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Supplement to

AN ADAPTIVE FINITE ELEMENT METHOD FOR TWO-PHASE STEFAN PROBLEMS IN TWO SPACE DIMENSIONS. PART I: STABILITY AND ERROR ESTIMATES

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S1. Supplement to Section 5.

LEMMA 5.1. *For all regular meshes* $\mathcal{S}$ *there exists a* C^∞*-function* $h\colon \bar{\Omega}\to\mathbf{R}^+$ *satisfying*

$$D^k h(x) = O(h_S^{1-k}), \qquad \forall\, x\in S\in\mathcal{S},\ k\geq 0. \tag{S1.1}$$

Proof. For each $S\in\mathcal{S}$ there exists a (finite) covering $\{B_i\}$ of S such that card $\{B_i\} = O(1)$, where $B_i := B(x_i, r_i)$, $x_i\in S$ and $r_i := ah_S/2$; thus (5.2) yields $\bar{B}_i := B(x_i, 2r_i)\subset \tilde{S}$. By virtue of (5.3), this gives rise to a covering $\{B_i\}_{i=1}^I$ of $\bar{\Omega}$ satisfying $b_S :=$ card $\{B_i : S\cap\bar{B}_i\neq\emptyset\} = O(1)$ for all $S\in\mathcal{S}$ (*nonoverlapping* property !). Let $\delta_0\in C_0^\infty(B(0,1))$ satisfy $\int_{\mathbf{R}^2}\delta_0 = 1, 0\leq\delta_0\leq 1$, $\delta_0(x) = 1$ for all $x\in B(0,1/2)$ and δ_0 is radially symmetric (mollifier function). Set $\mu_i(x) := \delta_0((x-x_i)/(2r_i))$ for all $x\in\bar{\Omega}$ and note that

$$\operatorname{supp}\mu_i = \bar{B}_i\subset\tilde{S}, \qquad 1\leq\sum_{i=1}^I \mu_i(x)\leq b, \quad \forall\, x\in\bar{\Omega},$$

where b depends only on the regularity of $\mathcal{S}$; we certainly have $b < b_S = O(1)$ for all $S\in\mathcal{S}$. Define now $h\in C^\infty(\bar{\Omega})$ to be

$$h(x) := \frac{a}{b}\sum_{i=1}^I r_i\mu_i(x), \qquad \forall\, x\in\bar{\Omega}.$$

Since $\rho_{S'} < h_S$ for all $S'\in\mathcal{S}_S$, it is easily seen from (5.1) that, if $x_i\in S$, then

$$\frac{a^2}{b}r_i = \frac{a^3}{2b}h_S \leq \frac{a^2}{2b}\min_{S'\in\mathcal{S}_S}\rho_{S'}\leq h(x)\leq\frac{a}{2}\max_{S'\in\mathcal{S}_S}\rho_{S'} < \frac{a}{2}h_S = r_i, \qquad \forall\, x\in B_i. \tag{S1.2}$$

Hence $B(x,h(x))\subset\bar{B}_i(\subset\tilde{S})$ for all $x\in B_i$. Since

$$D^k h(x) = \frac{a}{b}\sum_{i=1}^I r_i D^k\mu_i(x) = C\sum_{i=1}^I r_i^{1-k}D^k\delta_0((x-x_i)/(2r_i)), \qquad \forall\, x\in\Omega,$$

the desired result easily follows from (S1.2). ∎

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Proof of Lemma 5.2. Via a standard scaling argument, it suffices to derive the following form of (5.7) over the master element, for $1 \le p < \infty$,

$$\|\Pi\zeta(V) - \hat{\Pi}\zeta(V)\|_{L^p(\hat{S})} + \|\nabla[\Pi\zeta(V) - \hat{\Pi}\zeta(V)]\|_{L^p(\hat{S})} \le C(\Phi_{\mathcal{E}_{\hat{S}}} + \Psi_{\mathcal{S}_{\hat{S}}}), \tag{S1.3}$$

where, for convenience, we have set

$$\Phi_{\mathcal{E}_{\hat{S}}} := L_\zeta \left(\sum_{e \in \mathcal{E}_{\hat{S}}} h_e^2 D_e^p \right)^{1/p}, \qquad \Psi_{\mathcal{S}_{\hat{S}}} := L_{\zeta'} \left(\sum_{S \in \mathcal{S}_{\hat{S}}} h_S^2 d_S^{2p} \right)^{1/p}.$$

The case $p = \infty$ is similar. The proof of (S1.3) will be split into several steps. Let $\rho \in C_0^\infty$ be defined by

$$\rho(x,y) := h(x)^{-2} \delta_0((x-y)h(x)^{-1}), \qquad \forall\, x \in \hat{S}, \quad y \in \mathbf{R}^2,$$

where both h and δ_0 come from Lemma 5.1. Let $\{B_\imath\}_{\imath=1}^I$ be the covering of $\hat{S}$ used also in Lemma 5.1. By virtue of (5.1) and (S1.2), we have $B(x, h(x)) \subset \bar{B}_\imath (\subset \tilde{S})$ for all $x \in B_\imath$. We now introduce a regularization $\zeta(V)_\rho$ of $\zeta(V)$. Let $\zeta(V)_\rho$ be the C^∞-function defined by

$$\zeta(V)_\rho(x) := \int_{\mathbf{R}^2} \zeta(V(y))\rho(x,y)dy = \int_{B(x,h(x))} \zeta(V(y))\rho(x,y)dy, \qquad \forall\, x \in \hat{S},$$

where V is extended as a continuous piecewise linear function outside of $\hat{S}$, say in an obvious manner, whenever $\hat{S}$ has one edge on $\partial\Omega$. We claim that

$$|D^2\zeta(V)_\rho(x)| \le CL_\zeta \sum_{e \in \mathcal{E}_{\bar{B}_\imath}} D_e + CL_{\zeta'} \sum_{S \in \mathcal{S}_{\bar{B}_\imath}} d_S^2 \qquad \forall\, x \in B_\imath.$$

To prove this, we shall need the following properties of function ρ: for all $x \in \hat{S}$,

$$\int_{\mathbf{R}^2} \rho(x,y)dy = 1, \quad \int_{\mathbf{R}^2} (x-y)\rho(x,y)dy = 0,$$

from which

$$\int_{\mathbf{R}^2} D_x^k \rho(x,y)dy = 0, \quad k \ge 1,$$

$$\int_{\mathbf{R}^2} (x-y)_\imath D_{x_\jmath} \rho(x,y)dy = -\delta_{\imath\jmath}, \quad \imath,\jmath = 1,2, \qquad \int_{\mathbf{R}^2} (x-y) D_x^k \rho(x,y)dy = 0, \quad k \ge 2.$$

Here, $\delta_{\imath\jmath}$ stands for the Kronecker symbol. Moreover, in view of (S1.1), the following inequalities hold for all $x \in S (\in \mathcal{S}_{\hat{S}})$ as well:

$$\|\nabla_x \rho(x,\cdot)\|_{L^1(\mathbf{R}^2)} \le Ch(x)^{-1} \le Ch_S^{-1}, \quad \|D_x^2\rho(x,\cdot)\|_{L^1(\mathbf{R}^2)} \le Ch(x)^{-2} \le Ch_S^{-2}.$$

On making use of these properties of ρ, we easily get for all $x \in B_\imath$

$$\begin{aligned} D^2\zeta(V)_\rho(x) &= \int_{\mathbf{R}^2} \zeta(V(y)) D_x^2 \rho(x,y)dy \\ &= \int_{B(x,h(x))} [\zeta(V(y)) - \zeta(V(x)) + \nabla\zeta(V(x)) \cdot (x-y)] D_x^2\rho(x,y)dy \\ &= \int_{B(x,h(x))} I(y) D_x^2 \rho(x,y)dy, \end{aligned}$$

where

$$I(y) := \left(\int_0^1 \zeta'(V(sy + (1-s)x)\nabla V(sy+(1-s)x)ds - \zeta'(V(x))\nabla V(x) \right) \cdot (y-x).$$

But

$$\begin{aligned} I(y) = (y-x) \cdot \Bigg(&\int_0^1 [\zeta'(V(sy+(1-s)x) - \zeta'(V(x))]\nabla V(sy+(1-s)x)ds \\ &+ \zeta'(V(x)) \int_0^1 [\nabla V(sy+(1-s)x) - \nabla V(x)]ds \Bigg), \end{aligned}$$

whence

$$|I(y)| \le CL_\zeta \sum_{e \in \mathcal{E}_{B(x,f(x))}} h_e\delta_e + CL_{\zeta'} \sum_{S \in \mathcal{S}_{B(x,h(x))}} h_S^2 d_S^2.$$

Consequently, for all $x \in B_\imath$, we obtain

$$\begin{aligned} |D^2\zeta(V)_\rho(x)| &\le C \int_{\mathbf{R}^2} |D_x^2\rho(x,y)|dy \left(L_\zeta \sum_{e \in \mathcal{E}_{B(x,h(x))}} h_e\delta_e + L_{\zeta'} \sum_{S \in \mathcal{S}_{B(x,h(x))}} h_S^2 d_S^2 \right) \\ &\le CL_\zeta \sum_{e \in \mathcal{E}_{B(x,h(x))}} D_e + CL_{\zeta'} \sum_{S \in \mathcal{S}_{B(x,h(x))}} d_S^2 \le CL_\zeta \sum_{e \in \mathcal{E}_{\bar{B}_\imath}} D_e + CL_{\zeta'} \sum_{S \in \mathcal{S}_{\bar{B}_\imath}} d_S^2. \end{aligned}$$

In view of (5.3) and (S1.2) we have card $\mathcal{E}_{\bar{B}_\imath}$, card $\mathcal{S}_{\bar{B}_\imath} = O(1)$ which, in turn, implies the pointwise estimate $|D^2\zeta(V)_\rho(x)| \le CL_\zeta(\sum_{e \in \mathcal{E}_{\bar{B}_\imath}} D_e^p)^{1/p} + CL_{\zeta'}(\sum_{S \in \mathcal{S}_{\bar{B}_\imath}} d_S^{2p})^{1/p}$ for all $x \in B_\imath$, as results from Hölder's inequality. This inequality will serve to derive an L^p-bound. By virtue of the nonoverlapping property of $\{\bar{B}_\imath\}_{\imath=1}^I$, we have

$$\begin{aligned} \|D^2\zeta(V)_\rho\|_{L^p(\hat{S})} &\le \left(\sum_{\imath=1}^I \int_{B_\imath} |D^2\zeta(V)_\rho(x)|^p dx \right)^{1/p} \\ &\le C \left(\sum_{\imath=1}^I \left[L_\zeta^p \sum_{e \in \mathcal{E}_{\bar{B}_\imath}} h_e^2 D_e^p + L_{\zeta'}^p \sum_{S \in \mathcal{S}_{\bar{B}_\imath}} h_S^2 d_S^{2p} \right] \right)^{1/p} \le C(\Phi_{\mathcal{E}_{\hat{S}}} + \Psi_{\mathcal{S}_{\hat{S}}}), \end{aligned}$$

which is the desired bound. Standard interpolation theory [2, p. 115] then yields

$$\text{(S1.4)}\qquad \begin{aligned}\|\zeta(V)_\rho - \hat\Pi\zeta(V)_\rho\|_{L^p(\hat S)} &+ \|\nabla(\zeta(V)_\rho - \hat\Pi\zeta(V)_\rho)\|_{L^p(\hat S)} \\ &\le C\|D^2\zeta(V)_\rho\|_{L^p(\hat S)} \le C(\Phi_{\mathcal{E}_{\hat{\hat S}}} + \Psi_{\mathcal{S}_{\hat{\hat S}}}).\end{aligned}$$

At the same time, we can compare $\zeta(V)$ and $\zeta(V)_\rho$ as follows:

$$\nabla(\zeta(V) - \zeta(V)_\rho)(x) = \int_{B(x,h(x))} [\zeta(V(x)) - \zeta(V(y)) - \nabla\zeta(V(x))\cdot(x-y)]\nabla_x\rho(x,y)dy.$$

Hence, for all $x \in B_\iota$ we obtain

$$\begin{aligned}|\nabla(\zeta(V) - \zeta(V)_\rho)(x)| &\le C\int_{\mathbf{R}^2} |\nabla_x\rho(x,y)|dy \left(L_\zeta \sum_{e\in\mathcal{E}_{B(x,h(x))}} h_e\delta_e + L_{\zeta'} \sum_{S\in\mathcal{S}_{B(x,h(x))}} h_S^2 d_S^2 \right) \\ &\le CL_\zeta \sum_{e\in\mathcal{E}_{B_\iota}} h_e D_e + CL_{\zeta'} \sum_{S\in\mathcal{S}_{B_\iota}} h_S d_S^2.\end{aligned}$$

Arguing as before, we deduce that

$$\text{(S1.5)}\qquad \|\nabla(\zeta(V) - \zeta(V)_\rho)\|_{L^p(\hat S)} \le C(\Phi_{\mathcal{E}_{\hat{\hat S}}} + \Psi_{\mathcal{S}_{\hat{\hat S}}}).$$

A similar calculation for $(\zeta(V) - \zeta(V)_\rho)(x)$, valid for all $x \in B_\iota$, leads to

$$\begin{aligned}|(\zeta(V) - \zeta(V)_\rho)(x)| &= \left| \int_{B(x,h(x))} [\zeta(V(x)) - \zeta(V(y)) - \nabla\zeta(V(x))\cdot(x-y)]\rho(x,y)dy \right| \\ &\le CL_\zeta \sum_{e\in\mathcal{E}_{B_\iota}} h_e^2 D_e + CL_{\zeta'} \sum_{S\in\mathcal{S}_{B_\iota}} h_S^2 d_S^2,\end{aligned}$$

and, as a consequence, to

$$\text{(S1.6)}\qquad \|\zeta(V) - \zeta(V)_\rho\|_{L^p(\hat S)} \le C(\Phi_{\mathcal{E}_{\hat{\hat S}}} + \Psi_{\mathcal{S}_{\hat{\hat S}}}).$$

Moreover, since $\hat\Pi[\zeta(V) - \zeta(V)_\rho]$ is linear in $\hat S$, we see that

$$\|\hat\Pi[\zeta(V) - \zeta(V)_\rho]\|_{W^{1,p}(\hat S)} \le C\|\hat\Pi[\zeta(V) - \zeta(V)_\rho]\|_{L^\infty(\hat S)} \le C\|\zeta(V) - \zeta(V)_\rho\|_{L^\infty(\hat S)},$$

whence

$$\text{(S1.7)}\qquad \|\hat\Pi[\zeta(V) - \zeta(V)_\rho]\|_{W^{1,p}(\hat S)} \le C(\Phi_{\mathcal{E}_{\hat{\hat S}}} + \Psi_{\mathcal{S}_{\hat{\hat S}}}),$$

as results from the previous pointwise bound for $\zeta(V) - \zeta(V)_\rho$ and Hölder's inequality. To complete the argument, we make use again of standard interpolation estimates [2, p. 115] as follows:

$$\text{(S1.8)}\qquad \begin{aligned}\|\zeta(V) - \Pi\zeta(V)\|_{W^{1,p}(\hat S)} &\le C\left(\sum_{S\in\mathcal{S}_{\hat{\hat S}}} h_S^p \|D^2\zeta(V)\|_{L^p(S)}^p \right)^{1/p} \\ &\le CL_{\zeta'}\left(\sum_{S\in\mathcal{S}_{\hat{\hat S}}} h_S^{2+p} d_S^{2p} \right)^{1/p} \le C\Psi_{\mathcal{S}_{\hat{\hat S}}}.\end{aligned}$$

Combining the inequalities (S1.4)-(S1.8), we finally obtain the desired estimate (S1.3). ▯

Proof of Lemma 5.3. In view of (5.6) it is enough to prove the above estimates for $p = \infty$. We first examine $\zeta(V) - \Pi\zeta(V)$. Standard interpolation estimates yield

$$\|\zeta(V) - \Pi\zeta(V)\|_{L^\infty(\hat S)} \le \|\zeta(V) - \Pi\zeta(V)\|_{L^\infty(\hat{\hat S})} \le CL_{\zeta'} \max_{S\in\mathcal{S}_{\hat{\hat S}}} (h_S d_S)^2,$$

as well as

$$\|\nabla[\zeta(V) - \Pi\zeta(V)]\|_{L^\infty(\hat S)} \le \|\nabla[\zeta(V) - \Pi\zeta(V)]\|_{L^\infty(\hat{\hat S})} \le CL_{\zeta'} \max_{S\in\mathcal{S}_{\hat{\hat S}}} (h_S d_S^2).$$

We next deal with $\zeta(V) - \hat\Pi\zeta(V)$ and assume, without loss of generality, that $\hat S$ is the master triangle. A standard scaling argument will produce estimates (5.8) and (5.9) as soon as we have

$$\|\nabla[\zeta(V) - \hat\Pi\zeta(V)]\|_{L^\infty(\hat S)} \le CL_\zeta \max_{e\in\mathcal{E}_{\hat{\hat S}}} \delta_e + CL_{\zeta'} \max_{S\in\mathcal{S}_{\hat{\hat S}}} (h_S d_S^2).$$

Indeed, since $\zeta(V) - \hat\Pi\zeta(V)$ vanishes at the vertices of $\hat S$, we infer that

$$\|\zeta(V) - \hat\Pi\zeta(V)\|_{L^\infty(\hat S)} \le C\|\nabla[\zeta(V) - \hat\Pi\zeta(V)]\|_{L^\infty(\hat S)},$$

which, coupled with the fact that $h_S > h_{\hat S} = O(1)$ for all $S \in \mathcal{S}_{\hat S}$ as results from (5.1) and (5.5), leads to (5.8). Moreover, since

$$\min_{\hat S} \zeta(V)_x (\text{ resp. } \zeta(V)_y) \le \hat\Pi\zeta(V)_x (\text{ resp. } \hat\Pi\zeta(V)_y) \le \max_{\hat S} \zeta(V)_x (\text{ resp. } \zeta(V)_y),$$

we easily see that

$$\|\nabla[\zeta(V) - \hat\Pi\zeta(V)]\|_{L^\infty(\hat S)} \le \max_{x,y\in\hat S} |\nabla[\zeta(V(x)) - \zeta(V(y))]| =: \max_{x,y\in\hat S} I(x,y).$$

But

$$I(x,y) \le |\zeta'(V(y))\nabla[V(x)-V(y)]| + |[\zeta'(V(x))-\zeta'(V(y))]\nabla V(x)|$$
$$\le L_\zeta \sum_{e\in\mathcal{E}_{\hat{S}}} \delta_e + L_{\zeta'} \sum_{S\in\mathcal{S}_{\hat{S}}} (h_S d_S^2) \le CL_\zeta \max_{e\in\mathcal{E}_{\hat{S}}} \delta_e + CL_{\zeta'} \max_{S\in\mathcal{S}_{\hat{S}}} (h_S d_S^2),$$

because of (5.6). This provides the desired bound and completes the proof of the lemma. □

PROOF OF LEMMA 5.4. In view of (2.1), (2.2) and (2.12), we infer that

$$\|u_0 - U^0\|_{L^\infty(S)} \le C\tau, \qquad \forall\, S\in\mathcal{S}^0(=\mathcal{S}^1): \quad S\cap I_0 = \emptyset.$$

Since u_0 is discontinuous across I_0, the best we can say is $\|u_0 - U^0\|_{L^\infty(\mathcal{F}^0)} \le C$. The fact that I_0 has finite length leads now to

$$\|u_0 - U^0\|_{L^p(\mathcal{F}^0)} \le C\tau^{1/p}, \qquad \forall\, 1\le p<\infty.$$

Let $\varphi\in H_0^1(\Omega)$ be given. Then, using the Hölder inequality combined with a well-known 2-D Sobolev inequality, we get

$$|\langle u_0 - U^0, \varphi\rangle| \le \|u_0 - U^0\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)} \le Cq^{1/2}\tau^{1-1/q}\|\varphi\|_{H_0^1(\Omega)}.$$

The asserted estimate follows from taking $q = |\log\tau|$. □

We now demonstrate auxiliary trace estimates for the interpolation error $\Theta-\hat{\Theta}$. Let $\hat{\mathcal{E}}_0 = \{\hat{e}\in\hat{\mathcal{E}} : A_{\hat{e}} \ne \emptyset\}$, where $A_{\hat{e}} := (\hat{S}_1\cup\hat{S}_2)\cap\Omega_0$ and $\{\hat{S}_1,\hat{S}_2\} = \mathcal{A}_{\hat{e}}$; note that $\hat{\mathcal{E}}_0 \supset \hat{\mathcal{E}}\backslash(\mathcal{E}_F\cup\mathcal{E}_B)$. Let $\hat{\mathcal{E}}_2\subset\hat{\mathcal{E}}_0$ be the set of all those $\hat{e}$'s for which $\mathcal{A}_{\hat{e}}\cap\hat{\mathcal{S}}_2\ne\emptyset$; set $\hat{\mathcal{E}}_1 := \hat{\mathcal{E}}_0\backslash\hat{\mathcal{E}}_2$.

LEMMA 5.7. *Let $\hat{e}\in\hat{\mathcal{E}}_1$ Then*

$$\|\nabla(\Theta-\hat{\Theta})\|_{L^1(\hat{e})} \le C\sum_{e\in\mathcal{E}_{A_{\hat{e}}}} h_e^2 D_e + C\sum_{S\in\mathcal{S}_{A_{\hat{e}}}} (h_S d_S)^2. \tag{S1.9}$$

Proof. Let $\zeta=\beta$, $V=U$ and $p=1$ in Lemma 5.2. Let $\hat{S}\in\mathcal{A}_{\hat{e}}\cap\hat{\mathcal{S}}_0$. Recall the notation used in Lemma 5.2 and, in addition, the pointwise estimates

$$|D^2\beta(U)_\rho(x)| \le C\sum_{e\in\mathcal{E}_{B_\iota}} D_e + C\sum_{S\in\mathcal{S}_{B_\iota}} d_S^2,$$

and

$$|\nabla[\beta(U)-\beta(U)_\rho](x)| \le C\sum_{e\in\mathcal{E}_{B_\iota}} h_e D_e + C\sum_{S\in\mathcal{S}_{B_\iota}} h_S d_S^2,$$

for all $x\in B_\iota$. Hence,

$$\|\nabla[\beta(U)-\beta(U)_\rho]\|_{L^1(\hat{e})} \le C\sum_{\iota=1}^{I}\int_{e\cap B_\iota} |\nabla[\beta(U)-\beta(U)_\rho](x)|d\sigma(x)$$
$$\le C\sum_{e\in\mathcal{E}_{\hat{S}}} h_e^2 D_e + C\sum_{S\in\mathcal{S}_{\hat{S}}} h_S^2 d_S^2.$$

A well-known trace result can now be applied to arrive at

$$\|\nabla[\beta(U)_\rho - \hat{\Pi}\beta(U)_\rho]\|_{L^1(\hat{e})} \le Ch_{\hat{e}}^{-1}\|\nabla[\beta(U)_\rho - \hat{\Pi}\beta(U)_\rho]\|_{L^1(\hat{S})} + C\|D^2\beta(U)_\rho\|_{L^1(\hat{S})}$$
$$\le C\sum_{e\in\mathcal{E}_{\hat{S}}} h_e^2 D_e + C\sum_{S\in\mathcal{S}_{\hat{S}}} h_S^2 d_S^2.$$

At the same time, a local inverse inequality combined with (S1.7) leads to

$$\|\nabla[\hat{\Pi}\beta(U) - \hat{\Pi}\beta(U)_\rho]\|_{L^1(\hat{e})} \le Ch_{\hat{e}}^{-1}\|\nabla[\hat{\Pi}\beta(U) - \hat{\Pi}\beta(U)_\rho]\|_{L^1(\hat{S})}$$
$$\le C\sum_{e\in\mathcal{E}_{\hat{S}}} h_e^2 D_e + C\sum_{S\in\mathcal{S}_{\hat{S}}} h_S^2 d_S^2.$$

Finally, a standard L^∞-interpolation estimate yields

$$\|\nabla[\beta(U)-\Pi\beta(U)]\|_{L^1(e)} = \sum_{S\in\mathcal{S}_{\hat{e}}}\int_{\hat{e}\cap S} |\nabla[\beta(U)-\Pi\beta(U)](x)|d\sigma(x)$$
$$\le C\sum_{S\in\mathcal{S}_{\hat{e}}} h_S^2\|D^2\beta(U)\|_{L^\infty(S)} \le C\sum_{S\in\mathcal{S}_{\hat{S}}} h_S^2 d_S^2,$$

and concludes the proof of the lemma. □

LEMMA 5.8. *Let $\hat{e}\in\hat{\mathcal{E}}_2$. Then*

$$\|\nabla(\Theta-\hat{\Theta})\|_{L^1(\hat{e})} \le Ch_{\hat{e}}\left(\sum_{e\in\mathcal{E}_{A_{\hat{e}}}}\delta_e + \sum_{S\in\mathcal{S}_{A_{\hat{e}}}} h_S d_S^2\right). \tag{S1.10}$$

Proof. Let $\zeta=\beta$, $V=U$ and $p=\infty$ in Lemma 5.3. Let $\hat{S}\in\mathcal{A}_{\hat{e}}\cap\hat{\mathcal{S}}_0$. On using (4.10) in conjunction with (5.6), we readily get

$$\|\nabla(\Theta-\hat{\Theta})\|_{L^1(\hat{e})} \le Ch_{\hat{e}}\|\nabla(\Theta-\hat{\Theta})\|_{L^\infty(\hat{S})} \le Ch_{\hat{e}}\left(\sum_{e\in\mathcal{E}_{\hat{S}}}\delta_e + \sum_{S\in\mathcal{S}_{\hat{S}}} h_S d_S^2\right),$$

which yields the assertion. □

PROOF OF LEMMA 5.9. A bound on the first term follows from $\|\nabla\Theta^0\|_{L^\infty(\Omega)} \le \|\nabla\theta_0\|_{L^\infty(\Omega)}$ and (2.2). For the second term, we split $\mathcal{E}^1$ into the set $\mathcal{E}_F^1$ of those e's such that $A_e \cap I_0 \neq \emptyset$ and the complement, where $A_e := S_1 \cup S_2$, $\{S_1, S_2\} = \mathcal{A}_e$. We use again (2.2) to get $\delta_e \le C$ for all $e \in \mathcal{E}_F^1$, which, in view of (2.3), leads to

$$\sum_{e\in\mathcal{E}_F^1} h_e^2 D_e = \sum_{e\in\mathcal{E}_F^1} h_e\delta_e \le C \sum_{e\in\mathcal{E}_F^1} h_e \le C \text{ length}(I_0) \le C,$$

because I_0 has finite length. On the other hand, for all $e \notin \mathcal{E}_F^1$, we have

$$D_e = h_e^{-1}\delta_e \le h_e^{-1}\| \text{ osc } \nabla\theta_0\|_{L^\infty(A_e)} \le \|D^2\theta_0\|_{L^\infty(A_e)} \le C,$$

where "osc" stands for oscillation. This concludes the argument. □

PROOF OF LEMMA 5.10. Since $\hat\Theta|_{\hat S} = \Theta|_{\hat S}$ for all $\hat S \in (\mathcal{S}_B \cup \mathcal{S}_F)$, we only have to consider $\hat e \in \hat{\mathcal{E}}_0$. Set $\hat{\mathcal{E}}_2^1 := \{\hat e \in \hat{\mathcal{E}}_2 : A_{\hat e} \subset S, \text{ for some } S \in \mathcal{S}_0\}$ and $\hat{\mathcal{E}}_2^2 := \hat{\mathcal{E}}_2 \backslash \hat{\mathcal{E}}_2^1$. Let first $\hat e \in \hat{\mathcal{E}}_2^1$. It is easily seen that $D_{\hat e} \le C L_{\beta'} D_{S_{\hat e}}^2 \le C l_\beta^{-2} L_{\beta'} d_{S_{\hat e}}^2$, where $S_{\hat e} \supset A_{\hat e}$, and so

$$\sum_{\hat e\in\hat{\mathcal{E}}_2^1} h_{\hat e}^2 D_{\hat e} \le C \sum_{\hat e\in\hat{\mathcal{E}}_2^1} h_{\hat e}^2 d_{S_{\hat e}}^2 \le C \sum_{S\in\mathcal{S}_0} d_S^2 \sum_{\hat e\in\hat{\mathcal{E}}_S^2} h_{\hat e}^2 \le C \sum_{S\in\mathcal{S}_0} h_S^2 d_S^2,$$

where $\hat{\mathcal{E}}_S^2 := \{\hat e \in \hat{\mathcal{E}}_2^1 : A_{\hat e} \subset S\}$. For the two remaining cases, namely $\hat{\mathcal{E}}_2^2$ and $\hat{\mathcal{E}}_1$, we can obviously replace $D_{\hat e}$ by $h_{\hat e}^{-2}(\|\nabla(\hat\Theta|_{\hat S_1} - \Theta|_{\hat S_1})\|_{L^1(\hat e)} + \|\nabla(\hat\Theta|_{\hat S_2} - \Theta|_{\hat S_2})\|_{L^1(\hat e)})$. Let next $\hat e \in \hat{\mathcal{E}}_2^2$. Then, Lemma 5.8 implies

$$\begin{aligned}\sum_{\hat e\in\hat{\mathcal{E}}_2^2} h_{\hat e}^2 D_{\hat e} &\le C \sum_{\hat e\in\hat{\mathcal{E}}_2^2} h_{\hat e}\left(\sum_{e\in\mathcal{E}_{A_{\hat e}}} h_e D_e + \sum_{S\in\mathcal{S}_{A_{\hat e}}} h_S d_S^2\right)\\ &\le C\sum_{e\in\mathcal{E}} h_e D_e \sum_{\hat e\in\hat{\mathcal{E}}_e^2} h_{\hat e} + C\sum_{S\in\mathcal{S}} h_S d_S^2 \sum_{\hat e\in\hat{\mathcal{E}}_S^2} h_{\hat e} \le C\sum_{e\in\mathcal{E}} h_e^2 D_e + C\sum_{S\in\mathcal{S}} h_S^2 d_S^2,\end{aligned}$$

where $\hat{\mathcal{E}}_e^2 := \{\hat e \in \hat{\mathcal{E}}_2^2 : e \cap \hat e \neq \emptyset\}$ and $\hat{\mathcal{E}}_S^2 := \{\hat e \in \hat{\mathcal{E}}_2^2 : A_{\hat e} \cap \partial S \neq \emptyset\}$. Here we have argued as at the end of Lemma 5.6 to conclude that $\sum_{\hat e\in\hat{\mathcal{E}}_e^2} h_{\hat e} \le C h_e$ as well as $\sum_{\hat e\in\hat{\mathcal{E}}_S^2} h_{\hat e} \le C h_S$. Suppose now $\hat e \in \hat{\mathcal{E}}_1$. Lemma 5.7 finally yields

$$\sum_{\hat e\in\hat{\mathcal{E}}_1} h_{\hat e}^2 D_{\hat e} \le C \sum_{\hat e\in\hat{\mathcal{E}}_1}\left(\sum_{e\in\mathcal{E}_{A_{\hat e}}} h_e^2 D_e + \sum_{S\in\mathcal{S}_{A_{\hat e}}} h_S^2 d_S^2\right) \le C\sum_{e\in\mathcal{E}} h_e^2 D_e + C\sum_{S\in\mathcal{S}} h_S^2 d_S^2. \quad \square$$

PROOF OF LEMMA 5.11. Since $\hat\Theta|_S = \Theta|_{\hat S}$ for all $\hat S \in (\mathcal{S}_B \cup \mathcal{S}_F)$, we only have to prove the estimate in Ω_0. We obviously have

$$\int_{\Omega_0} |\nabla\hat\Theta|^2 = \int_{\Omega_0} |\nabla\Theta|^2 + \int_{\Omega_0} |\nabla(\hat\Theta - \Theta)|^2 + 2\int_{\Omega_0} \nabla(\hat\Theta - \Theta)\cdot\nabla\Theta.$$

Estimate (5.10), with $\zeta = \beta$, and (2.18), together with the fact that $\hat\Theta = \Theta$ for all $e \in \mathcal{E}_F \cup \mathcal{E}_B$, lead to

$$\begin{aligned}\int_{\Omega_0} \nabla(\hat\Theta - \Theta)\cdot\nabla\Theta &= \sum_{e\in\mathcal{E}_0} \langle\!\langle \hat\Theta - \Theta, [\![\nabla\Theta]\!]_e \cdot \nu_e \rangle\!\rangle_e\\ &\le C\|\hat\Theta - \Theta\|_{L^\infty(\Omega)} \sum_{e\in\mathcal{E}_0} h_e^2 D_e \le C\tau \sum_{e\in\mathcal{E}_0} h_e^2 D_e.\end{aligned}$$

In view of (5.11), with $\zeta = \beta$, and inequality $\sigma^2 \sum_{S\in\mathcal{S}} h_S^2 d_S^2 \le \|\nabla\Theta\|_{L^2(\Omega)}^2$, the desired estimate follows immediately. □

S2. Supplement to Section 6.

PROOF OF LEMMA 6.1. Let $C_0 := \|U^0\|_{L^\infty(\Omega)}$ and $f_0 := |f(0)|$. In view of (2.1) and (2.13) it is enough to prove that $\|U^n\|_{L^\infty(\Omega)} \le C_n$, where

$$C_n := C_0 \exp(n\tau L_\beta L_f) + \frac{f_0}{L_\beta L_f}\left(\exp(n\tau L_\beta L_f) - 1\right). \tag{S2.1}$$

Since (S2.1) is obviously valid for $n = 0$, assume, by induction, that it holds for $n-1$ as well. Let $x \in \Omega$ be a point at which U^n attains its maximum. Since U^n is piecewise linear, x is a node of $\mathcal{S}^n$ (say $x = x_j^n$). Function Θ^n attains also its maximum at x_j^n because of (2.13) and the monotonicity of β. Then, by virtue of (6.1) we can write

$$\begin{aligned}U^n(x_j^n)\int_\Omega \chi_j^n = \langle U^n, \chi_j^n\rangle^n &\le \langle \hat U^{n-1}, \chi_j^n\rangle^n + \tau\langle f(\hat\Theta^{n-1}), \chi_j^n\rangle^n\\ &= \left(\hat U^{n-1}(x_j^n) + \tau f(\hat\Theta^{n-1}(x_j^n))\right)\int_\Omega \chi_j^n,\end{aligned}$$

thus

$$U^n(x_j^n) \le \hat U^{n-1}(x_j^n) + \tau f(\hat\Theta^{n-1}(x_j^n)).$$

On using the induction assumption, together with the inequalities

$$\|\hat U^{n-1}\|_{L^\infty(\Omega)} \le \|U^{n-1}\|_{L^\infty(\Omega)}, \quad \|\hat\Theta^{n-1}\|_{L^\infty(\Omega)} \le L_\beta\|U^{n-1}\|_{L^\infty(\Omega)},$$

we see that

$$U^n(x_j^n) \le C_{n-1}(1 + \tau L_\beta L_f) + \tau f_0.$$

The fact that $1 + \tau L_\beta L_f \le \exp(\tau L_\beta L_f)$ results in $U^n(x_j^n) \le C_n$, where C_n is defined in (S2.1). This completes the induction argument. □

PROOF OF LEMMA 6.2. Since no confusion is possible, we remove the superscript n and simply use the following notation: $\mathcal{S} := \mathcal{S}^n$, $\Pi := \Pi^n$ and $k_{ij} := \int_\Omega \nabla\chi_i^n \cdot \nabla\chi_j^n$,

$\varphi_i := \varphi(x_i^n)$, $\alpha_i := \alpha(\varphi_i)$, for $1 \le i, j \le J := J^n$. The fact that $\mathcal{S}$ is weakly acute is reflected in

$$k_{ii} > 0, \quad k_{ij} \le 0 \quad (j \ne i), \quad \sum_{j=1}^{J} k_{ij} \ge 0, \qquad \forall\, 1 \le i \le J;$$

so the stiffness matrix $\mathbf{K} = (k_{ij})_{i,j=1}^{J}$ is an M-matrix. Hence,

$$\langle \nabla\varphi, \nabla\Pi\alpha(\varphi)\rangle = \sum_{i,j=1}^{J} \alpha_i \varphi_j k_{ij} = \sum_{i=1}^{J}\left(\sum_{j\ne i} \alpha_i\varphi_j k_{ij} + \alpha_i\varphi_i k_{ii}\right)$$
$$= \sum_{i=1}^{J}\left(\sum_{j\ne i} \alpha_i(\varphi_j - \varphi_i)k_{ij} + \alpha_i\varphi_i(k_{ii} + \sum_{j\ne i} k_{ij})\right).$$

Using the symmetry of $\mathbf{K}$, i.e. $k_{ij} = k_{ji}$, we observe that

$$\sum_{i=1}^{J}\sum_{j\ne i} \alpha_i(\varphi_j - \varphi_i)k_{ij} = \sum_{j=1}^{J}\sum_{i\ne j} \alpha_j(\varphi_i - \varphi_j)k_{ij},$$

and, as a consequence, that

$$L_\alpha \langle \nabla\varphi, \nabla\Pi\alpha(\varphi)\rangle = L_\alpha \sum_{i=1}^{J}\left(\sum_{j\ne i} \frac{1}{2}(\alpha_i - \alpha_j)(\varphi_j - \varphi_i)k_{ij} + \alpha_i\varphi_i(k_{ii} + \sum_{j\ne i} k_{ij})\right)$$
$$\ge \sum_{i=1}^{J}\left(\sum_{j\ne i} -\frac{1}{2}(\alpha_i - \alpha_j)^2 k_{ij} + \alpha_i^2(k_{ii} + \sum_{j\ne i} k_{ij})\right) = \sum_{i,j=1}^{J} \alpha_i\alpha_j k_{ij} = \|\nabla\Pi\alpha(\varphi)\|^2_{L^2(\Omega)},$$

because $\alpha(0) = 0$. The proof is thus complete. □

S3. Pointwise Error Estimates. Our goal now is to prove Lemma 7.1. Our approach is based on the method of local H^1-error estimates [21,22,23]. The crucial step here is a new version of the *energy* (or *Caccioppoli*) *estimate* of Lemma S3.1 which accounts for the lack of quasi-uniformity and improves upon the argument introduced in [21]. The only assumption on the triangulation, besides regularity, is $h_n \le C\rho_n^\gamma$ with $0 < \gamma \le 1$.

Let us start by adding some useful notation. Set $h := h_n, \rho := \rho_n, v := G\psi, v_h := G^n\psi$ and drop the super(sub)script n in the various symbols already used. In this spirit, $\{\chi_j\}_{j=1}^J$ and $\{x_j\}_{j=1}^J$ stand for the basis functions and nodes of $\mathcal{S}$, respectively, whereas $\mathbf{V}$ denotes the finite element space. Let B_r indicate the ball of radius $r \ge \Gamma h$ ($\Gamma > 1$ to be selected) and center $x_0 \in \Omega$. Set $\tilde r := r|\log h|$. Let ω denote a cutoff function with the following properties:

(S3.1) $\quad \omega = 1 \quad \text{in } \tilde B_{kr}, \quad \omega = 0 \quad \text{in } \Omega\backslash \underset{\sim}{B}_{(k+1)r}, \quad |D^i\omega| \le Cr^i \quad \text{in } A_k := \tilde B_{kr}\backslash \underset{\sim}{B}_{(k+1)r},$

where $\tilde W := \cup\{S \in \mathcal{S} : S \cap W \ne \emptyset\}$ and $\underset{\sim}{W} := \text{Interior}\,(\cup\{S \in \mathcal{S} : S \subset W\})$.

Lemma S3.1. *Let $\varphi \in \mathbf{V}$ be discrete harmonic in $B_{\tilde r}$, that is,*

(S3.2) $\qquad \langle \nabla\varphi, \nabla\chi_j\rangle = 0, \qquad$ *for all* $1 \le j \le J$ *such that* $x_j \in B_{\tilde r}$.

Then there exists $\Gamma > 1$ such that for $r \ge \Gamma h$ the following energy estimate holds:

(S3.3) $$\|\nabla\varphi\|_{L^2(B_r)} \le Cr^{-1}\|\varphi\|_{L^2(B_{\tilde r})}.$$

Proof. It is easily seen, from (S3.1) and (S3.2), that

$$\|\nabla\varphi\|^2_{L^2(B_{kr})} \le 2^{-1}\langle\varphi^2, \Delta\omega\rangle + \langle\nabla\varphi, \nabla(\omega^2\varphi - \chi)\rangle =: I + II, \qquad \forall\, \chi \in \mathbf{V}.$$

In view of (S3.1), we have $I \le Cr^{-2}\|\varphi\|^2_{L^2(A_k)}$. On the other hand, taking $\chi = \Pi(\omega^2\varphi) \in \mathbf{V}$ and using (S3.1) again yields

$$II \le C \sum_{S\in\mathcal{S}_{A_k}} h_S\|\nabla\varphi\|_{L^2(S)}\|D^2(\omega\varphi)\|_{L^2(S)}$$
$$\le C \sum_{S\in\mathcal{S}_{A_k}} h_S\|\nabla\varphi\|_{L^2(S)}\left(r^{-2}\|\varphi\|_{L^2(S)} + r^{-1}\|\nabla\varphi\|_{L^2(S)}\right)$$
$$\le Chr^{-1}\|\nabla\varphi\|^2_{L^2(A_k)} + Cr^{-2}\|\varphi\|^2_{L^2(A_k)}.$$

Hence,

$$\|\nabla\varphi\|^2_{L^2(B_{kr})} \le Chr^{-1}\|\nabla\varphi\|^2_{L^2(A_k)} + Cr^{-2}\|\varphi\|^2_{L^2(A_k)}, \qquad k \ge 1.$$

We now apply this inequality iteratively for $1 \le k \le K$ to arrive at

$$\|\nabla\varphi\|_{L^2(B_r)} \le (Chr^{-1})^{K/2}\|\nabla\varphi\|_{L^2(\underset{\sim}{B}_{Kr})} + Cr^{-1}\|\varphi\|_{L^2(\underset{\sim}{B}_{Kr})}.$$

Since there is no a priori structure of $\mathcal{S}$, triangles in B_{Kr} may have very disparate sizes. The best we can say then is

$$\|\nabla\varphi\|_{L^2(\underset{\sim}{B}_{Kr})} \le C\rho^{-1}\|\varphi\|_{L^2(B_{Kr})} \le Ch^{-1/\gamma}\|\varphi\|_{L^2(B_{Kr})},$$

as results from a standard local inverse inequality [2, p. 140]. Let Γ satisfy $\log(C\Gamma) \ge 2(1-\gamma)/\gamma$, provided $\gamma < 1$. Then, for $K = |\log h|$, we have

$$(Chr^{-1})^{K/2}h^{-1/\gamma} \le \Gamma r^{-1},$$

which in turn implies the assertion. In case $\gamma = 1$, two iterations suffice, namely $K = 2$. This concludes the proof. □

REMARK S3.1. It is worth noting the contrast between a quasi-uniform mesh (i.e., $\gamma = 1$) and the general situation for which $\tilde r/r = |\log h|$. This fact causes the extra powers of the logarithm in Lemma 7.1.

We are now in a position to derive our basic tool, namely, the *local H^1-error estimates*.

LEMMA S3.2. $\|v - v_h\|_{H^1(B_r)} \le C|\log h| \|\nabla(v - \Pi v)\|_{L^2(B_{2\tilde r})} + Cr^{-1}\|v - v_h\|_{L^2(B_{2\tilde r})}$.

Proof. Let ω be the cutoff function defined in (S3.1), this time subordinate to $B_{\tilde r}$. Set $\tilde v := \omega v$ and define $\tilde v_h \in \mathbf{V}$ to be the (local) H^1-projection of $\tilde v$ onto $\mathbf{V}$, namely,

$$\langle \nabla(\tilde v_h - \tilde v), \nabla \chi_j\rangle = 0, \quad \forall\ x_j \in \underset{\sim}{B}_{2\tilde r} \quad (1 \le j \le J); \qquad \tilde v_h(x) = 0, \qquad \forall\ x \in \Omega \backslash \underset{\sim}{B}_{2\tilde r}.$$

Then $v_h - \tilde v_h$ is discrete harmonic in $\underset{\sim}{B}_{2\tilde r}\ (\supset B_r)$. Hence, (S3.3) yields

$$\|\nabla(v_h - \tilde v_h)\|_{L^2(B_r)} \le Cr^{-1}\|v_h - \tilde v_h\|_{L^2(B_{\tilde r})} \le Cr^{-1}\left(\|v_h - v\|_{L^2(B_{\tilde r})} + \|\tilde v - \tilde v_h\|_{L^2(B_{\tilde r})}\right),$$

because $\tilde v = v$ in $B_{\tilde r}$. Since $\tilde v - \tilde v_h = 0$ on $\partial B_{2\tilde r}$, the Poincaré inequality implies

$$\begin{aligned}\|\tilde v - \tilde v_h\|_{L^2(B_{\tilde r})} &\le C\tilde r \|\nabla(\tilde v - \tilde v_h)\|_{L^2(B_{2\tilde r})} \\ &\le C\tilde r\|\nabla(\omega v)\|_{L^2(B_{2\tilde r})} \le C\tilde r\left(\|\nabla v\|_{L^2(B_{2\tilde r})} + \tilde r^{-1}\|v\|_{L^2(B_{2\tilde r})}\right).\end{aligned}$$

On the other hand, we obviously have

$$\|\nabla(\tilde v - \tilde v_h)\|_{L^2(B_r)} \le \|\nabla \tilde v\|_{L^2(B_{2\tilde r})} \le C\left(\|\nabla v\|_{L^2(B_{2\tilde r})} + \tilde r^{-1}\|v\|_{L^2(B_{2\tilde r})}\right).$$

Summarizing, we have obtained

$$\|v - v_h\|_{H^1(B_r)} \le C|\log h|\left(\|\nabla v\|_{L^2(B_{2\tilde r})} + r^{-1}\|v\|_{L^2(B_{2\tilde r})}\right) + Cr^{-1}\|v - v_h\|_{L^2(B_{2\tilde r})}.$$

The assertion finally follows upon writing $v - v_h = (v - \Pi v) - (v_h - \Pi v)$. $\square$

Let $g(x, y)$ be the Green's function corresponding to the Laplace operator, with vanishing Dirichlet boundary condition, and having the singularity located at $y \in \Omega$. Since $\partial\Omega \in C^{1,1}$, it is well known that

$$|D_x^2 g(x,y)| \le C|x-y|^{-2}, \qquad \forall\ x \ne y. \tag{S3.4}$$

PROOF OF LEMMA 7.1. Let $x_0 \in S_0$ be a point at which the maximum norm of $v - v_h$ is attained. Let $r = \Gamma h$. Then

$$\|v - v_h\|_{L^\infty(\Omega)} \le \|v - \Pi v\|_{L^\infty(S_0)} + \|\Pi v - v_h\|_{L^\infty(S_0)} \le \|v - \Pi v\|_{L^\infty(S_0)} + C\|\Pi v - v_h\|_{H^1(S_0)},$$

because $\Pi v - v_h \in \mathbf{V}$. Moreover, on using Lemma S3.2, we obtain

$$\begin{aligned}\|\Pi v - v_h\|_{H^1(S_0)} &\le \|v - \Pi v\|_{H^1(B_r)} + \|v - v_h\|_{H^1(B_r)} \\ &\le C|\log h|\|\nabla(v - \Pi v)\|_{L^2(B_{2\tilde r})} + Cr^{-1}\|v - v_h\|_{L^2(B_{2\tilde r})}.\end{aligned}$$

We claim that the rightmost term satisfies

$$\|v - v_h\|_{L^2(B_{2\tilde r})} \le Ch^2|\log h|^6\|\nabla(v - \Pi v)\|_{L^\infty(\Omega)}. \tag{S3.5}$$

Let us assume, for the moment, that (S3.5) is valid and complete the proof of Lemma 7.1. We clearly have

$$\|v - v_h\|_{L^\infty(\Omega)} \le Ch|\log h|^6\|\nabla(v - \Pi v)\|_{L^\infty(\Omega)} \le Ch^{2-2/p}|\log h|^6\|v\|_{W^{2,p}(\Omega)}.$$

The fact that $\partial\Omega \in C^{1,1}$ results in $\|v\|_{W^{2,p}(\Omega)} \le Cp\|\psi\|_{L^\infty(\Omega)}$; the precise dependence on p follows from tracing constants in the singular integrals involved. Then we choose $p = |\log h|$ in the above inequality to get the desired estimate.

To complete the argument we must prove (S3.5). To this end, we will use the Green's function g and the local energy estimates. Let $\varphi \in C_0^\infty(B_{2\tilde r})$ and $\zeta \in H_0^1(\Omega)$ satisfy $\Delta\zeta = -\varphi$. Let $\zeta_h \in \mathbf{V}$ be the H^1-projection of ζ onto $\mathbf{V}$, namely,

$$\langle \nabla(\zeta - \zeta_h), \nabla\chi_j\rangle = 0, \qquad \forall\ 1 \le j \le J.$$

Since

$$\langle v - v_h, \varphi\rangle = \langle \nabla(v - v_h), \nabla\zeta\rangle = \langle \nabla(v - \Pi v), \nabla(\zeta - \zeta_h)\rangle \le \|\nabla(v - \Pi v)\|_{L^\infty(\Omega)}\|\nabla(\zeta - \zeta_h)\|_{L^1(\Omega)},$$

all we have to show is

$$\|\nabla(\zeta - \zeta_h)\|_{L^1(\Omega)} \le Ch^2|\log h|^6\|\varphi\|_{L^2(\Omega)}. \tag{S3.6}$$

For $1 \le k \le K\ (\le C|\log h|)$, set

$$\begin{aligned}A_k &:= \{x \in \Omega : 4h|\log h|^k \le |x - x_0| < 4h|\log h|^{k+1}\}, \\ A_k^+ &:= \{x \in \Omega : \text{dist}\,(x, A_k) < 2h|\log h|^k\},\end{aligned}$$

and $A_0 := B_{4\tilde r}$. We claim that

$$\|\nabla\zeta\|_{L^2(\Omega)} \le C\tilde r|\log \tilde r|^{1/2}\|\varphi\|_{L^2(\Omega)}.$$

Indeed, using Hölder's inequality combined with a 2-D Sobolev inequality, we deduce that

$$\|\nabla\zeta\|^2_{L^2(\Omega)} = \langle\zeta, \varphi\rangle \le \|\zeta\|_{L^q(\Omega)}\|\varphi\|_{L^p(\Omega)} \le C\tilde r^{1-1/q}q^{1/2}\|\nabla\zeta\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.$$

The asserted estimate follows from taking $q = |\log \tilde r|$. Consequently,

$$\|\nabla(\zeta-\zeta_h)\|_{L^1(A_0)} \le C\tilde r \|\nabla(\zeta-\zeta_h)\|_{L^2(A_0)} \le C\tilde r\|\nabla\zeta\|_{L^2(\Omega)} \le Ch^2|\log h|^{5/2}\|\varphi\|_{L^2(\Omega)}.$$

We now examine the contribution due to A_k for $1 \le k \le K$. Let $B_\imath := B(x_\imath, h|\log h|^{k-1})$ be a covering of A_k with the nonoverlapping property

$$1 \le \sum_\imath \chi_{B_\imath}(x) \le C \text{ (independent of } h \text{ and } k), \qquad \forall\, x \in A_k,$$

where $\chi_{B_\imath}$ stands for the characteristic function of $B_\imath$. Set $B_\imath^+ := B(x_\imath, 2h|\log h|^k)$. As a consequence of Lemma S3.2, we see that

$$\|\nabla(\zeta-\zeta_h)\|_{L^2(B_\imath)} \le C|\log h|\ \|\nabla(\zeta-\Pi\zeta)\|_{L^2(B_\imath^+)} + C(h|\log h|^{k-1})^{-1}\|\zeta-\zeta_h\|_{L^2(B_\imath^+)}.$$

Since $\sum_\imath \chi_{\tilde B_\imath}(x) \le C|\log h|^2$, we infer that

$$\begin{aligned}\|\nabla(\zeta-\zeta_h)\|_{L^1(A_k)} &\le Ch|\log h|^{k+1}\|\nabla(\zeta-\zeta_h)\|_{L^2(A_k)}\\ &\le Ch|\log h|^{k+3}\|\nabla(\zeta-\Pi\zeta)\|_{L^2(A_k^+)} + C|\log h|^3\|\zeta-\zeta_h\|_{L^2(A_k^+)}.\end{aligned}$$

Using the representation formula $\zeta(x) = \int_{B_{2\tilde r}} g(x,y)\varphi(y)dy$ for $x \in \Omega$, in conjunction with (S3.4), we get

$$\|D^2\zeta\|_{L^2(\tilde A_k^+)} \le C(h|\log h|^k)^{-2}\text{ meas }(A_k^+)\|\varphi\|_{L^1(B_{2\tilde r})} \le C|\log h|^{2-k}\|\varphi\|_{L^2(\Omega)}.$$

Hence,

$$\|\nabla(\zeta-\Pi\zeta)\|_{L^2(A_k^+)} \le Ch\|D^2\zeta\|_{L^2(\tilde A_k^+)} \le Ch|\log h|^{2-k}\|\varphi\|_{L^2(\Omega)}.$$

Finally, it only remains to add over k for $1 \le k \le K$ $(\le C|\log h|)$, where C is such that $\{A_k\}_{k=0}^K$ form a covering of Ω, and apply (7.6) to conclude the whole argument. □

REMARK S3.2. We have deliberately omitted the technical details involved in the analysis of the discrepancy between Ω and its polygonal approximation, for which we refer to [23]. Lemma 7.1 can be viewed as a sharpening of the arguments in [21], where an interior $O(h^{2-\epsilon})$-rate of convergence is proved for convex polygons. We do not know whether the extra powers of the logarithm occurring in our pointwise estimate are really necessary, but we stress that it is still quasi-optimal.

REMARK S3.3. Suppose Ω is a rectangle, as in §8.3. It is easily seen, by means of reflection arguments, that the key properties $\|v\|_{W^{2,p}(\Omega)} \le Cp\|\psi\|_{L^\infty(\Omega)}$ and (S3.4) are still valid and so is Lemma 7.1. For general polygons let $y_x \in \partial\Omega$, for a given $x \in \Omega$, be the closest corner to x, $0 < \omega_x \le 2\pi$ be the corresponding internal angle at y_x and $\sigma_x := \pi/\omega_x$. Then, our technique, combined with that in [21], leads to the following *local* result

$$\text{(S3.7)} \qquad \|v-v_h\|_{L^\infty(B_r)} \le Ch|\log h|^6\|\nabla(v-\Pi v)\|_{L^\infty(B_{2r})} + Cr^{-1}\|v-v_h\|_{L^2(B_{2r})},$$

for $r \ge Ch^{1-\epsilon}$ $(\epsilon > 0)$, provided $h(x) \le Ch|x-y_x|^{\max(0,1-\sigma_x)}$, where $h(x)$ is the function introduced in Lemma 5.1. Details are omitted.

S4. Generalizations. So far we have assumed that triangles intersecting the discrete interface and "bad" triangles are kept fixed. We have also excluded the analysis of mushy regions and of variable conductivity as well as the treatment of polygonal domains. We now wish to investigate these issues in some detail. We will be using the notation of §§2,4.

S4.1. *Eliminating Fixed Triangles.* Recall that $\mathcal{S}_B$ is the set of all triangles on which either first or second derivatives exceed a certain tolerance, and that $\mathcal{S}_F$ are those traversed by the discrete interface F. All those elements were left fixed, which is nearly the best we can do in a general setting. We want to show that under the following assumptions on the behavior of the numerical method:

$$\text{(S4.1)} \qquad d_S \le C_1, \qquad \forall\, S \in \mathcal{S}_F \cup \mathcal{S}_B,$$

$$\text{(S4.2)} \qquad D_e \le C_2\tau^{-1}, \qquad \forall\, e \in \mathcal{E}_B \cup \mathcal{E}_F,$$

$$\text{(S4.3)} \qquad \text{length}(F) \le C_3, \qquad \text{meas}(\mathcal{B}) \le C_4\tau,$$

the constraint (4.8) can be removed, namely all elements of $\mathcal{S}$ become free. The entire mesh $\mathcal{S}$ can thus be regenerated, on replacing (4.2) and (4.3) by

$$\text{(S4.4)} \quad \hat h_e := \max\left(\lambda\tau, \mu_1\frac{\tau^{1/2}}{D_e^{1/2}}\right), \quad \forall\, e \in \mathcal{E}, \qquad \hat h_S := \max\left(\lambda\tau, \mu_1\frac{\tau^{1/2}}{d_S}\right), \quad \forall\, S \in \mathcal{S}.$$

Hence, what really matters in defining both $\hat h_e$ and $\hat h_S$ is local regularity because F does not appear explicitly in (S4.4). Condition (S4.1) excludes the formation of cusps but it is still quite reasonable in many circumstances of practical interest. Condition (S4.2) is also acceptable in light of (3.10) whereas (S4.3) is consistent with (4.11). In §8.3 we present a relevant example that fulfils (S4.1)-(S4.3) and compare the performance of this strategy and that of §4.

Let us start by proving the following crucial estimates:

$$\text{(S4.5)} \qquad \|\Pi\zeta(U) - \hat\Pi\zeta(U)\|_{L^p(\Omega)} \le C\tau^{1/p}, \qquad \forall\, 1 \le p < \infty, \quad \zeta' \in W^{1,\infty}(\mathbf{R}),$$

$$\text{(S4.6)} \qquad \|\Theta - \hat\Theta\|_{L^\infty(\Omega)} \le C\tau, \qquad \|\nabla(\Theta-\hat\Theta)\|_{L^2(\Omega)} \le C\tau^{1/2}.$$

Set

$$\hat{\mathcal{F}} := \cup\{\hat S \in \hat{\mathcal{S}} : \mathcal{S}_{\hat S} \cap \mathcal{S}_F \ne \emptyset\}, \qquad \hat{\mathcal{B}} := \cup\{\hat S \in \hat{\mathcal{S}} : \mathcal{S}_{\hat S} \cap \mathcal{S}_B \ne \emptyset\}.$$

Since $h_{\hat{S}}, h_S = O(\tau)$ for all $\hat{S} \subset \hat{\mathcal{F}} \cup \hat{\mathcal{B}}$ and $S \in \mathcal{S}_{\hat{S}}$, we have $\text{card}(\mathcal{S}_{\hat{S}}) = O(1)$. Also, (S4.3) implies $\text{meas}(\hat{\mathcal{F}} \cup \hat{\mathcal{B}}) \le C\tau$. Since U satisfies (6.2), we get

$$\|\Pi\zeta(U) - \hat{\Pi}\zeta(U)\|_{L^p(\hat{\mathcal{F}}\cup\hat{\mathcal{B}})} \le C\|U\|_{L^\infty(\Omega)} \left(\text{meas}(\hat{\mathcal{F}} \cup \hat{\mathcal{B}})\right)^{1/p} \le C\tau^{1/p}.$$

This estimate, together with (5.10) in $\Omega\backslash(\hat{\mathcal{F}} \cup \hat{\mathcal{B}})$, yields (S4.5), which in turn is used with $p = 1$ to demonstrate Lemma 6.3. The estimates in (S4.6) are the analogues of (5.10) and (5.11) for $\zeta = \beta$. We only have to consider $\hat{S} \subset \hat{\mathcal{F}} \cup \hat{\mathcal{B}}$, as before. Note that Lemma 5.3, with $V = U$ and $\zeta = \beta$, cannot be applied because of the lack of $W^{2,\infty}$-regularity of $\beta(U)$ in $\mathcal{F}$. However we just modify the proof of Lemma 5.3, by splitting $\Theta - \hat{\Theta}$ as follows:

$$\Theta - \hat{\Theta} = \Theta - \hat{\Pi}\Theta + \hat{\Pi}[\Pi\beta(U) - \beta(U)].$$

Using now Lemma 5.3, with $V = \Theta$ and $\zeta =$ Identity, combined with (S4.2), we get

$$\|\Theta - \hat{\Pi}\Theta\|_{L^\infty(\hat{S})} + \|\nabla(\Theta - \hat{\Pi}\Theta)\|_{L^2(\hat{S})} \le Ch_{\hat{S}} \max_{e\in\mathcal{E}_{\hat{S}}} h_e D_e \le C\tau.$$

On the other hand, in view of (4.6) we have $\delta_S = h_S d_S \le C\hat{h}_S d_S$, and thus (7.12) and standard interpolation estimates yield

$$\begin{aligned}\|\hat{\Pi}[\Pi\beta(U) - \beta(U)]\|_{L^\infty(\hat{S})} + \|\nabla\hat{\Pi}[\Pi\beta(U) - \beta(U)]\|_{L^2(\hat{S})} &\le C\|\Pi\beta(U) - \beta(U)\|_{L^\infty(\hat{S})} \\ &\le C \max_{S\in\mathcal{S}_{\hat{S}}\cap(\mathcal{S}_F\cup\mathcal{S}_B)} h_S d_S + CL_{\beta'} l_\beta^{-1} \max_{S\in\mathcal{S}_{\hat{S}}\backslash(\mathcal{S}_F\cup\mathcal{S}_B)} \hat{h}_S^2 d_S^2 \le C\tau,\end{aligned}$$

as results from (S4.1) and (S4.4). This proves the first estimate in (S4.6). For the second one, the fact that $h_S = O(\tau)$ in $\hat{\mathcal{F}} \cup \hat{\mathcal{B}}$, coupled with the two preceding estimates, yields

$$\|\nabla(\Theta - \hat{\Theta})\|^2_{L^2(\hat{\mathcal{F}}\cup\hat{\mathcal{B}})} \le C \sum_{\hat{S}\subset\hat{\mathcal{F}}\cup\hat{\mathcal{B}}} \|\nabla(\Theta - \hat{\Theta})\|^2_{L^2(\hat{S})} \le C \sum_{\hat{S}\subset\hat{\mathcal{F}}\cup\hat{\mathcal{B}}} h_{\hat{S}}^2 \le C \text{ meas}(\hat{\mathcal{F}} \cup \hat{\mathcal{B}}) \le C\tau.$$

The same argument leads to $\|\nabla(\Theta - \hat{\Theta})\|_{L^1(\hat{e})} \le C\tau$ for all $\hat{e} \in \hat{\mathcal{E}}$ such that $\mathcal{S}_{A_{\hat{e}}} \cap (\mathcal{S}_F \cup \mathcal{S}_B) \ne \emptyset$. The above estimates for $\Theta - \hat{\Theta}$ are the basic ingredients to demonstrate Lemmas 5.10 and 5.11, from which Lemmas 6.4 and 6.5 follow.

It only remains to comment upon the error estimate (7.10). The only novelty arises from treating term V. In view of (S4.5), we resort to a 2-D Sobolev inequality, with $q = |\log\tau|$, to conclude

$$\begin{aligned}|V| &\le \sum_{n=1}^m \|U^{n-1} - \hat{U}^{n-1}\|_{L^p(\Omega)} \|Ge_u^n\|_{L^q(\Omega)} \\ &\le Cq^{1/2} \max_{1\le n\le m} \|Ge_u^n\|_{H_0^1(\Omega)} \sum_{n=1}^m \|U^{n-1} - \hat{U}^{n-1}\|_{L^p(\Omega)} \\ &\le C\tau^{1/2-1/q} q^{1/2} \max_{1\le n\le m} \|e_u^n\|_{H^{-1}(\Omega)} \le \eta \max_{1\le n\le m} \|e_u^n\|^2_{H^{-1}(\Omega)} + C\eta^{-1}\tau|\log\tau|,\end{aligned}$$

because the number of mesh changes is bounded by $O(\tau^{-1/2})$. This ends the discussion.

S4.2. *Mushy Regions.* It is well known for the continuous problem (2.8) that in the absence of initial mushy regions they can only develop when $f \ne 0$ [20]. They are a striking manifestation of degeneracy in that the underlying PDE becomes $u_t = f(0)$, at least formally. Our objective now is to motivate the proper adjustments needed by the mesh selection algorithm to handle such a situation.

We allow the initial interface I_0 to degenerate into a mushy region, namely meas $(I_0) > 0$, at the expense of the further assumptions

(S4.7) $\qquad \partial I_0 \cap \Omega \quad$ is a Lipschitz curve,

(S4.8) $\qquad u_0 \in W^{1,\infty}(I_0).$

To determine the initial mesh $\mathcal{S}^1$, we only have to make sure that the initial refined region contains I_0. Hence, in view of (S4.7) and (S4.8), Lemma 5.4 still holds.

We distinguish between two opposite situations according to the role played by heat conduction. In the first case, heat reaches the interface $\partial I(t)$ at x by conduction. Consequently, the flux satisfies $\nabla\theta^{+,-}(x,t) = [\![\nabla\theta(x,t)]\!] \ne 0$ for $x \in \partial I(t)\cap\Omega$ and (2.11) provides a useful expression for the interface velocity $V(x)$. Moreover, the mush is *contracting* at x in the sense that the adjacent phase moves toward the mushy region. In the other case, instead, the internal heat production (or consumption), represented by f, is solely responsible for the interface motion. Since there is no heat conduction at $x \in \partial I(t) \cap \Omega$, namely $\nabla\theta^{+,-}(x,t) = [\![\nabla\theta(x,t)]\!] = 0$, (2.11) no longer provides information on $V(x)$. It says, however, that enthalpy u is continuous at $x \in \partial I(t) \cap \Omega$, or equivalently $[\![u(x,t)]\!] = 0$, if $V(x) \ne 0$. The mush can either be contracting or expanding at x; we say that the mushy region is *degenerate* at x. Both opposite situations can occur at the same time [16].

A *discrete mushy region* $\mathcal{M}$ is defined as the union of all triangles $S \in \mathcal{S}$ such that $\Theta|_S$ vanishes. We say that $\mathcal{M}$ is *degenerate* on a side $e \subset \partial\mathcal{M}$ if

(S4.9) $\qquad d_S \le \mu_4 h_e, \qquad \forall\ S \in \mathcal{S}_{\hat{e}},$

where $\mu_4 > 0$ is a suitable constant. Let $\partial_d\mathcal{M}$ denote the union of all those e's and set $\partial_n\mathcal{M} := \partial\mathcal{M}\backslash\partial_d\mathcal{M}$, which stands for the nondegenerate part of $\partial\mathcal{M}$.

We can certainly infer that the evolution of a degenerate mushy boundary cannot be predicted by means of the Stefan condition, as we did before. Since that was our basic tool in constructing the *new* refined region $\hat{\mathcal{R}}$, where the discrete interface is supposed to belong to, we may wonder whether the entire mush should lie in $\hat{\mathcal{R}}$. We want to show that our algorithm is robust in the sense that the *structural* assumptions

(S4.10) $\qquad \|\nabla U\|_{L^1(\Omega)} \le C \sum_{S\in\mathcal{S}} h_S^2 D_S \le C, \qquad \text{length}(\partial\mathcal{M}) \le C,$

lead to stability and accuracy provided the *new* refined region $\hat{\mathcal{R}}$ contains the current mushy region $\mathcal{M}$. In other words, a degenerate mushy boundary may exit $\mathcal{R}$ without causing troubles, simply because the enthalpy is regular.

We now would like to motivate the left estimate in (S4.10). To this end, let $\check{U} := \hat{U}^{n-2}$ and assume that such a bound is valid for $\check{U}$. Observe first that the assumption length$(\partial\mathcal{M}) \le C$, together with $D_S \le Ch_S^{-1}$ for all $S \in \mathcal{S}$, leads to

$$\sum_{S\in\mathcal{S}_{\partial\check{\mathcal{M}}}} h_S^2 D_S \le C \sum_{S\in\mathcal{S}_{\partial\check{\mathcal{M}}}} h_S \le C \text{ length}(\partial\mathcal{M}) \le C.$$

Let now $x_j \in \mathcal{M}$ satisfy $\tilde{S}_j \subset \mathcal{M}$. From (2.20), it is easily seen that $U(x_j) = \check{U}(x_j) + \tau f(\beta(\check{U}(x_j)))$. We next subtract this expression from a similar one for x_i, generic node adjacent to x_j, to arrive at

$$|U(x_i) - U(x_j)| \le (1 + \tau L_f L_\beta)|\check{U}(x_i) - \check{U}(x_j)|,$$

where we have used (2.1) and (2.4). Hence,

$$\sum_S h_S^2 D_S \le (1 + \tau L_f L_\beta) \sum_S h_S^2 \check{D}_S \le C,$$

where the summation is carried over all $S \in \mathcal{S}$ having a vertex x_j as above. It only remains to consider S's belonging to either the solid or liquid phase, where U and Θ turn out to be equivalent (see (4.10). By virtue of (6.6), which does not entail the absence of mush, we conclude that

$$\sum_{S\in\mathcal{S}_{\Omega\setminus\check{\mathcal{M}}}} h_S^2 D_S \le C \sum_{S\in\mathcal{S}_{\Omega\setminus\check{\mathcal{M}}}} h_S^2 d_S \le C\|\nabla\Theta\|_{L^2(\Omega)} \le C.$$

Our argument is not completely satisfactory, though, in that we cannot derive the global bound $\|\nabla U\|_{L^1(\Omega)} \le C$ depending on the initial enthalpy only , because $\mathcal{S}$ is not translation invariant (it is not even quasi-uniform!). We refer to [11] for the continuous problem.

Assume (S4.1)-(S4.3) and (S4.10). The *mesh selection algorithm* of §4.2 is then modified as follows. The first test on the interface location is replaced by $\partial_n\mathcal{M} \subset \mathcal{R}$, whereas the other two, namely (4.6) and (4.7), remain unchanged; the last refers to $\partial_n\mathcal{M}$ only. Thus $\partial_d\mathcal{M}$ may exit $\mathcal{R}$. In addition to (4.2), (4.3) and (4.5), the last being enforced on $\partial_n\mathcal{M}$ only, the following local parameter is used to determine the local meshsize within $\mathcal{M}$:

(S4.11) $$\hat{h}_S := \max\left(\lambda\tau, \mu_5\tau/D_S^{1/3}\right), \qquad \forall\, S \in \mathcal{S}_{\mathcal{M}\setminus\partial_n\mathcal{M}},$$

where $\mu_5 > 0$ is a suitable constant; set $\hat{h}_e := \infty$ for all $e \subset \mathcal{M}$. If $\mathcal{S}$ is to be discarded because a test of §4.2 fails, then all its elements are free, which means that $\mathcal{S}$ can be completely regenerated, but we impose the constraints (4.9) and $\mathcal{M} \subset \hat{\mathcal{R}}$. Note that (S4.11) comes from combining $\mathcal{M} \subset \hat{\mathcal{R}}$ with (S4.10) and the fact that the local meshsize within $\hat{\mathcal{R}}$ is $O(\tau)$.

To prove stability, we have to verify (S4.5) and (S4.6). We first use Lemma 5.3, with $V = U$, for all $\hat{S} \in \hat{\mathcal{S}}_{\mathcal{M}}\setminus\hat{\mathcal{S}}_{\hat{\partial}_n\mathcal{M}}$, where $\hat{\partial}_n\mathcal{M} = \cup\{\hat{S} \in \hat{\mathcal{S}} : \mathcal{S}_{\hat{S}} \cap \mathcal{S}_{\partial_n\mathcal{M}} \ne \emptyset\}$. Since $h_{\hat{S}} = O(\tau)$ for all $\hat{S} \in \hat{\mathcal{S}}_{\mathcal{M}}$ and $D_S \le Ch_S^{-1}$, $h_{\hat{S}} \le Ch_S$ for all $S \in \mathcal{S}_{\hat{S}}$, (S4.10) yields

$$\|\Pi\zeta(U) - \hat{\Pi}\zeta(U)\|_{L^p(\mathcal{M}\setminus\hat{\partial}_n\mathcal{M})} \le C \left(\sum_{\hat{S}\in\hat{\mathcal{S}}_{\mathcal{M}}\setminus\hat{\mathcal{S}}_{\hat{\partial}_n\mathcal{M}}} h_{\hat{S}}^2 \max_{S\in\mathcal{S}_{\hat{S}}}(h_S^p D_S^p + h_S^{2p} D_S^{2p})\right)^{1/p}$$
$$\le C\tau^{1/p} \left(\sum_{S\in\mathcal{S}_{\check{\mathcal{M}}}\setminus\mathcal{S}_{\partial_n\mathcal{M}}} h_S^2 D_S\right)^{1/p} \le C\tau^{1/p}, \quad \forall\, 1 \le p < \infty.$$

This estimate, together with (5.10), (6.2) and property meas $(\hat{\partial}_n\mathcal{M}) \le C\tau$, yields (S4.5). In deriving (S4.6) we only have to examine what happens on $\partial_d\mathcal{M}$, because $\Theta = 0$ in $\mathcal{M}$. In view of (S4.9), (S4.6) follows as in §S4.1, for instance.

Regarding accuracy, novelties arise from treating terms II_3 and III because properties $h_S = O(\tau)$ for all $S \in \mathcal{S}^n_{\mathcal{R}}$ as well as $F^{n-1}, F^n \subset \mathcal{R}^n$, used in Theorem 7.1, are no longer valid. For term II_3, we observe first that $\beta(U^n) = 0$ for all $S \in \mathcal{S}_{\mathcal{M}^n}$ and next use that length$(\partial\mathcal{M}^n) \le C$ and $d_S \le C$ to conclude

$$\sum_{S\in\mathcal{S}_{\partial\mathcal{M}^n}} \|\beta(U^n) - \Pi^n\beta(U^n)\|_{L^1(S)} \le \sum_{S\in\mathcal{S}_{\partial\mathcal{M}^n}} h_S^3 d_S \le C\tau \text{ length}(\partial\mathcal{M}^n) \le C\tau.$$

Term III can be treated by virtue of property $(U^n - \hat{U}^{n-1})(x) = O(\tau)$ for all $x \in S$ such that either $\tilde{S} \subset \mathcal{M}^n$ or $S \in \mathcal{S}^n_{\partial_d\mathcal{M}^n}$ (use (2.20)). We finally proceed as in §S4.1, first to treat term V, and next to derive (7.10).

S4.3. *Variable Conductivity.* We now turn our attention to the more realistic situation of temperature-dependent conductivity. Let $k : \mathbf{R} \to [1, \infty)$ satisfy

(S4.12) $$k \in W^{1,\infty}(\mathbf{R}).$$

Since the discrete maximum principle was a fundamental tool in our stability analysis, we transform (1.1) so as to preserve such a property. This is accomplished as follows. Let $\tilde{k} : \mathbf{R} \to [1, \infty)$ denote a piecewise linear approximation of k that verifies

(S4.13) $$\|k - \tilde{k}\|_{L^\infty(\mathbf{R})} \le C\tau^{1/2}, \qquad \|\tilde{k}\|_{W^{1,\infty}(\mathbf{R})} \le C\|k\|_{W^{1,\infty}(\mathbf{R})}.$$

Let $K, \tilde{K} : \mathbf{R} \to \mathbf{R}$ be the following Kirchhoff transformations:

(S4.14) $$K(z) := \int_0^z k(s)ds, \qquad \tilde{K}(z) := \int_0^z \tilde{k}(s)ds,$$

and set $\alpha := K \circ \beta$, $\tilde{\alpha} := \tilde{K} \circ \beta$. Note that $\tilde{K}$, and so $\tilde{\alpha}$, are easy to evaluate in practice. Since (1.1) becomes

$$u_t - \Delta v = f(\theta), \qquad \theta = \beta(u), \quad v = K(\theta) = \alpha(u),$$

we propose the following discrete scheme:

$$\text{(S4.15)} \qquad \tau^{-1}\langle U^n - \hat{U}^{n-1}, \chi\rangle^n + \langle \nabla V^n, \nabla \chi\rangle = \langle f(\hat{\Theta}^{n-1}), \chi\rangle^n, \qquad \forall\, \chi \in \mathbf{V}^n,$$

where $V^n := \Pi^n[\tilde{\alpha}(U^n)] = \Pi^n[\tilde{K}(\Theta^n)]$. It is easily seen that (S4.15) bears the same advantages and stability properties as (2.15), because $\tilde{\alpha}$ and $f \circ \tilde{K}^{-1}$ verify (2.1) and (2.4), respectively. The functions V^n and $\hat{V}^{n-1} := \Pi^n[\tilde{\alpha}(U^{n-1})]$ are to be considered in place of Θ^n and $\hat{\Theta}^{n-1}$ in the stability estimates of §6. It only remains to analyze accuracy. More specifically, we just have to examine term II in (7.11), namely, $II = \sum_{n=1}^m \int_{I^n} \langle v(t) - V^n, e_u^n\rangle dt$, which is further decomposed as follows:

$$\begin{aligned} II_1 + \cdots + II_4 :=& \sum_{n=1}^m \int_{I^n} \langle v(t) - V^n, u(t^n) - u(t)\rangle dt + \sum_{n=1}^m \int_{I^n} \langle \tilde{\alpha}(u(t)) - \tilde{\alpha}(U^n), u(t) - U^n\rangle dt \\ &+ \sum_{n=1}^m \int_{I^n} \langle \tilde{\alpha}(U^n) - \Pi^n \tilde{\alpha}(U^n), e_u(t)\rangle dt + \sum_{n=1}^m \int_{I^n} \langle K(\theta(t)) - \tilde{K}(\theta(t)), e_u(t)\rangle dt. \end{aligned}$$

In view of (S4.13) and (2.5), we get

$$\begin{aligned} |II_4| &= \left| \sum_{n=1}^m \int_{I^n} \langle \int_0^{\theta(t)} [k(s) - \tilde{k}(s)] ds, e_u(t)\rangle dt \right| \\ &\leq \sum_{n=1}^m \int_{I^n} \|k(\theta(t)) - \tilde{k}(\theta(t))\|_{L^\infty(\mathbf{R})} \|\nabla\theta(t)\|_{L^2(\Omega)} \|e_u(t)\|_{H^{-1}(\Omega)} dt \\ &\leq C\tau^{1/2} \|\nabla\theta\|_{L^1(0,t^m,L^2(\Omega))} \|e_u\|_{L^\infty(0,t^m,H^{-1}(\Omega))} \leq \eta \|e_u\|^2_{L^\infty(0,t^m;H^{-1}(\Omega))} + C\eta^{-1}\tau. \end{aligned}$$

We next continue as in Theorem 7.1 to derive the rate of convergence (7.10) for the errors $\tilde{K}(\theta(t)) - \Pi^n[\tilde{K}(\Theta^n)]$ and e_u. Finally, (S4.13) and (7.12) lead to the desired estimate for $\|e_\theta\|_{L^2(0,T,L^2(\Omega))}$.

It is worth noting that, by virtue of (6.2), the approximation preprocess (S4.13) must be carried out on a bounded interval only.

S4.4. *Polygonal Domains.* The only limitation of our approach to handle polygonal domains arises in the proof of Theorem 7.1 and, more precisely, in the analysis of term IV in (7.11). In fact, we can no longer use a global pointwise estimate because of the corner singularities.

For convex domains we split IV into two terms according to what happens within $\mathcal{R}$ and outside. The second contribution is easy to tackle in view of (6.7) and (7.6). The first one, instead, requires the use of (S3.7) with r being a fixed number less than the smallest distance between corners and the center x_0 of B_r verifying $x_0 \in \mathcal{R}$. Suppose 0 is a corner and set $\sigma := \pi/\omega$, where $\omega < \pi$ is the internal angle. The singularity of $v = G\psi$ is of the form $v(x) \approx |x|^\sigma$ for $\sigma \neq$ integer; otherwise a logarithm occurs [7]. Since

$$\|v - v_h\|_{L^\infty(B_r)} \leq C\tau^{1/2} |\log\tau|^6 \|\nabla(v - \Pi v)\|_{L^\infty(B_{2r})} + C\tau,$$

and we need $\|v - v_h\|_{L^\infty(B_r)} \leq C\tau|\log\tau|^7 \|\psi\|_{L^\infty(\Omega)}$, we only have to consider $1 < \sigma < 2$. Assume

$$\text{(S4.16)} \qquad \text{dist}\,(0, B_{2r}) \geq C \max(\tau, \tau^{1/2(2-\sigma)}).$$

Then, the local meshsize $h(x)$ introduced in Lemma 5.1 must satisfy

$$\lambda\tau \leq h(x) \leq C\tau^{1/2}|x|^{2-\sigma}, \qquad \forall\, x \in B_{2r},$$

which is possible because $|x| \geq C\tau^{1/2(2-\sigma)}$. When (S4.16) is violated, or equivalently $\mathcal{R}$ comes too close to a corner, (S3.7) does not seem to be good enough as compared to what interpolation theory suggests. This is due to the appearance of $h = O(\tau^{1/2})$ on the right-hand side rather than the local meshsize. A result of such a flavor was derived in [21,22] under the structural constraint that all triangles at about the same distance from a corner have nearly the same size, which may not be true in our setting. Such a case thus constitutes an open problem.

For concave polygonal domains, we need further refinements near the concave corners to eliminate the pollution effect in H^1, namely $h(x) \approx \tau^{1/2}|x|^{1-\sigma+\varepsilon}$ for $1/2 \leq \sigma < 1, \varepsilon > 0$ [7], and assume that $\mathcal{R}$ remains far away from the corners. We can therefore apply (7.6) and (S3.7) to prove Theorem 7.1. The general situation is also an open question.