be remarked that the modules 27, 27^*, and 78 can be realized over $\mathrm{GF}(5)$, and that the decomposition

$$26 \otimes 26 = 325 \oplus 351_0$$

is a direct sum of two indecomposable modules. For 325, this follows from its projectivity; the fact that 351_0 is indecomposable can be seen by looking at Green correspondents, and will be proved in Corollary 4.3.

So we are left with three pairs of modules $(351, 351^*)$, $(460, 460')$, and $(593, 593')$. Two of them can be found by working out tensor products as above. Looking at ordinary characters, we have the decompositions

$$26 \otimes 27 = \chi_{17}, \qquad 27 \otimes 27 = \chi_5 + \chi_8 + \chi_9.$$

The first product yields the module 593 (in the same way, the product $26 \otimes 27^*$ gives the module $593'$); in the second product we find the module 351 as a composition factor, and so we also get its dual 351^*. As a matter of fact, it suffices to calculate the skew tensor product $27 \wedge 27$ to get the representation 351. This shows that the modules 351 and 351^* can be realized over GF(5). All the constructions in this section have been worked out using the CAYLEY version of the Meat-Axe, the modules 593 and $593'$ requiring a large amount of CPU-time. For the modules 460 and $460'$, we have to use another approach.

Since the order of $L_3(3){:}\,2$ is prime to 5, its trivial representation $I_{L_3(3):2}$ over a field of characteristic 5 is projective. Inducing it up to ${}^2F_4(2)'$, we get the permutation representation of degree 1600 of §2, which is therefore projective, too. Looking at the decomposition matrix of ${}^2F_4(2)'$ [4], we see that this is just the projective indecomposable module with the trivial representation in its socle, and that the modules 460 and $460'$ are composition factors of it. But a dimension of 1600 seems to be too hard for the current CAYLEY version of the Meat-Axe, so we have to use the Assembler version. Routines for the arithmetic in GF(25) have been written in the 032 Assembler for the IBM RT 6150 to work out the composition factors of the above permutation module and to get the modules 460 and $460'$. This ends the construction of the 5-modular absolutely irreducible representations of ${}^2F_4(2)'$ in the principal block, and we are ready to look for their Green correspondents and sources in the next section.

4. Green correspondents and sources

Having constructed the 5-modular irreducible representations of ${}^2F_4(2)'$ in the principal block, we are now interested in their Green correspondents and sources. For definitions and proofs from modular representation theory, the reader is referred again to the book of Landrock [5]; the programs are described in Schneider [9].

The first question we have to answer is the question of vertices. But the Sylow-5-subgroup S of ${}^2F_4(2)'$ is elementary abelian of order 25, and by a result of Knörr (Landrock [5, p. 244]) we get

Theorem 4.1. *All irreducible modules of ${}^2F_4(2)'$ in the principal block have vertex S.*

Since we chose our generators for ${}^2F_4(2)'$ in an elegant way, we have no difficulties to restrict our modules to the Sylow-5-normalizer N. From theory we know that for all modules M in question the restriction to N decomposes as $M\downarrow_N = f(M) \oplus M_1 \oplus \cdots \oplus M_r$, with $r \geq 0$, where $f(M)$ is the Green correspondent of M and has vertex N again. The vertices of the modules M_i are of the form $S^g \cap N$, $g \notin N$. Since S is a Sylow-5-subgroup of ${}^2F_4(2)'$, every 5-element in $S^g \cap N$ would be contained in S, contradicting the trivial intersection property. But a module has vertex $\{1\}$ if and only if it is projective. So all the M_i are projective. In particular, their dimensions are divisible by

25, whereas the dimension of $f(M)$ cannot be divided by 25, since $\dim(M)$ is not divisible by 25. This will be of great help when we work out the Green correspondents. For this purpose we use the CAYLEY system again, together with programs of Schneider, for the calculation of endomorphism rings. After the construction of the Green correspondents we will be interested in their socle series. To get these, we need a complete list of the 5-modular irreducible representations of the Sylow-5-normalizer $N \simeq 5^2{:}4A_4$. This is fairly easy, and N turns out to have 14 such modules, namely

$$I,\ 1,\ 1_a,\ 1_a^*,\ 1_b,\ 1_b^*,\ 2_a,\ 2_a^*,\ 2_b,\ 2_b^*,\ 2_c,\ 2_c^*,\ 3_a,\ 3_b,$$

where I denotes the trivial module and *, as usual, the dual one. With this information and some other programs of Schneider, the socle series of a given module can be computed.

Applying these methods, we get the following results.

Theorem 4.2. *The Green correspondents of the modules* 26 *and* 26^* *are*

$$f(26) = 1_a^*, \qquad f(26^*) = 1_a.$$

Proof. The computation of the endomorphism ring yields a so-called Fitting-element, which gives a decomposition

$$26\downarrow_N = 1_a^* \oplus M,$$

where $\dim(M) = 25$, so M has to be projective and $f(26) = 1_a^*$. It can be shown that $M = \Pi_{1_a}$, the projective indecomposable module with 1_a in the socle. From duality, the second statement follows. □

Corollary 4.3. *The decomposition*

$$26 \otimes 26 = 325 \oplus 351_0$$

is the direct sum of two indecomposable modules.

Proof. Since 325 is projective indecomposable, there is nothing to show. Now look at 351_0 and write it as

$$351_0 = M_1 \oplus \cdots \oplus M_r$$

for some $r \geq 1$, where the M_i are indecomposable. Restricting the tensor product to N yields

$$(26 \otimes 26)\downarrow_N = 325\downarrow_N \oplus M_1\downarrow_N \oplus \cdots \oplus M_r\downarrow_N .$$

By looking at the decomposition matrix, we see that none of the M_i can be projective. So each of the $M_i\downarrow_N$ has at least one nonprojective direct summand N_i, and we get

$$(26 \otimes 26)\downarrow_N = N_1 \oplus \cdots \oplus N_r \oplus \{\text{some other summands}\} \oplus \{\text{projectives}\}.$$

On the other hand, we have

$$\begin{aligned}(26 \otimes 26)\downarrow_N = 26\downarrow_N \otimes 26\downarrow_N &= (1_a^* \oplus \Pi_{1_a}) \otimes (1_a^* \otimes \Pi_{1_a}) \\ &= 1_b \oplus \{\text{projectives}\}.\end{aligned}$$

So $r = 1$, and 351_0 is indecomposable with $f(351_0) = 1_b$. □

Theorem 4.4. *The modules* 27 *and* 27^* *are absolutely indecomposable when restricted to* N, *so their Green correspondents are*

$$f(27) = 27\downarrow_N, \qquad f(27^*) = 27^*\downarrow_N .$$

They have the following socle series:

$$f(27) = \begin{array}{ccccc} & & 1 & & \\ & & 2_a & & \\ & 3_b & & 3_b & \\ 2_a^* & & 2_b & & 2_c^* \\ 1_a & & 1_a^* & & 3_a \\ & 2_b^* & & 2_c & \\ & & 3_b & & \end{array}, \qquad f(27^*) = \begin{array}{ccccc} & & 3_b & & \\ & 2_b & & 2_c^* & \\ 1_a & & 1_a^* & & 3_a \\ & 2_b^* & & 2_c & \\ & & 3_b & & \\ & 2_a & & 2_a^* & \\ & 1 & & 3_b & \end{array}.$$

Proof. First we compute the endomorphism ring of $27\downarrow_N$, and it can be shown that it has no nontrivial idempotents, so $27\downarrow_N$ is indecomposable. With the programs of Schneider it is no problem to prove the stated socle series for $f(27)$. The result for $f(27^*)$ follows in the same way. □

Remark 4.5. As in Corollary 4.3, it follows that the tensor products $26 \otimes 27$ and $26 \otimes 27^*$ are indecomposable, because the products of the Green correspondents $f(26) \otimes f(27)$ and $f(26) \otimes f(27^*)$ are indecomposable.

Theorem 4.6. *The module* 78 *has a* 28*-dimensional Green correspondent. Its socle series is*

$$f(78) = \begin{array}{ccccc} & & 2_a^* & & \\ & & 3_a & & \\ 2_a & & 2_b^* & & 2_c \\ 1_b & & 1_b^* & & 3_b \\ & 2_b & & 2_c^* & \\ & I & & 3_a & \\ & 2_a & & 2_a^* & \end{array}.$$

Proof. As in the proof of Theorem 4.1, we compute the endomorphism ring and get a decomposition

$$78\downarrow_N = 28 \oplus 50,$$

where the second module has to be projective. Further calculations prove the indecomposability of the first one, and we get the socle series as in the proof of Theorem 4.4. □

In the same way, but with more and more expensive computer work, we get the following result.

Theorem 4.7. *The modules* 109 *and* $109'$ *have the following Green correspondents*:

$$f(109) = \begin{matrix} 1_b \\ 2_b \\ 3_a \\ 2_b^* \\ 1_b^* \end{matrix}, \qquad f(109') = \begin{matrix} 1_b^* \\ 2_c^* \\ 3_a \\ 2_c \\ 1_b \end{matrix}.$$

For the CAYLEY version of the endomorphism ring program, a dimension of 300 seems to be the limit. So we have to proceed differently in the other cases; but first we want to work out the sources for the modules of dimension up to 109.

For the question of sources, we have to restrict our modules further down to S. This is no problem, because we know generators s and t for S as words in the generators u and v for N, and we can concentrate on the restriction of the Green correspondents to S. We can use the same methods as above and prove the indecomposability of these restrictions in all cases. Since S is a 5-group, it has only one irreducible representation, namely the trivial one I, and the socle series of an S-module can be given as a sequence $(s_1, s_2, \dots, s_r)$ of integers, where s_i is the multiplicity of I as a direct summand in the ith socle layer of the module.

Theorem 4.8. *For the modules* $26, 26^*, 27, 27^*, 78, 109,$ *and* $109'$ *the restriction of the Green correspondent to* S *is indecomposable, hence equal to the source, and the socle series of the sources are given by*

$$\begin{aligned}
\operatorname{socser}(\operatorname{src}(26)) &= (1), \\
\operatorname{socser}(\operatorname{src}(26^*)) &= (1), \\
\operatorname{socser}(\operatorname{src}(27)) &= (3, 4, 5, 6, 6, 2, 1), \\
\operatorname{socser}(\operatorname{src}(27^*)) &= (4, 4, 3, 4, 5, 4, 3), \\
\operatorname{socser}(\operatorname{src}(78)) &= (4, 4, 4, 5, 6, 3, 2), \\
\operatorname{socser}(\operatorname{src}(109)) &= (1, 2, 3, 2, 1), \\
\operatorname{socser}(\operatorname{src}(109')) &= (1, 2, 3, 2, 1).
\end{aligned}$$

Now we are left with three pairs of modules $(351, 351^*)$, $(460, 460')$, and $(593, 593')$. Let M be one of these modules. By Mackey decomposition we know that

$$\operatorname{src}(M)\uparrow^N\downarrow_S = \bigoplus_{i=1}^{r} (\operatorname{src}(M) \otimes x_i)$$

is the direct sum of conjugates of $\operatorname{src}(M)$, not necessarily different, where $x_1, \dots, x_r \in N$. But the Green correspondent $f(M)$ is a direct summand of the induced module $\operatorname{src}(M)\uparrow^N$, so the restriction of M to S decomposes as

$$\begin{aligned} M\downarrow_S &= (M\downarrow_N)\downarrow_S = (f(M)\oplus\{\text{projectives}\})\downarrow_S \\ &= \bigoplus_{j=1}^{k}(\operatorname{src}(M)\otimes y_j)\oplus\{\text{projectives}\}, \end{aligned}$$

and the dimension of $f(M)$ is equal to the dimension of the nonprojective part of $M\downarrow_S$.

With this information we proceed in the following way:

(i) Restrict M to S and compute the socle series of $M\downarrow_S$.
(ii) Look at the highest socle layer of $M\downarrow_S$ to get the exact number of projective summands of $M\downarrow_S$.
(iii) Use the socle series of the projective indecomposable module of S to deduce the socle series of $f(M)\downarrow_S$, the nonprojective part of $M\downarrow_S$.
(iv) Compute the restriction $M\downarrow_N$ and its socle series.
(v) Use the socle series of all the projective indecomposable modules of N and the information about $\dim(f(M))$ and the dimension of the projective part of $M\downarrow_N$ to deduce the socle series of $f(M)$.

This method enables us to state the socle series of some Green correspondents without their explicit construction.

What is left is the question of the sources of these modules. We know that $f(M)\downarrow_S$ is the direct sum of conjugates of $\operatorname{src}(M)$ and every conjugate has the same socle series. Therefore, $f(M)\downarrow_S$ can only decompose into more than one direct summand, if for every socle layer of $f(M)\downarrow_S$ the multiplicity of the trivial representation I is divisible by the same integer m. If this is not the case, $f(M)\downarrow_S$ is indecomposable, hence equal to the source $\operatorname{src}(M)$. This will answer our questions for the next two pairs of modules, and we get the following results.

Theorem 4.9. *The Green correspondents and sources of* 351 *and* 351^* *are* 76-*dimensional. Their socle series are given by*

$$f(351) = \begin{array}{ccccccccccccccc}
 & & & & & & & 3_a & & & & & & & \\
 & & & & & 2_a & & 2_b^* & & 2_c & & & & & \\
 & & & I & & 1_b & & 1_b^* & & 3_b & & 3_b & & & \\
2_a^* & & 2_a^* & & 2_b & & 2_b & & 2_b & & 2_c^* & & 2_c^* & & 2_c^*, \\
 & 1 & & 1_a & & 1_a^* & & 3_a & & 3_a & & 3_a & & 3_a & \\
 & & 2_a & & 2_a & & 2_b^* & & 2_b^* & & 2_c & & 2_c & & \\
 & & & I & & 1_b & & 1_b^* & & 3_b & & 3_b & & & \\
 & & & & & 2_a^* & & 2_b & & 2_c^* & & & & &
\end{array}$$

$$f(351^*) = \begin{array}{ccccccccccccc}
 & & & & 2_a & & 2^*_b & & 2_c & & & & \\
 & & I & & 1_b & & 1^*_b & & 3_b & & 3_b & & \\
 & 2^*_a & & 2^*_a & & 2_b & & 2_b & & 2^*_c & & 2^*_c & \\
1 & & 1_a & & 1^*_a & & 3_a & & 3_a & & 3_a & & 3_a \\
 & 2_a & & 2_a & & 2^*_b & & 2^*_b & & 2_c & & 2_c & \\
 & & I & & 1_b & & 1^*_b & & 3_b & & 3_b & & \\
 & & & & 2^*_a & & 2_b & & 2^*_c & & & & \\
 & & & & 2^*_b & & 2_c & & 3_a & & & &
\end{array},$$

$$\operatorname{socser}(\operatorname{src}(351)) = (6, 9, 12, 15, 16, 9, 6, 3),$$
$$\operatorname{socser}(\operatorname{src}(351^*)) = (7, 6, 9, 12, 15, 12, 9, 6).$$

Theorem 4.10. *The modules* 593 *and* $593'$ *have the following Green correspondents and sources*:

$$f(593) = \begin{array}{ccc}
 & 3_a & \\
2_a & & 2^*_b \\
I & & 3_b \\
2^*_a & & 2_b \\
 & 3_a &
\end{array}, \qquad f(593') = \begin{array}{ccc}
 & 3_a & \\
2_a & & 2_c \\
I & & 3_b \\
2^*_a & & 2^*_c \\
 & 3_a &
\end{array},$$

$$\operatorname{socser}(\operatorname{src}(593)) = (3, 4, 4, 4, 3),$$
$$\operatorname{socser}(\operatorname{src}(593')) = (3, 4, 4, 4, 3).$$

The above method fails for the restriction $f(460)\downarrow_S$, because its socle series is $(4, 8, 12, 12, 12, 8, 4)$, so all parts are divisible by 4. But we are able to split off all projective indecomposable summands of $460\downarrow_S$; there are exactly 16 of them, obtained by the CAYLEY system, and so we can construct the restriction $f(460)\downarrow_S$, a module of dimension 60. Therefore, it is no problem to use the endomorphism ring program to decompose this module into indecomposable summands, and we finish with the last result about the Green correspondents and sources of 460 and $460'$.

Theorem 4.11. *The Green correspondents of the modules* 460 *and* $460'$ *have dimension* 60 *and the following socle series*:

$$f(460) = \begin{array}{cccccc}
 & & 2^*_a & 2_b & & \\
 & 1 & 1_a & 3_a & 3_a & \\
2_a & 2_a & 2^*_b & 2^*_b & 2_c & 2_c \\
I & 1_b & 1^*_b & 3_b & 3_b & 3_b \\
2^*_a & 2^*_a & 2_b & 2_b & 2^*_c & 2^*_c \\
 & 1 & 1^*_a & 3_a & 3_a & \\
 & & 2_a & 2^*_b & &
\end{array},$$

$$f(460') = \begin{array}{cccccc} & & 2_a^* & 2_c^* & & \\ & 1 & 1_a^* & 3_a & 3_a & \\ 2_a & 2_a & 2_b^* & 2_b^* & 2_c & 2_c \\ I & 1_b & 1_b^* & 3_b & 3_b & 3_b \\ 2_a^* & 2_a^* & 2_b & 2_b & 2_c^* & 2_c^* \\ & 1 & 1_a & 3_a & 3_a & \\ & & 2_a & 2_c & & \end{array}.$$

The sources have dimension 15, *and their socle series are*

$$\mathrm{socser}(\mathrm{src}(460)) = (1, 2, 3, 3, 3, 2, 1),$$
$$\mathrm{socser}(\mathrm{src}(460')) = (1, 2, 3, 3, 3, 2, 1).$$

All the calculations in the last three sections have been done on an IBM RT 6150 at the University of Essen.

Appendix

The following matrices generate a 26-dimensional representation of ${}^2F_4(2)'$ over GF(25). For simplicity, only the exponents of a primitive element ω of GF(25) have been printed, where ω is a root of the polynomial $X^2 + X + 2$ over GF(5).

Thus: . means 0, 0 means $\omega^0 = 1$, i means ω^i for $1 \le i \le 23$.

```
A =  3  1 14  3 16 10  5 18  .  7  6  6 17 19  1  0  5 10  1  .  . 22 18 17  9  9
     0  4 12  5  7  7 17 21  3 20  8 13  9  7  6 12 12 23 23 11 22  2 10  . 19 13
    22 19 16  8 12  3 10 11 19 23 13  2 19  1 17 13 12 18  7  9 20 10 19  5 22 16
    17 11 12 22 18 11  8 10  7 16 17 19  2 15 17  0  2 13 22  8 19  1  .  2  3 16
     0  1 10 20 22 22 16 22 12 19 17  9 14  8 17  0  3 14 10  9 23  7  4  9  3 15
    20 23  8 13 17  8 22 23 16  1 21  3 13 11 18  3  6  4  4  0 14  1  0  6  4  7
    16 23 13 12 23 17  8  5  0 18 18  4 13  .  8 16 16  3  8 23 22  9  7  6 16  2
    17  . 14 18 21 11 14 10 15 19 13  2 19 13 22 18 23 21 23  9 13  8 14 13 20  1
     3 17 14 20 23  1  8 16 14  9  .  6  4  8 10 12 20 22 22 10  7  1 14 17 15  6
    15  2 11  2  1 17  7  1 13  9  3 18 15 22  . 19  6  2  9 16  8 19 12 21  .  2
    15 21 16 12 20  .  5  2  9 19 15 21  6  6 10  4 17 12 19 20 22 18 21  8  8  7
    18  6  8 23 14 12  .  2  8 19 23 17  6 12 10 12  8 21 11  0 18 15  3 12 12 15
     9 15  . 19 18  4 20  3  5  7 18 11  . 18  0  6  . 18  0  0  . 12  .  .  .  .
    11  9  9  6 10  9 21 11 23 11 22  6  0 12 18  . 12  .  .  0 12 12 12  0 12  0
    10 22 15  7 17 22  7  6  9 16 12 11  . 18 12  0  . 12  0  .  .  .  0  .  0  .
     4 23  3  1  7  8 19 21 13 19 17 23  0  0 12 12 12  0  .  0 12  .  .  0  .  0
    16  8  3 10  2 17  9  8 19 10 21 12 18  .  .  6  6 18  .  .  .  .  .  .  .  .
    23 20  . 23 12 15  2 20 18 17 22 20  6  6  0  6 18 18  .  .  .  0  0  .  0  .
    13  9  5  1 17 17  4  5 14 13  2 15 12 12 12 12  6  6 18 18  6  6  . 18  . 18
     6  6  7 11  0  3  3  3 10  7 10  3  0  .  .  6 18 18 18  6 18  . 18  6 18  6
    11 11  7  3 15 15 13 15 13  6 19 19 18  .  . 18  0  0 18  . 18  6  .  6  .  6
    10  6  .  8 16 23  5  8  1 18  7 16  6  . 12  6 18 12 18 18  .  . 18  . 18  .
    23 14  0  5 16 13  3  7 18 23  6  4  6 18  0 12  6 12  0  0 18  0 12 12 12 12
    18 16 21 13  8  7  4 23  3  0 13  6 12  0 18  6 12  6 12  6  0 12  0 18  0 18
     5 12 18 22 20 15 14 16  . 12  1  5 18  6 12  0 18  0 12 12  6 12  0  0  0  0
     3 18  6  4 22  3 15  3  3  3  8  9  0 12  6 18  0 18  0 18 12  0 12  6 12  6
```

$$
B = \begin{matrix}
8 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 16 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 16 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & 16 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 8 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 8 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 0 & . & 6 & 15 & 1 & 23 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 5 & 2 & . & 1 & 18 & 23 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 15 & 0 & 5 & 11 & 12 & 12 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 2 & 8 & 16 & . & 8 & 11 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 11 & 18 & 13 & 15 & 1 & 21 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 22 & 1 & 23 & . & 0 & 10 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 12 & 0 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 12 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 12 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 12 & . & 0 & . & 0 & 12 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & 12 & 12 & 0 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & 12 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 0
\end{matrix}
$$

$$
C = \begin{matrix}
14 & 5 & 23 & 18 & 11 & 7 & 8 & 12 & 12 & 3 & 12 & 21 & 16 & 11 & 7 & 18 & 16 & 0 & 6 & 9 & 4 & 13 & . & 22 & 22 & 16 \\
9 & 8 & 18 & 0 & 23 & 22 & 23 & 0 & 5 & 14 & 15 & 8 & 8 & 4 & 8 & 22 & 22 & 8 & 18 & 19 & 9 & 1 & 14 & 21 & 22 & 7 \\
4 & 14 & 22 & 23 & 4 & 19 & 16 & 10 & 6 & 0 & 20 & 3 & 21 & 21 & 1 & 16 & 8 & 12 & 23 & 6 & 13 & 23 & 12 & 19 & 2 & 4 \\
19 & 19 & 18 & 1 & 8 & 3 & 3 & . & 9 & 9 & 13 & 16 & . & 6 & 10 & 18 & 23 & 16 & 21 & 20 & 3 & 11 & 3 & 14 & 10 & 8 \\
5 & 5 & 18 & 22 & . & 0 & 2 & 15 & 6 & 8 & 12 & 16 & 0 & 18 & 14 & 7 & 21 & 11 & 20 & 19 & 21 & 22 & 23 & 21 & 9 & 18 \\
13 & 16 & 8 & . & 18 & 7 & 16 & 17 & 23 & 21 & 9 & 18 & 19 & 22 & 18 & 3 & 8 & 10 & 7 & 8 & 19 & 17 & 6 & 15 & 1 & 8 \\
5 & 22 & 8 & 23 & 8 & 19 & 12 & 23 & 18 & 10 & 11 & 19 & 18 & 4 & 14 & 5 & 22 & 9 & 10 & 14 & 12 & 23 & 23 & 6 & 8 & 21 \\
. & 4 & 7 & 18 & 15 & 22 & 0 & 5 & 0 & 13 & 8 & 14 & 18 & 15 & 6 & 12 & 14 & 0 & 23 & 10 & 23 & 21 & 2 & 16 & 12 & 1 \\
4 & 19 & 14 & 3 & 20 & 8 & 1 & 16 & 7 & 12 & 12 & 13 & 1 & 4 & 18 & 17 & 6 & 22 & 12 & 11 & 14 & 9 & . & 11 & 4 & 10 \\
22 & 20 & 1 & 22 & 21 & 17 & 10 & 11 & 18 & 6 & 16 & 15 & 12 & 22 & 19 & 23 & 1 & . & 11 & . & 20 & 10 & 21 & 9 & 3 & 12 \\
5 & 8 & 22 & 23 & 2 & 7 & 14 & 23 & 13 & 15 & 22 & 5 & 0 & 7 & 9 & 20 & 15 & 8 & 9 & 21 & 23 & 6 & 11 & 0 & 12 & 1 \\
0 & 15 & 20 & 17 & 9 & 12 & 19 & 9 & 19 & 7 & . & 3 & 18 & 9 & 6 & 22 & 16 & 6 & 23 & 4 & 18 & . & 14 & . & 16 & 13 \\
23 & 11 & 4 & 0 & . & 23 & 15 & 8 & 6 & 22 & 17 & 11 & . & 6 & 18 & 0 & 0 & 12 & . & 12 & 0 & . & 12 & 6 & . & . \\
6 & 3 & 3 & 10 & 9 & 5 & 0 & 12 & 10 & 20 & 16 & 22 & . & 6 & 12 & 12 & 6 & 0 & 12 & 18 & 0 & 0 & . & . & . & . \\
18 & 14 & 13 & 4 & 2 & 23 & 0 & 21 & 16 & 13 & 16 & . & . & . & 12 & . & 6 & 12 & 0 & 0 & . & . & 12 & . & 0 & 12 \\
17 & 20 & 9 & 14 & 4 & 6 & 15 & 4 & 21 & 20 & 11 & 1 & . & 12 & 0 & 6 & 18 & 12 & 0 & . & 0 & 0 & . & . & . & . \\
4 & 4 & 3 & 18 & 12 & 9 & 18 & 15 & 14 & 0 & 13 & 13 & . & 12 & 18 & 0 & 6 & . & . & 12 & 0 & 12 & . & . & . & . \\
20 & 12 & 11 & 2 & 4 & 1 & 20 & 17 & 6 & 1 & 19 & 13 & . & 18 & 0 & 6 & 6 & 0 & 0 & 0 & . & . & 0 & 12 & 0 & 12 \\
3 & 21 & 16 & 21 & 13 & 5 & 8 & 5 & 3 & 12 & 22 & 3 & . & 18 & . & 12 & . & . & 6 & . & . & 18 & . & . & 6 & 6 \\
. & 15 & 15 & 0 & 2 & 3 & 1 & 22 & 21 & 22 & 11 & 2 & . & 0 & . & 6 & . & . & . & . & 6 & . & 0 & 6 & 18 & 18 \\
. & 17 & 23 & 11 & 4 & 10 & 10 & 8 & 19 & 10 & 1 & 7 & . & 18 & . & 12 & 0 & . & . & 6 & . & . & 18 & 6 & 0 & . \\
21 & 20 & 11 & 1 & 11 & 1 & 20 & 23 & 0 & 15 & 7 & 7 & . & 6 & . & 6 & 12 & . & . & . & . & 18 & 18 & . & 6 & 18 \\
15 & 7 & 5 & . & 1 & 2 & 21 & 9 & 2 & 4 & 0 & 4 & 18 & 18 & . & 18 & 0 & . & 18 & 18 & . & . & . & 6 & 18 & 6 \\
18 & 8 & 12 & 12 & 20 & 1 & 16 & 6 & 14 & 10 & 4 & 21 & 0 & 0 & 6 & 0 & 6 & 18 & . & . & . & . & . & 6 & . & . \\
3 & . & 21 & 6 & 14 & 22 & 12 & . & 4 & 4 & 4 & 7 & 12 & 6 & 12 & 0 & 18 & 6 & 6 & . & 18 & . & . & . & 6 & . \\
7 & 21 & 18 & 0 & 12 & 13 & 4 & . & 7 & 18 & 0 & 15 & 0 & 18 & 6 & 12 & 6 & 6 & 12 & . & . & 18 & 6 & 18 & 6 & .
\end{matrix}
$$

D =	0	.	18	3	.	12	6	3	6	13	22	14	22	12	0	11	2	2	10	15	13	5	22	17	3	12
	2	15	19	11	21	21	7	19	23	16	19	1	8	16	1	21	4	3	5	18	11	11	4	11	9	0
	8	15	2	2	10	0	5	13	.	4	12	4	22	6	9	15	16	13	3	23	11	8	3	13	6	19
	12	.	12	2	14	3	18	14	20	5	9	3	1	14	10	21	4	17	14	17	21	15	5	5	16	22
	23	14	21	23	23	14	22	11	10	1	4	22	11	8	19	9	3	23	11	14	19	2	20	11	20	17
	17	10	13	11	6	3	4	7	23	3	13	5	9	8	21	11	15	6	2	.	8	5	0	4	20	5
	20	14	1	.	6	19	0	0	10	12	16	3	7	9	22	8	2	9	15	6	20	5	1	6	20	21
	.	13	9	2	17	20	10	0	20	3	8	15	9	10	11	23	1	3	2	22	10	1	18	21	22	11
	4	4	9	3	7	7	6	11	.	19	21	6	2	11	.	5	5	7	11	22	7	18	5	8	9	1
	0	18	.	14	7	21	19	5	23	23	19	2	19	18	9	1	10	14	3	22	16	12	15	17	18	1
	23	8	18	15	15	15	17	5	21	12	14	2	0	4	14	7	10	.	4	13	12	3	17	20	17	0
	2	15	4	7	0	9	14	18	4	21	21	11	16	23	17	11	12	4	21	10	3	19	3	5	.	6
	20	2	.	2	19	.	2	16	19	.	2	11	21	1	16	19	16	14	20	11	22	8	13	20	22	10
	5	4	22	11	12	18	14	16	5	19	23	6	14	22	11	13	11	3	10	9	8	13	10	10	8	20
	5	13	7	13	7	8	15	5	6	14	8	5	15	3	14	16	1	20	0	13	0	8	22	0	.	22
	1	0	23	6	3	13	.	4	12	17	17	15	10	5	8	2	9	.	6	3	6	11	14	3	3	14
	0	.	16	13	1	2	23	23	11	8	18	23	13	2	9	14	8	7	18	16	18	14	18	23	5	23
	10	12	4	11	20	2	17	17	20	8	12	7	10	1	7	10	22	15	5	7	23	15	12	5	5	12
	17	17	18	16	7	13	13	3	12	0	4	5	19	10	20	16	7	12	23	9	23	12	4	0	6	6
	9	10	15	0	14	20	7	2	21	0	13	9	10	3	7	2	9	13	19	15	22	14	21	0	18	0
	3	10	21	21	11	1	23	20	21	14	20	9	4	16	4	23	13	8	11	0	11	21	18	20	1	21
	15	19	4	0	20	16	9	21	17	15	13	10	7	19	6	2	16	13	22	15	22	0	20	18	18	12
	.	5	0	19	.	.	5	20	12	5	5	16	20	8	23	5	7	10	2	2	4	22	15	4	22	19
	7	18	16	13	9	.	5	12	1	0	8	2	16	11	0	18	19	11	7	16	3	16	14	3	16	13
	1	7	17	18	16	7	22	.	.	22	.	7	13	23	9	10	3	0	16	16	13	8	14	3	16	13
	11	13	16	0	14	18	6	3	12	9	12	22	3	15	22	21	9	20	19	4	15	4	12	1	4	1

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