VARIATIONAL FORMULATION OF A
MODEL FREE–BOUNDARY PROBLEM

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ABSTRACT. The purpose of this work is to present an error analysis of the numerical approximation by a finite element method of a free-surface problem. The analysis has been done in an abstract model which has many of the features of a free-surface problem for a viscous liquid. We study in this paper how the numerical approximation of the free boundary affects the approximation of the other variables of the problem and vice versa.

We present the numerical analysis of a free-boundary problem that is intended to incorporate many of the difficulties found in a class of models of fluid-flow phenomena with free surfaces. One such phenomenon which motivates the current work is the flow of a liquid constrained only partly by a container, that is, in which a part of the boundary of the domain filled by the liquid is an interface with another liquid of much smaller density, and for which surface tension plays a significant role in determining the shape of the free surface. One model for the behavior of such liquids is based on the assumption that the surface tension between the two liquids is proportional to the curvature of the free surface; the constant of proportionality is a physical property of the two fluids. This model has been studied extensively in recently years, both experimentally (cf. Jean and Pritchard [15] and Pritchard [19]), theoretically (cf. Allain [2], Beale [4], Bemelmans [5], Jean [14], Pukhnachov [20], and Solonnikov [27]), asymptotically (Keller and Miksis [16]) and computationally (cf. Cuvelier [10], Ryskin and Leal [23], and Saito and Scriven [24]). Our purpose here is to establish a framework for the analysis of convergence properties of the computational techniques being used. The only previous work that we are aware of in this direction is by Nitsche [18].

In the first section of the paper, we define our model problem in classical terms. In the second section, we construct a variational formulation for the problem that has two new features. One is that it allows the existence of a solution to be proved with weaker assumptions on the data than has been possible

before. But more importantly, it provides a framework in which a finite element method can be defined and analyzed rigorously. The third section of the paper carries out this analysis. Our variational approach is similar to one used in the code FIDAP [28]. Moreover, our analysis of this class of free-surface problems has much in common with the theoretical studies listed in the previous paragraph, except that the Hölder spaces used there are replaced by appropriate Sobolev spaces. This has the advantage of offering a framework in which finite element spaces that are typically used can be viewed as "conforming" to the theory. Unfortunately, the nature of the problem apparently precludes the use of the traditional Hilbert space theory; we require estimates for a power higher than two of the gradient of the finite element approximation. This is a consequence of the nonlinear relationship between the gradient of the field variable (which corresponds to the velocity variable in a flow problem) and the free surface.

We shall make use of the Lebesgue and Sobolev spaces $L_p(\Omega)$ and $W_p^m(\Omega)$, respectively, $1 \leq p \leq \infty$, $m \in \mathbb{N}$, defined for a bounded open set, Ω, in $\mathbb{R}^n$, $n = 1$ or 2. These spaces are provided with the usual norm $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W_p^m(\Omega)}$, respectively. $\overset{\circ}{W}{}_p^1(\Omega)$ is the space of those functions in $W_p^1(\Omega)$ which vanish on the boundary of Ω in the generalized sense. The inner product in L_2 is denoted by $\langle \cdot, \cdot \rangle$.

1. PROBLEM FORMULATION

Consider a function $\gamma \in W_\infty^1(0, 1)$ such that $\|\gamma\|_{W_\infty^1(0,1)} < 1$, and related to this function define the following sets:

$$\Omega_\gamma = \{(x, y) \mid 0 < x < 1, 0 < y < 1 + \gamma(x)\}$$

and

$$\Gamma_\gamma = \{(x, y) \mid 0 < x < 1, y = 1 + \gamma(x)\}.$$

Consider also another function $g \in W_p^1(\Omega^*)$, where Ω^* is some fixed domain such that $\Omega_\gamma \subset \Omega^*$ for all γ under consideration, and suppose $g(x, y) = 0$ for $y \geq 1/2$. (In the sequel, we shall choose $p > 2$; it will also be seen to be sufficient to define $\Omega^* = [0, 1] \times [0, 2]$.) With these assumptions we define the following

Model free-boundary problem. Find γ and u such that

$$\Delta u = 0 \quad \text{in } \Omega_\gamma,$$

$$u = g \quad \text{on } \partial\Omega_\gamma,$$

(1.1)
$$\frac{-s\gamma''(x)}{[1 + \gamma'(x)^2]^{1/2}} = \frac{\partial u(x, 1 + \gamma(x))}{\partial \vec{n}} \quad \forall x \in [0, 1],$$

$$\gamma(0) = \gamma(1) = 0.$$

The parameter $s > 0$ plays the role of the "surface tension" and $\vec{n}$ is the outward normal vector to Γ_γ. Note that u is a scalar field and that in the balance of terms on the interface we are using a nonlinear term which is not the curvature of γ. This is done in order to simplify the variational formulation of the problem. In §4, we describe the simple modifications necessary to obtain results pertaining to models involving the curvature operator.

In §2 a weak formulation of this problem is proposed which is more suitable for discretization via variational techniques. It is proved in Corollary 2.1 that the weak formulation admits a unique solution $(\gamma, u) \in W^1_\infty(0, 1) \times W^1_p(\Omega_\gamma)$ for small values of the norm of g, provided that $p \in (2, P)$ for some $P > 2$. In §3 we study the discrete approximation of this problem by a finite element method using piecewise linear functions. In Theorem 3.1 we prove that the discrete problem admits a unique solution, and in Theorem 3.2 and Corollary 3.1 we prove the convergence of this method. One of the conclusions of this work is that it is appropriate to use the same order of piecewise polynomials to approximate u and γ in order to obtain the optimal order of convergence.

2. The weak formulation of problem (1.1)

A preliminary weak formulation of problem (1.1) can be given in the usual way by converting the two elliptic problems for u and γ, respectively, as follows. First introduce bilinear forms

$$a_\gamma(u, v) := \int_{\Omega_\gamma} \nabla u \cdot \nabla v \, dx,$$

$$b(\gamma, \chi) := s \int_0^1 \gamma'(x) \chi'(x) \, dx.$$

Then our preliminary variational formulation is: Find $\gamma \in \mathring{W}^1_\infty(0, 1)$ and $u - g \in \mathring{W}^1_p(\Omega_\gamma)$ such that

$$a_\gamma(u, v) = 0 \quad \forall v \in \mathring{W}^1_q(\Omega_\gamma),$$

$$b(\gamma, \chi) = \int_0^1 \frac{\partial u}{\partial \vec{n}}(x, 1 + \gamma(x))\chi(x)[1 + \gamma'(x)^2]^{1/2}dx \quad \forall \chi \in \mathring{W}^1_1(0, 1).$$

The shortcoming of this variational formulation is its reference to a quantity, $\partial u / \partial \vec{n}$, that is undefined for arbitrary u in the space in which we seek the solution, u. We now address this issue. (Our reasons for choosing non-Hilbert spaces in the variational formulation will be clarified subsequently.)

If u is a sufficiently regular solution of problem (1.1) then Green's formula implies

(2.1)
$$\int_0^1 \frac{\partial u}{\partial \vec{n}}(x, 1 + \gamma(x)) \chi(x)[1 + \gamma'(x)^2]^{1/2}dx$$
$$= \int_{\Gamma_\gamma} \frac{\partial u}{\partial \vec{n}} v \, ds = \int_{\Omega_\gamma} \nabla u \cdot \nabla v \, dx,$$

provided that $v(x, 1 + \gamma(x)) = \chi(x)$ and $v = 0$ on $\partial\Omega_\gamma \backslash \Gamma_\gamma$.

Lemma 2.0. *For all* $\chi \in W_1^1(0, 1)$ *it is possible to construct an extension* $E_\gamma \chi \in$ $W_q^1(\Omega_\gamma)$ *whenever* $q < 2$, *such that* $E_\gamma \chi|_{\Gamma_\gamma} = \chi$ *and* $E_\gamma \chi|_{\partial \Omega_\gamma \setminus \Gamma_\gamma} = 0$.

Proof. First, by one of Sobolev's imbedding theorems, $\chi \in W_1^1(0, 1)$ implies $\chi \in W_q^{1-1/q}(0, 1)$ for $q < 2$. If $\chi \in W_q^{1-1/q}(0, 1)$ for $q < 2$, we can extend χ by zero to $\partial \Omega_0$, where $\Omega_0 = [0, 1] \times [0, 1]$, and assure that this extension $\tilde{\chi} \in W_q^{1-1/q}(\partial \Omega_0)$. By the surjectivity property of the trace operator, there is a $\hat{v} \in W_q^1(\Omega_0)$ such that $\hat{v}|_{\partial \Omega_0} = \tilde{\chi}$. Let $\widehat{E}\chi$ denote this extension, $\hat{v}$. Since the embedding and trace are continuous operators, there exist $C, C_E < \infty$, independent of χ, such that

$$(2.2) \qquad \|\widehat{E}\chi\|_{W_q^1(\Omega_0)} \le C \|\chi\|_{W_q^{1-1/q}(\partial \Omega_0)} \le C_E \|\chi\|_{W_1^1(0, 1)}.$$

(For details, see Grisvard [13] and Arnold, Scott and Vogelius [3].)

Now we transform the domain Ω_0 to Ω_γ by the following change of variables:

$$(2.3) \qquad (\xi, \eta) \in \Omega_0 \longrightarrow (x, y) \in \Omega_\gamma, \qquad x = \xi \text{ and } y = (1 + \gamma(\xi))\eta.$$

This induces a mapping on functions, v, on Ω_γ to functions, $\hat{v}$, on Ω_0 via

$$(2.4) \qquad \hat{v}(\xi, \eta) = v(\xi, (1 + \gamma(\xi))\eta).$$

If $\|\gamma\|_{W_\infty^1(0, 1)} < 1$, then the mappings (2.3) are invertible, and for any $\hat{v} \in$ $W_q^1(\Omega_0)$ there is a $v \in W_q^1(\Omega_\gamma)$ such that

$$(2.5) \qquad v(x, y) = \hat{v}(x, y/(1 + \gamma(x))).$$

Moreover, we have

$$(2.6) \qquad \begin{aligned} \|v\|_{W_q^1(\Omega_\gamma)} &\le \mu(\|\gamma\|_{W_\infty^1(0, 1)}) \|\hat{v}\|_{W_q^1(\Omega_0)}, \\ \|\hat{v}\|_{W_q^1(\Omega_0)} &\le \mu(\|\gamma\|_{W_\infty^1(0, 1)}) \|v\|_{W_q^1(\Omega_\gamma)}, \end{aligned}$$

where $\mu(t)$ is a continuous function for $0 \le t < 1$ and q may be any real number in the range $1 \le q \le \infty$. Now we can identify χ with its extension $E_\gamma \chi$ in Ω_γ by $\widehat{E_\gamma \chi} := \widehat{E}\chi$, i.e.,

$$(2.7) \qquad E_\gamma \chi(x, y) = \widehat{E}\chi(x, y/(1 + \gamma(x))),$$

where $\widehat{E}\chi$ was defined above extending χ to Ω_0. From (2.6) and (2.2) we conclude

$$(2.8) \qquad \begin{aligned} \|E_\gamma \chi\|_{W_q^1(\Omega_\gamma)} &\le \mu(\|\gamma\|_{W_\infty^1(0, 1)}) \|\widehat{E}\chi\|_{\overset{\circ}{W}_q^1(\Omega_0)} \\ &\le C_E \mu(\|\gamma\|_{W_\infty^1(0, 1)}) \|\chi\|_{W_1^1(0, 1)}. \qquad \square \end{aligned}$$

Remark. No extension is possible for $q \ge 2$, even for $\chi \in \overset{\circ}{W}_1^1(0, 1)$. To prove this, pick a sequence ϕ_j such that $\phi_j = 1$ in $[\frac{1}{4}, \frac{3}{4}]$, $\phi_j = 0$ outside

$[\frac{1}{4} - 1/j, \frac{3}{4} + 1/j]$ and ϕ_j is linear in $[\frac{1}{4} - 1/j, \frac{1}{4}]$ and $[\frac{3}{4}, \frac{3}{4} + 1/j]$. This sequence is bounded in $\overset{\circ}{W}{}^1_1(0, 1)$. However, it is not bounded in $H^{\frac{1}{2}}(0, 1)$. This can be seen either by direct computation using Fourier transforms or by observing that the ϕ_j's tend to the characteristic function of $[\frac{1}{4}, \frac{3}{4}]$ and that the latter is not in $H^{\frac{1}{2}}(0, 1)$. Since the ϕ_j's do not remain bounded in $H^{\frac{1}{2}}(0, 1)$, they cannot be extended boundedly to $H^1(\Omega_0)$ [6].

Now equation (2.1) can be inserted in our preliminary weak formulation to yield our final weak formulation of problem (1.1) (for $p > 2$ and $1/p + 1/q = 1$).

Variational free-boundary problem. Find $\gamma \in \overset{\circ}{W}{}^1_\infty(0, 1)$ and $u \in g \oplus \overset{\circ}{W}{}^1_p(\Omega_\gamma)$ such that

(2.9)
$$a_\gamma(u, v) = 0 \qquad \forall v \in \overset{\circ}{W}{}^1_q(\Omega_\gamma),$$
$$b(\gamma, \chi) = a_\gamma(u, E_\gamma\chi) \qquad \forall \chi \in \overset{\circ}{W}{}^1_1(0, 1).$$

Remark. The problem (2.9) does not depend on the extension E_γ because

$$a_\gamma(u, E^1_\gamma\chi - E^2_\gamma\chi) = 0 \quad \forall \chi \in \overset{\circ}{W}{}^1_1(0, 1)$$

for two such extensions, since $E^1_\gamma\chi - E^2_\gamma\chi \in \overset{\circ}{W}{}^1_q(\Omega_\gamma)$. Moreover, $a_\gamma(u, E_\gamma\chi)$ can be evaluated via (2.1), that is

$$a_\gamma(u, E_\gamma\chi) = \int_{\Omega_\gamma} \nabla u \cdot \nabla E_\gamma\chi \, dx = \int_{\Gamma_\gamma} \frac{\partial u}{\partial \vec{n}} E_\gamma\chi \, ds$$

$$= \int_0^1 \frac{\partial u}{\partial \vec{n}}(x, 1 + \gamma(x))\chi(x)[1 + \gamma'(x)^2]^{1/2} dx,$$

provided u and χ are regular enough.

To establish the existence of a solution (γ, u) of (2.9), we shall use a contraction argument. Before presenting this result, some a priori estimates will be proved. First define the following seminorms in $W^1_p(0, 1)$:

$$|\gamma|_{W^1_p(0,1)} = \|\gamma'\|_{L^p(0,1)} \quad \forall \gamma \in W^1_p(0, 1), \qquad 1 \le p \le \infty.$$

The following is a simple consequence of Rellich's theorem; cf. Brezis [7].

Proposition 2.1. $|\cdot|_{W^1_p(0,1)}$ *is a norm in* $\overset{\circ}{W}{}^1_p(0, 1)$ *equivalent to* $\|\cdot\|_{W^1_p(0,1)}$ *for* $1 \le p \le \infty$.

Proposition 2.2. *The bilinear form,* b, *is continuous from* $W^1_\infty(0, 1) \times W^1_1(0, 1)$ $\rightarrow \mathbb{R}$. *Furthermore, there exists a constant* $\beta < \infty$ *such that for all* $\gamma \in \overset{\circ}{W}{}^1_\infty(0, 1)$

(2.10)
$$\|\gamma\|_{W^1_\infty(0,1)} \le \beta \sup_{0 \ne \chi \in \overset{\circ}{W}{}^1_1(0,1)} \frac{b(\gamma, \chi)}{\|\chi\|_{W^1_1(0,1)}}.$$

Proof. The continuity of b is a consequence of Hölder's inequality. On the other hand, Proposition 2.1 implies that for all $\gamma \in \overset{\circ}{W}{}^1_\infty(0, 1)$

$$\|\gamma\|_{W^1_\infty(0,1)} \leq C \|\gamma'\|_{L^\infty(0,1)} = C \sup_{\substack{f \in L^1(0,1) \\ f \neq 0}} \frac{\langle \gamma', f \rangle}{\|f\|_{L^1(0,1)}}$$

$$= C \sup_{\substack{f \in L^1(0,1) \\ f \neq 0}} \frac{\langle \gamma', f - \overline{f} \rangle}{\|f\|_{L^1(0,1)}} \leq 2C \sup_{\substack{f \in L^1(0,1) \\ f - \overline{f} \neq 0}} \frac{\langle \gamma', f - \overline{f} \rangle}{\|f - \overline{f}\|_{L^1(0,1)}},$$

where $\overline{f} := \int_0^1 f(x)\, dx$ and we have used the simple inequality $\|f - \overline{f}\|_{L^1(0,1)} \leq 2\|f\|_{L^1(0,1)}$. For any $f \in L^1(0, 1)$, there is a $\chi \in \overset{\circ}{W}{}^1_1(0, 1)$ such that $\chi'(x) = f(x) - \overline{f}$. Using again the fact that $|\cdot|_{W^1_1(0,1)}$ and $\|\cdot\|_{W^1_1(0,1)}$ are equivalent norms in $\overset{\circ}{W}{}^1_1(0, 1)$, we have

$$\|\gamma\|_{W^1_\infty(0,1)} \leq C \sup_{0 \neq \chi \in \overset{\circ}{W}{}^1_1(0,1)} \frac{\int_0^1 \gamma'(x)\chi'(x)dx}{\|\chi'\|_{L^1(0,1)}}$$

$$\leq \frac{C}{s} \sup_{0 \neq \chi \in \overset{\circ}{W}{}^1_1(0,1)} \frac{b(\gamma, \chi)}{\|\chi\|_{W^1_1(0,1)}},$$

which proves inequality (2.10) with $\beta = \dfrac{C}{s}$. $\square$

Remark. In the previous proof and in subsequent estimates, C denotes a generic constant that may increase from line to line but is dependent only on the stated quantities.

Proposition 2.3. *Suppose* $\|\gamma\|_{W^1_\infty(0,1)} < 1/2$. *Then* a_γ *is a continuous bilinear form from* $W^1_p(\Omega_\gamma) \times W^1_q(\Omega_\gamma) \to \mathbb{R}$, *where* $1/p + 1/q = 1$. *Moreover, there are constants* $\alpha < \infty$ *and* $Q < 2 < P$ *such that for all* $u \in \overset{\circ}{W}{}^1_p(\Omega_\gamma)$

$$(2.11) \qquad \|u\|_{W^1_p(\Omega_\gamma)} \leq \alpha \sup_{0 \neq v \in \overset{\circ}{W}{}^1_q(\Omega_\gamma)} \frac{a_\gamma(u, v)}{\|v\|_{W^1_q(\Omega_\gamma)}},$$

whenever $Q \leq p \leq P$.

Proof. The first part of the theorem is easily proved using Hölder's inequality. The second part will be proved using a result of Meyers [17] concerning second-order elliptic divergence equations with bounded, measurable coefficients. To use this result, we transform the domain Ω_γ to Ω_0, using the change of variables (2.3). Thus, for all $u \in W^1_p(\Omega_\gamma)$ and $v \in W^1_q(\Omega_\gamma)$, we have $\hat{u} \in W^1_p(\Omega_0)$ and $\hat{v} \in W^1_q(\Omega_0)$ such that

$$(2.12) \qquad a_\gamma(u, v) = \int_{\Omega_0} \sum_{i,j=1}^2 A_{ij}(\xi, \eta; \gamma)\, \partial_i \hat{u}\, \partial_j \hat{v}\, d\xi d\eta =: \hat{a}(\hat{u}, \hat{v}; \gamma),$$

where $\partial_1 u = \partial u / \partial \xi$ and $\partial_2 u = \partial u / \partial \eta$, and

$$(2.13) \qquad A_{11}(\xi, \eta; \gamma) = 1 + \gamma(\xi), \qquad A_{22}(\xi, \eta; \gamma) = \frac{1 + \gamma'(\xi)^2 \eta^2}{1 + \gamma(\xi)},$$

$$A_{12}(\xi, \eta; \gamma) = A_{21}(\xi, \eta; \gamma) = -\gamma'(\xi) \eta.$$

Note that $A = (DD^T)^{-1} |\det D|$, where D is the Jacobian of the mapping (2.3). For all $\gamma \in W_\infty^1(0, 1)$ such that $\|\gamma\|_{W_\infty^1(0, 1)} < 1/2$, A_{ij} is well defined a.e. in Ω_0 for $1 \leq i, j \leq 2$, and the following holds: A is a real, symmetric matrix such that

$$(2.14) \quad \exists \Lambda > 0 \text{ such that } A_{ij}(\xi, \eta) z_j z_i \geq \Lambda z_i^2 \quad \forall z \in \mathbb{R}^2, \text{ a.a. } (\xi, \eta) \in \Omega_0,$$

$$(2.15) \qquad \exists M < \infty \text{ such that } \max_{1 \leq i, j \leq 2} \|A_{ij}\|_{L^\infty(\Omega_0)} \leq M.$$

Indeed, (2.15) is a consequence of the assumption $\|\gamma\|_{W_\infty^1(0, 1)} < 1/2$, and (2.15) implies (2.14) as follows. The eigenvalues of $(A_{ij}(\xi, \eta))$ satisfy $\lambda_{\min}(\xi, \eta) \leq \lambda_{\max}(\xi, \eta) \leq 2M$. Since $\det A = 1$, we have $\lambda_{\min} = \lambda_{\max}^{-1} \geq 1/2M =: \Lambda$, which implies (2.14).

From Meyers [17] it follows that there are real numbers $Q < 2 < P$ and a finite constant α such that

$$(2.16) \qquad \|\hat{u}\|_{W_p^1(\Omega_0)} \leq \alpha \sup_{0 \neq \hat{v} \in \overset{\circ}{W}_q^1(\Omega_0)} \frac{\hat{a}(\hat{u}, \hat{v}; \gamma)}{\|\hat{v}\|_{W_q^1(\Omega_0)}}$$

whenever $Q < p < P$. In view of (2.12), the result follows from (2.16) and the equivalence of norms (2.6). $\square$

Existence results. For $\varepsilon > 0$, consider the following set

$$V_\varepsilon = \{(\gamma, u) \in \overset{\circ}{W}_\infty^1(0, 1) \times W_p^1(\Omega_\gamma) \colon \|\gamma\|_{W_\infty^1(0, 1)} < \tfrac{1}{2}, \ \|u\|_{W_p^1(\Omega_\gamma)} < \varepsilon\},$$

and let T be the following mapping: given $(\gamma, u) \in V_\varepsilon$, let

$$T(\gamma, u) = (T_1(\gamma, u), T_2(\gamma, u)) = (\tilde{\gamma}, \tilde{u}),$$

where $(\tilde{\gamma}, \tilde{u})$ is determined as follows. First, $\tilde{\gamma} \in \overset{\circ}{W}_\infty^1(0, 1)$ is found satisfying

$$(2.17) \qquad b(\tilde{\gamma}, \chi) = a_\gamma(u, E_\gamma(\chi)) \quad \forall \chi \in \overset{\circ}{W}_1^1(0, 1).$$

Then $\tilde{u} - g \in W_p^1(\Omega_{\tilde{\gamma}})$ solves

$$(2.18) \qquad a_{\tilde{\gamma}}(\tilde{u}, v) = 0 \quad \forall v \in \overset{\circ}{W}_q^1(\Omega_{\tilde{\gamma}}).$$

Lemma 2.1. *There exist two positive real numbers δ and ε and a real number $P > 2$ such that, if $\|g\|_{W_p^1(\Omega^*)} < \delta$, then T is a mapping from $V_\varepsilon \to V_\varepsilon$ whenever $p \in (2, P)$.*

Proof. Let $(\gamma, u) \in V_\varepsilon$. We prove first that problem (2.17) admits a unique solution $\tilde{\gamma} \in \overset{\circ}{W}_\infty^1(0, 1)$. Uniqueness follows from Proposition 2.2. Existence

can be verified as follows. Let φ be the linear functional from $W_1^1(0, 1)$ to $\mathbb{R}$ defined by $\varphi(\chi) = a_\gamma(u, E_\gamma \chi)$. Then φ is continuous from $W_1^1(0, 1) \to \mathbb{R}$ in view of (2.8):

$$|\varphi(\chi)| = |a_\gamma(u, E_\gamma \chi)| \leq C \|u\|_{W_q^1(\Omega_\gamma)} \|E_\gamma \chi\|_{W_q^1(\Omega_\gamma)}$$

$$\leq C\mu(\|\gamma\|_{W_\infty^1(0,1)}) \|u\|_{W_p^1(\Omega_\gamma)} \|\chi\|_{W_1^1(0,1)} \leq C\varepsilon \|\chi\|_{W_1^1(0,1)},$$

where the last constant is of the form $C\mu(1/2)$. Since $\varphi \in W_1^1(0, 1)' \subset W_2^1(0, 1)'$, existence of a solution, $\tilde{\gamma}$, in $W_2^1(0, 1)$ is standard; cf. Brezis [7]. The fact that this $\tilde{\gamma}$ resides in $W_\infty^1(0, 1)$ follows from Proposition 2.2 via a density argument. Applying Proposition 2.2 again yields

$$\|\tilde{\gamma}\|_{W_\infty^1(0,1)} \leq \beta \sup_{0 \neq \chi \in \overset{\circ}{W}_1^1(0,1)} \frac{b(\tilde{\gamma}, \chi)}{\|\chi\|_{W_1^1(0,1)}}$$

(2.19)

$$= \beta \sup_{0 \neq \chi \in \overset{\circ}{W}_1^1(0,1)} \frac{a_\gamma(u, E_\gamma(\chi))}{\|\chi\|_{W_1^1(0,1)}} \leq C\varepsilon\beta.$$

Choose ε such that

$$\varepsilon \leq (2C\beta)^{-1}.$$

Thus, $\|\tilde{\gamma}\|_{W_\infty^1(0,1)} < 1/2$.

A similar argument now shows that Proposition 2.3 implies that there is a unique solution $w \in \overset{\circ}{W}_p^1(\Omega_{\tilde{\gamma}})$ solving

$$a_{\tilde{\gamma}}(w, v) = -a_{\tilde{\gamma}}(g, v) \quad \forall v \in \overset{\circ}{W}_q^1(\Omega_{\tilde{\gamma}}),$$

and it is bounded by

$$\|w\|_{W_p^1(\Omega_{\tilde{\gamma}})} \leq \alpha \sup_{0 \neq \hat{v} \in \overset{\circ}{W}_q^1(\Omega_\gamma)} \frac{a_{\tilde{\gamma}}(g, v)}{\|v\|_{W_q^1(\Omega_{\tilde{\gamma}})}} \leq \alpha \|g\|_{W_p^1(\Omega^*)}.$$

Therefore, $\tilde{u} = g + w$ satisfies equation (2.18) and

$$\|\tilde{u}\|_{W_p^1(\Omega_{\tilde{\gamma}})} \leq (1 + \alpha)\|g\|_{W_p^1(\Omega^*)}.$$

Now choose δ such that $\delta \leq \varepsilon/(1+\alpha)$. Hence, $\|g\|_{W_p^1(\Omega^*)} < \delta$ implies $(\tilde{\gamma}, \tilde{u}) \in V_\varepsilon$. $\square$

Although the set V_ε is appropriate from a physical point of view, it is not convenient for measuring the difference between two solutions because it is not based on a linear space. For this reason, we introduce the set

$$\widehat{V}_\varepsilon = \{(\gamma, \hat{u}) \in \overset{\circ}{W}_\infty^1(0, 1) \times \overset{\circ}{W}_p^1(\Omega_0): \|\gamma\|_{W_\infty^1(0,1)} < \tfrac{1}{2}, \|\hat{u}\|_{W_p^1(\Omega_0)} < \varepsilon\}.$$

There is a natural mapping between V_ε and $\widehat{V}_\varepsilon$ induced by (2.3), namely $(\gamma, v) \to (\gamma, \hat{v})$, where $\hat{v}$ is defined by (2.4). The inverse mapping is well defined in view of (2.6). A mapping $\widehat{T}$ can be defined correspondingly via

$$\widehat{T}(\gamma, \hat{v}) := (T_1(\gamma, v), T_2\widehat{(\gamma, v)}).$$

However, note that the "^" on v refers to the mapping (2.3) generated by γ, whereas the one on $T_2(\gamma, v)$ refers to the one generated by $\tilde{\gamma} = T_1(\gamma, v)$. The result is a mapping related to T that maps a subset of a linear space, $\widehat{V}_\varepsilon$, into itself.

Theorem 2.1. *Let P and δ be as in Lemma 2.1. For all $p \in (2, P)$ and all $g \in W_p^1(\Omega^*)$ such that $\|g\|_{W_p^1(\Omega^*)} < \delta$, the mapping $\widehat{T}$ is a contraction from $\widehat{V}_\varepsilon \to \widehat{V}_\varepsilon$ with respect to the norm*

$$(2.20) \qquad |||(\gamma, \hat{u})||| = \varepsilon \|\gamma\|_{W_\infty^1(0, 1)} + \|\hat{u}\|_{W_p^1(\Omega_0)}$$

for ε sufficiently small.

Proof. By Lemma 1.1 we have that $\widehat{T}: \widehat{V}_\varepsilon \to \widehat{V}_\varepsilon$ for ε sufficiently small. In order to prove that $\widehat{T}$ is a contraction, take $\tilde{\gamma}^j = T_1(\gamma^j, u^j)$ and $\tilde{u}^j = T_2(\gamma^j, u^j)$, where (γ^j, u^j) is any element of V_ε, $j = 1, 2$. Let $\gamma = \tilde{\gamma}^1 - \tilde{\gamma}^2$. Then $\gamma \in \overset{\circ}{W}_\infty^1(0, 1)$, and for all $\chi \in \overset{\circ}{W}_1^1(0, 1)$, γ satisfies

$$\begin{aligned}
b(\gamma, \chi) &= b(\tilde{\gamma}^1 - \tilde{\gamma}^2, \chi) = \hat{a}(\hat{u}^1, \widehat{E}\chi; \gamma^1) - \hat{a}(\hat{u}^2, \widehat{E}\chi; \gamma^2) \\
&= \int_{\Omega_0} \sum_{i, j=1}^2 A_{ij}(\xi, \eta; \gamma^1)\, \partial_j \hat{u}^1 \partial_i \widehat{E}\chi \, d\xi d\eta \\
&\quad - \int_{\Omega_0} \sum_{i, j=1}^2 A_{ij}(\xi, \eta; \gamma^2)\, \partial_j \hat{u}^2 \partial_i \widehat{E}\chi \, d\xi d\eta \\
&= \int_{\Omega_0} \sum_{i, j=1}^2 [A_{ij}(\xi, \eta; \gamma^1) - A_{ij}(\xi, \eta; \gamma^2)]\, \partial_j \hat{u}^1 \partial_i \widehat{E}\chi \, d\xi \, d\eta \\
&\quad + \int_{\Omega_0} \sum_{i, j=1}^2 A_{ij}(\xi, \eta; \gamma^2)\, [\partial_j \hat{u}^1 - \partial_j \hat{u}^2] \partial_i \widehat{E}\chi \, d\xi d\eta.
\end{aligned}$$

In view of (2.6) and (2.13), there exists $C < \infty$ such that

$$\begin{aligned}
|b(\gamma, \chi)| &\le C[\, \|\hat{u}^1\|_{W_p^1(\Omega_0)} \|\gamma^1 - \gamma^2\|_{W_\infty^1(0, 1)} \\
&\quad + \|\gamma^2\|_{W_\infty^1(0, 1)} \|\hat{u}^1 - \hat{u}^2\|_{W_p^1(\Omega_0)}]\, \|\chi\|_{W_1^1(0, 1)} \\
&\le C[\, \varepsilon\, \|\gamma^1 - \gamma^2\|_{W_\infty^1(0, 1)} + \|\hat{u}^1 - \hat{u}^2\|_{W_p^1(\Omega_0)}]\, \|\chi\|_{W_1^1(0, 1)},
\end{aligned}$$

since $(\gamma^1, u^1) \in V_\varepsilon$ and $(\gamma^2, u^2) \in V_\varepsilon$ implies that $\|\gamma^2\|_{W_\infty^1(0, 1)} < 1/2$ and $\|\hat{u}^1\|_{W_p^1(\Omega_0)} < \varepsilon$. From Proposition 2.2, we conclude that

$$(2.21) \qquad \|\gamma\|_{W_\infty^1(0, 1)} \le C\beta(\varepsilon\, \|\gamma^1 - \gamma^2\|_{W_\infty^1(0, 1)} + \|\hat{u}^1 - \hat{u}^2\|_{W_p^1(\Omega_0)}).$$

To bound the difference between $\tilde{u}^1$ and $\tilde{u}^2$, we do the following. Note first that $\widehat{\tilde{u}^1}$ and $\widehat{\tilde{u}^2}$ satisfy

$$\hat{a}(\widehat{\tilde{u}^1}, \hat{v}; \tilde{\gamma}^1) = 0 \quad \forall \hat{v} \in \overset{\circ}{W}^1_q(\Omega_0),$$

$$\hat{a}(\widehat{\tilde{u}^2}, \hat{v}; \tilde{\gamma}^2) = 0 \quad \forall \hat{v} \in \overset{\circ}{W}^1_q(\Omega_0).$$

Because we assumed that $g(x, y) = 0$ for $y \geq 1/2$ and $\tilde{\gamma}^j(i) = 0$ for $i = 0, 1$ and $j = 1, 2$, we have $\hat{g}^1 - \hat{g}^2 \in \overset{\circ}{W}^1_p(\Omega_0)$, where the different "hats" refer to different mappings (2.3). Thus, $\hat{u} := \widehat{\tilde{u}^1} - \widehat{\tilde{u}^2} \in \overset{\circ}{W}^1_p(\Omega_0)$ and satisfies

$$\begin{aligned}
\hat{a}(\hat{u}, \hat{v}; \tilde{\gamma}^1) &= \hat{a}(\widehat{\tilde{u}^1}, \hat{v}; \tilde{\gamma}^1) - \hat{a}(\widehat{\tilde{u}^2}, \hat{v}; \tilde{\gamma}^1) \\
&= -\hat{a}(\widehat{\tilde{u}^2}, \hat{v}; \tilde{\gamma}^1) = \hat{a}(\widehat{\tilde{u}^2}, \hat{v}; \tilde{\gamma}^2) - \hat{a}(\widehat{\tilde{u}^2}, \hat{v}; \tilde{\gamma}^1) \\
&= \int_{\Omega_0} \sum_{i,j=1}^{2} [A_{ij}(\xi, \eta; \tilde{\gamma}^2) - A_{ij}(\xi, \eta; \tilde{\gamma}^1)] \, \partial_j \widehat{\tilde{u}^2} \, \partial_i \hat{v} \, d\xi d\eta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\hat{a}(\hat{u}, \hat{v}; \tilde{\gamma}^1)| &\leq C \|\tilde{\gamma}^1 - \tilde{\gamma}^2\|_{W^1_\infty(0,1)} \|\widehat{\tilde{u}^2}\|_{W^1_p(\Omega_0)} \|\hat{v}\|_{W^1_q(\Omega_0)} \\
&\leq C\varepsilon \, \|\tilde{\gamma}^1 - \tilde{\gamma}^2\|_{W^1_\infty(0,1)} \|\hat{v}\|_{W^1_q(\Omega_0)} \\
&= C\varepsilon \, \|\gamma\|_{W^1_\infty(0,1)} \|\hat{v}\|_{W^1_q(\Omega_0)}.
\end{aligned}$$

Inequality (2.16) then yields

$$\|\hat{u}\|_{W^1_p(\Omega_0)} \leq \alpha \sup_{0 \neq \hat{v} \in \overset{\circ}{W}^1_q(\Omega_0)} \frac{\hat{a}(\hat{u}, \hat{v}; \tilde{\gamma}^1)}{\|\hat{v}\|_{W^1_q(\Omega_0)}} \leq C\varepsilon \, \alpha \|\gamma\|_{W^1_\infty(0,1)}.$$

Applying inequality (2.21) gives

$$\|\hat{u}\|_{W^1_p(\Omega_0)} \leq C\alpha\beta\varepsilon[\varepsilon \, \|\gamma^1 - \gamma^2\|_{W^1_\infty(0,1)} + \|\hat{u}^1 - \hat{u}^2\|_{W^1_p(\Omega_0)}].$$

Inserting this inequality and inequality (2.21) in the definition of $||| \cdot |||$, we have

$$|||\widehat{T}(\gamma^1, \widehat{u^1}) - \widehat{T}(\gamma^2, \widehat{u^2})||| \leq C(1+\alpha)\beta\varepsilon |||(\gamma^1, \widehat{u^1}) - (\gamma^2, \widehat{u^2})|||$$

Choosing $\varepsilon < (C(1+\alpha)\beta)^{-1}$ we conclude that $\widehat{T}$ is a contraction from $\widehat{V}_\varepsilon \to \widehat{V}_\varepsilon$. □

Corollary 2.1. *If $\|g\|_{W^1_p(\Omega^*)}$ is sufficiently small, then for all $p \in (2, P)$ problem (2.9) admits a unique solution $(\gamma, u) \in V_\varepsilon$ which can be calculated by a fixed*

point iteration,

$$(\gamma^{i+1}, u^{i+1}) = T(\gamma^i, u^i),$$

starting with any $(\gamma^0, u^0) \in V_\varepsilon$.

One choice of γ^0 and u^0 can be the following: $\gamma^0 \equiv 0$ and u^0 the solution of

$$(2.22) \qquad \begin{aligned} \Delta u &= 0 \quad \text{in } \Omega_0, \\ u &= g \quad \text{on } \partial\Omega_0. \end{aligned}$$

Clearly, $\|u^0\|_{W_p^1(\Omega_0)} \leq C\|g\|_{W_p^1(\Omega^*)}$, so choosing δ as was required in the proof of Lemma 2.1 yields $(\gamma^0, u^0) \in V_\varepsilon$.

We conclude this section with some remarks regarding its results.

(1) We have not proved global uniqueness, and there may indeed be solutions of larger norm even for small data.

(2) Regularity of solutions can be studied using the techniques in the works cited at the beginning that use Hölder norms. The solutions obtained by those techniques, when applicable, are of course identical to the weak solutions obtained here.

3. THE VARIATIONAL FORMULATION OF THE DISCRETE PROBLEM

Let π_h^0, $0 < h \leq h_0 < 1$, be a quasi-uniform triangulation of Ω_0, i.e., such that the following regularity condition is satisfied: the triangles $\widetilde{K} \in \pi_h^0$ meet only in entire common sides or in vertices; each triangle $\widetilde{K} \in \pi_h^0$ contains a circle of radius $c_1 h$ and is contained in a circle of radius $c_2 h$, where the constants $c_1 > 0$ and $c_2 < \infty$ do not depend on $\widetilde{K}$ or h.

We denote by π_h^γ the triangulation obtained by transforming the vertices of π_h^0 via the mapping in (2.3). More precisely, let F^γ denote the continuous, piecewise affine interpolant of the mapping (2.3) with respect to the mesh π_h^0, and let π_h^γ be the image of π_h^0 with respect to F^γ. Denote by Ω_γ^h the image of Ω_0 with respect to F^γ. For each triangle $\widetilde{K}$ in π_h^0, denote its vertices by $\tilde{\mathbf{n}}_{K,j} = (\xi_{K,j}, \eta_{K,j})$ for $j = 1, 2, 3$, and let $\mathbf{n}_{K,j} := F^\gamma(\tilde{\mathbf{n}}_{K,j})$. The mapping F^γ is given on each triangle by

$$F^\gamma|_{\widetilde{K}}(\xi, \eta) = D_K(\xi, \eta) + \vec{d}_K \quad \forall(\xi, \eta) \in \widetilde{K},$$

where D_K is a 2×2 matrix and $\vec{d}_K \in \mathbb{R}^2$. When it is not confusing to do so, we shall refer to the vertices by a global node number $k = k(K, j)$.

Proposition 3.1. *If* $\gamma \in W_\infty^1(0, 1)$ *and* $\|\gamma\|_{W_\infty^1(0,1)} < 1$, *then the mapping* F^γ *is invertible and* π_h^γ *is a quasi-uniform triangulation. Furthermore, the entries of the Jacobians,* D_K, *of* $F^\gamma|_{\widetilde{K}}$, *together with their inverses, are all bounded by* $C(1 - \|\gamma\|_{W_\infty^1(0,1)})^{-1}$.

Proof. Let $c = \|\gamma\|_{W^1_\infty(0,1)} < 1$. Let $k = k(K, j)$ and $k' = k'(K, j')$ denot
node indices of two distinct vertices ($j \neq j'$) of a given triangle. We have

$$
\begin{aligned}
|y_k - y_{k'}| &= |\eta_k - \eta_{k'} + \gamma(\xi_k)\eta_k - \gamma(\xi_{k'})\eta_{k'}| \\
&= |(\eta_k - \eta_{k'})(1 + \gamma(\xi_{k'})) + \eta_k(\gamma(\xi_k) - \gamma(\xi_{k'}))| \\
&\geq |(\eta_k - \eta_{k'})(1 + \gamma(\xi_{k'}))| - |\eta_k(\gamma(\xi_k) - \gamma(\xi_{k'}))| \\
&\geq (1-c)|\eta_k - \eta_{k'}| - c|\eta_k(\xi_k - \xi_{k'})| \\
&\geq (1-c)|\eta_k - \eta_{k'}| - c|\xi_k - \xi_{k'}|.
\end{aligned}
$$

Since $x_k = \xi_k$, we have

$$
|x_k - x_{k'}| + |y_k - y_{k'}| \geq (1-c)\left(|\eta_k - \eta_{k'}| + |\xi_k - \xi_{k'}|\right).
$$

Similarly, we can estimate that

$$
|x_k - x_{k'}| + |y_k - y_{k'}| \leq (1+c)\left(|\eta_k - \eta_{k'}| + |\xi_k - \xi_{k'}|\right).
$$

Thus each triangle is mapped onto a triangle of comparable dimensions, so tha
π_h^γ is a quasi-uniform triangulation. Moreover, this also shows that each F_i^γ
and hence F^γ, is invertible.

Since $F^\gamma(\tilde{\mathbf{n}}_{i,3}) = \mathbf{n}_{i,3}$, we may write

$$
F^\gamma|_{\widetilde{K}}((\xi, \eta) + \tilde{\mathbf{n}}_{i,3}) = D_K(\xi, \eta) + \mathbf{n}_{i,3}.
$$

Since $F^\gamma(\tilde{\mathbf{n}}_{i,j}) = \mathbf{n}_{i,j}$ for $j = 1, 2$, we must have

$$
D_K \tilde{C}^j = C^j \quad \forall j = 1, 2,
$$

where $C^j := \mathbf{n}_{i,j} - \mathbf{n}_{i,3}$ and $\tilde{C}^j := \tilde{\mathbf{n}}_{i,j} - \tilde{\mathbf{n}}_{i,3}$. If we let C (resp. $\tilde{C}$) denote th
matrix with columns C^j (resp. $\tilde{C}^j$), then we have $D_K \tilde{C} = C$ or $D_K = C\tilde{C}^{-1}$
Thus, we also have $D_K^{-1} = \tilde{C}C^{-1}$. From these representations and the bound:
above, the claimed estimates on the entries of D_K and D_K^{-1} follow easily. □

As a consequence of Proposition 3.1, if $\|\gamma\|_{W^1_\infty(0,1)} < 1$, then the mapping F^γ
induces an isomorphism, $v \to \tilde{v} := v \circ F^\gamma$, of the spaces $W^1_q(\Omega_\gamma^h) \to W^1_q(\Omega_0)$
Moreover, we have the estimates

$$
\begin{aligned}
\|v\|_{W^1_q(\Omega_\gamma^h)} &\leq \mu(\|\gamma\|_{W^1_\infty(0,1)})\|\tilde{v}\|_{W^1_q(\Omega_0)}, \\
\|\tilde{v}\|_{W^1_q(\Omega_0)} &\leq \mu(\|\gamma\|_{W^1_\infty(0,1)})\|v\|_{W^1_q(\Omega_\gamma^h)},
\end{aligned}
\qquad 1 \leq q \leq \infty.
$$

(3.1)

Note that the transformation $\Omega_0 \to \Omega_\gamma^h$ is different from the change of vari-
ables (2.3) that we used in the continuous case; to distinguish one from the
other, we denote by $(\tilde{x}, \tilde{y})$ the point in Ω_0 related to $(x, y) \in \Omega_\gamma^h$ by

$$
F^\gamma(\tilde{x}, \tilde{y}) = (x, y),
$$

whereas we denoted (2.3) via $(\hat{x}, \hat{y}) \to (x, y)$. The transformation F^γ is
introduced in order to transform the triangles $\widetilde{K}$ into triangles K and the
space $\mathbf{P}^1(\widetilde{K})$ into $\mathbf{P}^1(K)$. This would no longer be true if we applied the
transformation (2.3) to the triangulation π_h^0.

Related to the triangulations π_h^0 and π_h^γ, respectively, we define the following spaces:

$$V_h^\gamma = \{v \in C^0(\Omega_\gamma^h): v|_K \in \mathbf{P}^1(K) \quad \forall K \in \pi_h^\gamma\},$$

$$\overset{\circ}{V}{}_h^\gamma = \{v \in V_h^\gamma: v(\mathbf{n}_i) = 0 \quad \forall \mathbf{n}_i \in \partial\Omega_\gamma^h\},$$

$$V_h^0 = \{\tilde{v} \in C^0(\Omega_0): \tilde{v}|_{\widetilde{K}} \in \mathbf{P}^1(\widetilde{K}) \quad \forall \widetilde{K} \in \pi_h^0\},$$

$$\overset{\circ}{V}{}_h^0 = \{\tilde{v} \in V_h^0: \tilde{v}(\tilde{\mathbf{n}}_i) = 0 \quad \forall \tilde{\mathbf{n}}_i \in \partial\Omega_0\}.$$

Let $\{\xi_i \in [0, 1]: 1 \leq i \leq l_h\}$ denote all x-coordinates such that $(\xi_i, 1)$ is a vertex of a triangle $\widetilde{K} \in \pi_h^0$, i.e., $\xi_i = \xi_k$ for some k. Decompose $[0, 1] = \bigcup_{i=1}^{l_h}[\xi_{i-1}, \xi_i]$, and relate to this mesh the following discrete space:

$$S_h = \{\gamma \in C^0(0, 1): \gamma|_{[\xi_{i-1}, \xi_i]} \in \mathbf{P}^1([\xi_{i-1}, \xi_i]), \ 0 \leq i \leq l_h + 1,$$
$$\text{and } \gamma(0) = \gamma(1) = 0\}.$$

These are the spaces to be used in the variational approximation of (2.9).

Piecewise linear interpolants. Given $\gamma \in \overset{\circ}{W}{}_\infty^1(0, 1)$, $1 \leq p \leq \infty$, we define $I_h\gamma$ to be the usual piecewise linear interpolant, based on the knots $\{\xi_i: 1 \leq i \leq l_h\}$, from $C^0(0, 1)$ to $\{\gamma \in C^0(0, 1): \gamma|_{[\xi_{i-1}, \xi_i]} \in \mathbf{P}^1([\xi_{i-1}, \xi_i])\}$. If $\gamma \in W_\infty^2(0, 1)$ $\cap \overset{\circ}{W}{}_\infty^1(0, 1)$, the following estimate holds:

$$(3.2) \qquad \|\gamma - I_h\gamma\|_{W_\infty^1(0, 1)} \leq Ch \, \|\gamma\|_{W_\infty^2(0, 1)}.$$

For functions $v \in W_p^1(\Omega_\gamma^h)$ with $p > 2$ we shall use the usual piecewise linear interpolant $I_h^\gamma: W_p^1(\Omega_\gamma^h) \to V_h^\gamma$, namely

$$(3.3) \qquad I_h^\gamma v = \sum_{k=1}^{m_h} v(\mathbf{n}_k) \, v_k^h,$$

where $\{v_k^h: 1 \leq k \leq m_h\}$ is the usual Lagrange basis of V_h^γ defined by

$$v_k^h(\mathbf{n}_{k'}) = \delta_{kk'}, \qquad 1 \leq k, k' \leq m_h.$$

Note that, for any $v \in W_p^1(\Omega_\gamma^h)$,

$$(3.4) \qquad \widetilde{I_h^\gamma v} = I_h^0 \hat{v} = I_h^0 \tilde{v}.$$

As usual, we have

$$(3.5) \qquad \|w - I_h^0 w\|_{W_p^1(\Omega_0)} \leq Ch\|w\|_{W_p^2(\Omega_0)}.$$

Also note that $I_h(v|_{\{(x, 1):0 \leq x \leq 1\}}) = (I_h^0 v)|_{\{(x, 1):0 \leq x \leq 1\}}$, so our notations for the boundary and interior interpolants do not conflict. Finally, let $g_h^\gamma = I_h^\gamma g$; if $g \in W_p^2(\Omega^*)$ for $p > 2$, then

$$(3.6) \qquad \|g_h^\gamma\|_{W_p^1(\Omega_\gamma^h)} \leq C\|g\|_{W_p^1(\Omega^*)}.$$

Let $\chi_h \in S_h$. As $\chi_h \in W_p^1(0,1)$, for $1 \leq p \leq \infty$, we can consider the extension, $E\chi_h$, of χ_h to Ω_0 defined in the proof of Lemma 2.0. We know from the continuous case, (2.2), that $E\chi_h \in W_q^1(\Omega_0)$ for $q < 2$. In order to construct a piecewise linear interpolant of this extension to V_h^0, we shall use a variant of a piecewise linear interpolant for "rough functions," which was introduced by Clement [9]. The presentation that appears in Scott and Zhang [25] will be used. They show that there is an interpolant, J_h, that is given by

$$J_h^\gamma v = \sum_{k=1}^{m_h} L_k(v)\, v_k^h,$$

where L_k is a linear functional given by a weighted integral along an edge in the triangulation having $\mathbf{n}_k$ as a node. We can pick these edges arbitrarily, so that for all boundary nodes, $\mathbf{n}_k \in \{(x,1): 0 \leq x \leq 1\}$, we can require all the edges to lie on the boundary segment $\{(x,1): 0 \leq x \leq 1\}$. Moreover, J_h is a projection that reproduces any piecewise linear function locally; in particular, $J_h E\chi$ is an extension of χ to V_h^0. Finally, it is shown in [25] that

(3.7)
$$\|J_h v\|_{W_q^1(\Omega_0)} \leq C\, \|v\|_{W_q^1(\Omega_0)},$$
$$\|v - J_h v\|_{W_q^1(\Omega_0)} \leq C\, h\, \|v\|_{W_q^2(\Omega_0)},$$

where C is a constant independent of h or v, and $1 \leq q \leq \infty$. Now we can define the "discrete extension" $E_\gamma^h \chi_h$ of χ_h to the whole domain Ω_γ^h by

$$\widetilde{E_\gamma^h \chi_h} = J_h E\chi_h.$$

Clearly, $E_\gamma^h \chi_h \in V_h^\gamma$, and from inequalities (3.7) and (3.1) we can assert the existence of a positive constant C, independent of h and χ_h, such that

(3.8) $$\|E_\gamma^h \chi_h\|_{W_q^1(\Omega_\gamma^h)} \leq C\mu(\|\gamma\|_{W_\infty^1(0,1)}) \|\chi_h\|_{W_1^1(0,1)}.$$

The discrete free-boundary problem. We pose the following discrete problem associated with the quasi-uniform triangulation π_h^0.

Find $\gamma_h \in S_h$ and $u_h \in g_h^{\gamma_h} \oplus V_h^{\gamma_h}$ such that

(3.9a) $$a_{\gamma_h}(u_h, v_h) = 0 \quad \forall v_h \in \mathring{V}_h^{\gamma_h},$$

(3.9b) $$b(\gamma_h, \chi) = a_{\gamma_h}(u_h, E_{\gamma_h}^h \chi) \quad \forall \chi \in S_h.$$

Remark. In view of (3.9a), the right-hand side of equation (3.9b) can be written as

$$a_{\gamma_h}(u_h, E_{\gamma_h}^h \chi) = a_{\gamma_h}(u_h, E_h \chi)$$

for any extension E_h, e.g., we can take

$$E_h \chi = \sum_{\mathbf{n}_k \in \{(x,1):\, 0 \leq x \leq 1\}} \chi(\xi_k) v_k^h,$$

since $E_h\chi - E_\gamma^h\chi \in \overset{\circ}{W}{}_q^1(\Omega_\gamma)$. Thus, the right-hand side of equation (3.9b) can be easily computed.

To prove the existence of a solution of the discrete problem (3.9), we need to prove a discrete version of Propositions 2.2 and 2.3.

Proposition 3.2. *There exists a positive constant β, independent of h, such that for any $\gamma_h \in S_h$*

(3.10)
$$\|\gamma_h\|_{W_\infty^1(0,1)} \le \beta \sup_{0 \ne \chi_h \in S_h} \frac{b(\gamma_h, \chi_h)}{\|\chi_h\|_{W_1^1(0,1)}}.$$

Proof. Let P_h denote the projection from $W_1^1(0,1)$ to S_h defined by

$$b(P_h\chi - \chi, \sigma) = 0 \quad \forall \sigma \in S_h.$$

The results of Douglas, Dupont, and Wahlbin [11] imply that for any $\chi \in W_p^1(0,1)$

(3.11)
$$\|P_h\chi\|_{W_p^1(0,1)} \le C \|\chi\|_{W_p^1(0,1)}, \qquad 1 \le p \le \infty.$$

For $\gamma_h \in \overset{\circ}{W}{}_\infty^1(0,1)$, inequalities (2.10) and (3.11) with $p = 1$ imply that

(3.12)
$$\|\gamma_h\|_{W_\infty^1(0,1)} \le \beta \sup_{0 \ne \chi \in \overset{\circ}{W}{}_1^1(0,1)} \frac{b(\gamma_h, \chi)}{\|\chi\|_{W_1^1(0,1)}} = \beta \sup_{0 \ne \chi \in \overset{\circ}{W}{}_1^1(0,1)} \frac{b(\gamma_h, P_h\chi)}{\|\chi\|_{W_1^1(0,1)}}$$
$$\le C\beta \sup_{0 \ne \chi \in \overset{\circ}{W}{}_1^1(0,1)} \frac{b(\gamma_h, P_h\chi)}{\|P_h\chi\|_{W_1^1(0,1)}} = C\beta \sup_{0 \ne \chi_h \in S_h} \frac{b(\gamma_h, \chi_h)}{\|\chi_h\|_{W_1^1(0,1)}}.$$

Choosing the new $\beta = C\beta$ gives us inequality (3.10). □

We now prove an inequality analogous to (2.11) for the discrete case. Since this result is a special case of a result of independent interest, namely a stability bound for the Ritz projection with respect to a bilinear form having only bounded, measurable coefficients, we give the more general version first.

Proposition 3.3. *Suppose that Ω_0 is a convex polygon and that the bilinear form*

$$a(u,v) = \int_{\Omega_0} \sum_{i,j=1}^{2} A_{ij}(\xi,\eta)\, \partial_i u\, \partial_j v\, d\xi d\eta$$

satisfies the conditions (2.14) and (2.15). Let $\overset{\circ}{V}{}_h^0$ denote piecewise linear functions, on any quasi-uniform mesh, that vanish on the boundary. Then there are constants $\alpha < \infty$, $h_0 > 0$ and $2 < P$ such that for all $0 < h \le h_0$ and $u_h \in \overset{\circ}{V}{}_h^0$

$$\|u_h\|_{W_p^1(\Omega_0)} \le \alpha \sup_{0 \ne v_h \in \overset{\circ}{V}{}_h^0} \frac{a(u_h, v_h)}{\|v_h\|_{W_q^1(\Omega_0)}},$$

whenever $2 \le p \le P$ (q is the dual index to p, i.e., $q = p/p - 1$).

Let $R_h u$ be the Ritz projection with respect to the bilinear form $a(u, v)$ of an element $u \in \overset{\circ}{W}\,^1_2(\Omega_0)$ on $\overset{\circ}{V}\,^0_h$, i.e., $R_h u$ is the unique element of $\overset{\circ}{V}\,^0_h$ which satisfies

$$a(u - R_h u, v) = 0 \quad \forall\, v \in \overset{\circ}{W}\,^1_2(\Omega_0).$$

Corollary 3.0. *Under the assumptions of Proposition* 3.3, *the Ritz projection is stable in* $W^1_p(\Omega_0)$, *i.e., there is a positive constant* C, *independent of* h *and* u, *such that*

$$\|R_h u\|_{W^1_p(\Omega_0)} \leq C \,\|u\|_{W^1_p(\Omega_0)}, \quad 2 \leq p \leq P.$$

This follows from Proposition 3.3 by applying the definition of R_h and Hölder's inequality:

$$
\begin{aligned}
\|R_h u\|_{W^1_p(\Omega_0)} &\leq \alpha \sup_{0 \neq v_h \in \overset{\circ}{V}\,^0_h} \frac{a(R_h u, v_h)}{\|v_h\|_{W^1_q(\Omega_0)}} \\
&= \alpha \sup_{0 \neq v_h \in \overset{\circ}{V}\,^0_h} \frac{a(u, v_h)}{\|v_h\|_{W^1_q(\Omega_0)}} \leq C \,\|u\|_{W^1_p(\Omega_0)}.
\end{aligned}
$$

In view of the discussion in the proof of Proposition 2.3, the following is an immediate consequence of Proposition 3.3.

Proposition 3.4. *Suppose* $\|\gamma\|_{W^1_\infty(0,1)} < 1/2$. *Then there are constants* $\alpha < \infty$ *and* $2 < P$, *independent of* h, *such that for all* $u_h \in \overset{\circ}{V}\,^0_h$

$$(3.13) \qquad \|u_h\|_{W^1_p(\Omega_0)} \leq \alpha \sup_{0 \neq v_h \in \overset{\circ}{V}\,^0_h} \frac{\hat{a}(u_h, v_h; \gamma)}{\|v_h\|_{W^1_q(\Omega_0)}},$$

whenever $2 \leq p \leq P$.

Proof of Proposition 3.3. The proof is based on the ideas of Meyers [17]. First we establish the corresponding inequality in the case that the bilinear form in question is much simpler. Consider the Ritz projection $R^*_h : \overset{\circ}{W}\,^1_2 \to \overset{\circ}{V}\,^0_h$ defined by

$$\langle \nabla(R^*_h u - u), \nabla v_h \rangle = 0 \quad \forall v_h \in \overset{\circ}{V}\,^0_h,$$

where $\langle \cdot, \cdot \rangle$ denotes the vector-L^2 inner product on Ω_0. Rannacher and Scott [21] proved that

$$(3.14) \qquad \|R^*_h u\|_{\overset{\circ}{W}\,^1_p(\Omega_0)} \leq C^* \|u\|_{\overset{\circ}{W}\,^1_p(\Omega_0)}, \qquad 2 \leq p \leq \infty.$$

An argument similar to (3.12), using (2.11) or (2.16) with $\gamma \equiv 0$, shows that

$$(3.15) \qquad \|u_h\|_{\overset{\circ}{W}\,^1_p(\Omega_0)} \leq K(p) \sup_{0 \neq v_h \in \overset{\circ}{V}\,^0_h} \frac{\langle \nabla u_h, \nabla v_h \rangle}{\|v_h\|_{\overset{\circ}{W}\,^1_q(\Omega_0)}}$$

for all $1 \leq p \leq \infty$, where

$$K(p) := \alpha_p \|R_h^*\|_{\overset{\circ}{W}_p^1(\Omega_0) \to \overset{\circ}{W}_p^1(\Omega_0)} = \alpha_p \sup_{0 \neq u \in \overset{\circ}{W}_p^1(\Omega_0)} \frac{\|R_h^* u\|_{\overset{\circ}{W}_p^1(\Omega_0)}}{\|u\|_{\overset{\circ}{W}_p^1(\Omega_0)}}.$$

Here, we recall that

$$\|u\|_{\overset{\circ}{W}_p^1(\Omega_0)} := \left(\int_{\Omega_0} |\nabla u|^p \, dx \right)^{1/p},$$

and we note that the constant α_p from (2.16) was observed to be log-convex as a function of $1/p$ by Meyers [17].

We now wish to show that $\|R_h^*\|_{\overset{\circ}{W}_p^1(\Omega_0) \to \overset{\circ}{W}_p^1(\Omega_0)}$ is continuous as a function of p. To use Banach space interpolation theory, we view R_h^* as inducing a mapping, $\mathscr{R}_h^* : \nabla u \longrightarrow \nabla u_h$, of $L^p(\Omega)^2$ to itself. More precisely, given $\mathbf{F} \in L^p(\Omega)^2$, let $u_h(\mathbf{F}) \in \overset{\circ}{V}_h^0$ solve

$$\langle \nabla u_h, \nabla v_h \rangle = \langle \mathbf{F}, \nabla v_h \rangle \quad \forall v_h \in \overset{\circ}{V}_h^0.$$

Then $\mathscr{R}_h^*(\mathbf{F}) := \nabla u_h$. Similarly, we can let $u \in \overset{\circ}{W}_p^1(\Omega_0)$ solve

$$\langle \nabla u, \nabla v \rangle = \langle \mathbf{F}, \nabla v \rangle \quad \forall v \in \overset{\circ}{W}_q^1(\Omega_0),$$

and we have from [16] that

$$\|u\|_{\overset{\circ}{W}_p^1(\Omega_0)} \leq \alpha_p \|\mathbf{F}\|_{L^p(\Omega)^2}.$$

Therefore,

$$\|\mathscr{R}_h^* \mathbf{F}\|_{L^p(\Omega)^2} \leq C(p) \|\mathbf{F}\|_{L^p(\Omega)^2},$$

where $C(p) \leq C^* \alpha_p$ and $C(2) = 1$.

Using operator interpolation, we conclude that

$$\|\mathscr{R}_h^* \mathbf{F}\|_{[L^2(\Omega)^2, L^p(\Omega)^2]_\theta} \leq C(P)^\theta \|\mathbf{F}\|_{[L^2(\Omega)^2, L^p(\Omega)^2]_\theta}.$$

Whether using the real (with appropriate second index) or complex interpolation method [6],

$$c(P)^{-1} \|\mathbf{F}\|_{L^{2(1-\theta)+P\theta}(\Omega)^2} \leq \|\mathbf{F}\|_{[L^2(\Omega)^2, L^p(\Omega)^2]_\theta} \leq c(P) \|\mathbf{F}\|_{L^{2(1-\theta)+P\theta}(\Omega)^2},$$

where $c(2) = 1$ and $c(p)$ is a continuous function of p near $p = 2$. Therefore,

$$\|R_h^*\|_{\overset{\circ}{W}_p^1(\Omega_0) \to \overset{\circ}{W}_p^1(\Omega_0)} \leq c(P)^2 C(P)^\theta, \qquad p = 2(1-\theta) + P\theta.$$

Thus, for all $\varepsilon > 0$ there exists $P > 2$ such that $K(p) \leq 1 + \varepsilon$ for $2 < p < P$, where ε and P are independent of h.

We now consider the general case via a perturbation argument. Define a bilinear form $\mathbb{B} : W_p^1(\Omega_0) \times W_q^1(\Omega_0) \to \mathbb{R}$ by

$$\mathbb{B}(u, v) := \langle \nabla u, \nabla v \rangle - \frac{1}{M} a(u, v),$$

where M satisfies (2.15). It follows from (2.14)-(2.15) that

$$\mathbb{B}(u, v) = \int_{\Omega_0} \sum_{i,j=1}^{2} B_{ij}(\xi, \eta) \, \partial_i u \, \partial_j v \, d\xi d\eta,$$

where the eigenvalues of B are in $[0, 1 - \Lambda/M]$ a.e. and Λ satisfies (2.14). Note that $\Lambda/M = 1/M^2 \le 1/2$. Therefore, Hölder's inequality implies

$$(3.16) \qquad |\mathbb{B}(u_h, v_h)| \le \kappa_p \left(1 - \frac{\Lambda}{M} \right) \|u_h\|_{\mathring{W}_p^1(\Omega_0)} \|v_h\|_{\mathring{W}_q^1(\Omega_0)},$$

where κ_p is a smooth function of p (near $p = 2$) satisfying $\kappa_2 = 1$. Using the identity

$$\langle \nabla u, \nabla v \rangle = \mathbb{B}(u, v) + \frac{1}{M} a(u, v),$$

together with estimates (3.15) and (3.16), yields

$$\left(\frac{1}{K(p)} - \kappa_p \left(1 - \frac{\Lambda}{M} \right) \right) \|u_h\|_{\mathring{W}_p^1(\Omega_0)} \le \frac{1}{M} \sup_{0 \ne v_h \in \mathring{V}_h^0} \frac{a(u_h, v_h)}{\|v_h\|_{\mathring{W}_q^1(\Omega_0)}}.$$

In view of the continuity of K and κ and the fact that $K(2) = \kappa_2 = 1$, there exists $P > 2$ such that for any $p \in [2, P]$, $1/K(p) - \kappa_p(1 - \Lambda/M) > 0$. Choosing

$$\alpha(p) = \frac{K(p)}{M[1 - \kappa_p(1 - \Lambda/M)K(p)]},$$

we complete the proof using the equivalence of norms

$$\| \cdot \|_{W_p^1(\Omega_0)} \simeq \| \cdot \|_{\mathring{W}_p^1(\Omega_0)} \quad \text{and} \quad \| \cdot \|_{W_q^1(\Omega_0)} \simeq \| \cdot \|_{\mathring{W}_q^1(\Omega_0)}$$

for functions vanishing on the boundary. $\square$

Proposition 3.5. *Suppose* $\|\gamma_h\|_{W_\infty^1(0,1)} < 1/2$. *Then there are constants* $\alpha < \infty$, $2 < P$ *and* $h_0 > 0$ *such that for all* $0 < h \le h_0$ *and for any* $u_h \in \mathring{V}_h^{\gamma_h}$

$$(3.17) \qquad \|u_h\|_{W_p^1(\Omega_{\gamma_h}^h)} \le \alpha \sup_{0 \ne v_h \in \mathring{V}_h^{\gamma_h}} \frac{a_{\gamma_h}(u_h, v_h)}{\|v_h\|_{W_q^1(\Omega_{\gamma_h}^h)}},$$

whenever $2 \le p \le P$.

Proof. Transform the bilinear form a_{γ_h} to a bilinear form in Ω_0, that is,

(3.18)
$$a_{\gamma_h}(u_h, v_h) = \sum_{i=1}^{N_h} \int_K \nabla u_h|_K \cdot \nabla v_h|_K \, dxdy$$
$$= \sum_{i=1}^{N_h} \int_{\widetilde{K}} \sum_{l,m=1}^{2} A_{lm}^h(\xi, \eta; \gamma_h) \, \partial_l \tilde{u}_h \partial_m \tilde{v}_h \, d\xi d\eta$$
$$=: \tilde{a}_h(\tilde{u}_h, \tilde{v}_h; \gamma_h),$$

where $A^h|_K = (D_K D_K^T)^{-1} |\det D_K|$ (cf. Proposition 3.1). In view of Proposition 3.1, we may apply Proposition 3.3 again. The proposition follows from the equivalence, (3.1), of the norms $\|v\|_{W_q^1(\Omega_\gamma^h)}$ and $\|\tilde{v}\|_{W_q^1(\Omega_0)}$. $\quad\square$

Define
$$V_\varepsilon^h = \{(\gamma_h, u_h) \in S_h \times V_h^{\gamma_h}: \|\gamma_h\|_{W_\infty^1(0,1)} < 1/2, \ \|u_h\|_{W_p^1(\Omega_{\gamma_h}^h)} < \varepsilon\},$$

and consider the transformation T_h that associates with an element $(\gamma_h, u_h) \in V_\varepsilon^h$ an element (γ_h^*, u_h^*) which satisfies

(3.19)
$$b(\gamma_h^*, \chi_h) = a_{\gamma_h}(u_h, E_{\gamma_h}^h \chi_h) \quad \forall \chi_h \in S_h,$$
$$u_h^* \in g_h^{\gamma_h^*} \oplus \mathring{V}_h^{\gamma_h} \quad \text{such that} \quad a_{\gamma_h^*}(u_h^*, v_h) = 0 \quad \forall v_h \in \mathring{V}_h^{\gamma_h}.$$

Proposition 3.6. *Let p be as in Theorem 2.1. There exist two strictly positive constants δ and ε such that if*
$$\|g\|_{W_p^1(\Omega^*)} < \delta,$$

then T_h is a mapping from $V_\varepsilon^h \to V_\varepsilon^h$.

Proof. Let $(\gamma_h, u_h) \in V_\varepsilon^h$. The variational problem, to find $\gamma_h^* \in S_h$ such that
$$b(\gamma_h^*, \chi_h) = a_{\gamma_h}(u_h, E_{\gamma_h}^h \chi_h) \quad \forall \chi_h \in S_h,$$

admits a unique solution in S_h because b is a positive-definite, bilinear form in $S_h \times S_h$ and the functional
$$\phi(\chi_h) = a_{\gamma_h}(u_h, E_{\gamma_h}^h \chi_h)$$

is continuous from S_h to $\mathbb{R}$. From inequalities (3.10) and (3.8) we have the following estimate on γ_h^*:

(3.20)
$$\|\gamma_h^*\|_{W_\infty^1(0,1)} \leq C\beta\mu(\|\gamma_h\|_{W_\infty^1(0,1)}) \|u_h\|_{W_p^1(\Omega_{\gamma_h}^h)} \leq C\varepsilon.$$

Choose ε such that $C\varepsilon < 1/2$. Hence $\|\gamma_h^*\|_{W_\infty^1(0,1)} < 1/2$.

Next we bound u_h^*. As $a_{\gamma_h^*}$ is a positive-definite, continuous bilinear form on $V_h(\Omega_{\gamma_h^*})$, there exists a unique $w_h \in \mathring{V}_h(\Omega_{\gamma_h^*})$ such that
$$a_{\gamma_h^*}(w_h, v_h) = -a_{\gamma_h^*}(g_h^{\gamma_h^*}, v_h) \quad \forall v_h \in \mathring{V}_h^{\gamma_h^*}.$$

In view of inequality (3.20) for γ_h^*, Proposition 3.5 can be used to bound $\|w_h\|_{W_p^1(\Omega_{\gamma_h^*}^h)}$:

$$\|w_h\|_{W_p^1(\Omega_{\gamma_h^*}^h)} \le C\alpha \, \|g_h^{\gamma_h^*}\|_{W_p^1(\Omega_{\gamma_h^*}^h)}.$$

Let $u_h^* = w_h + g_h^{\gamma_h^*}$; then

$$\|u_h^*\|_{W_p^1(\Omega_{\gamma_h^*}^h)} \le (1 + C\alpha) \, \|g_h^{\gamma_h^*}\|_{W_p^1(\Omega_{\gamma_h^*}^h)}.$$

From inequalities (3.6) and (3.20) and the assumptions that $(\gamma_h, u_h) \in V_\varepsilon^h$ and $\|g\|_{W_p^1(\Omega^*)} < \delta$, we find that

$$\|u_h^*\|_{W_p^1(\Omega_{\gamma_h^*})} \le C\delta.$$

Finally, choose δ such that $C\delta < \varepsilon$. Hence $(\gamma_h^*, u_h^*) \in V_\varepsilon^h$. $\square$

Theorem 3.1. *If the norm of g is small enough, T_h is a contraction from $\widetilde{V}_\varepsilon^h \to \widetilde{V}_\varepsilon^h$ for the norm $\|\|\cdot\|\|$.*

The definition of $\widetilde{V}_\varepsilon^h$ and the proof of this result are similar to the continuous case.

Corollary 3.1. *If $\|g\|_{W_p^1(\Omega^*)}$ is sufficiently small, problem (3.9) admits a unique solution in V_ε^h which can be calculated by a fixed point iteration,*

$$(\gamma_h^{i+1}, u_h^{i+1}) = T(\gamma_h^i, u_h^i),$$

whenever $(\gamma_h^0, u_h^0) \in V_\varepsilon^h$.

As initial data (γ_h^0, u_h^0), we can choose $\gamma_h^0 \equiv 0$ in $[0, 1]$ and u_h^0 as the Ritz projection, $R_h^* u^0$, of the solution, u^0, of problem (2.22). Clearly, for h small enough, $(u_h^0, \gamma_h^0) \in V_\varepsilon^h$.

Convergence results.

Theorem 3.2. *If the small-norm solution $(\gamma, u) \in V_\varepsilon$ of problem (2.9) for $p > 2$ satisfies $\gamma \in W_\infty^2(0, 1)$ and ε is sufficiently small, there is a constant $C < \infty$ and a strictly positive h_0 such that for all $0 < h \le h_0$*

(3.21)
$$\|\gamma - \gamma_h\|_{W_p^1(0, 1)} + \|\hat{u} - \tilde{u}_h\|_{W_p^1(\Omega_0)}$$
$$\le C\Big(h\|\gamma\|_{W_\infty^2(0, 1)} + \inf_{\tilde{v} \in \tilde{g}_h^\gamma \oplus \overset{\circ}{V}_h^0} \|\hat{u} - \tilde{v}\|_{W_p^1(\Omega_0)} \Big),$$

where (u_h, γ_h) is the small-norm solution of the discrete problem (3.9).

Proof. Let $(\gamma, u) \in V_{\varepsilon_1}$ be the solution of problem (2.9), and let $(\gamma_h, u_h) \in V_{\varepsilon_2}^h$ be the solution of problem (3.9). Since $E\chi_h - J_h E\chi_h \in \overset{\circ}{W}_q^1(\Omega_0)$, we have

(3.22)
$$
\begin{aligned}
b(\gamma - \gamma_h, \chi_h) &= a_\gamma(u, E_\gamma \chi_h) - a_{\gamma_h}(u_h, E_{\gamma_h}^h h\chi_h) \\
&= \hat{a}(\hat{u}, E\chi_h; \gamma) - \tilde{a}_h(\tilde{u}_h, J_h E\chi_h; \gamma_h) \\
&= \hat{a}(\hat{u}, J_h E\chi_h; \gamma) - \tilde{a}_h(\tilde{u}_h, J_h E\chi_h; \gamma_h) \\
&= \hat{a}(\hat{u}, J_h E\chi_h; \gamma) - \tilde{a}_h(\hat{u}, J_h E\chi_h; \gamma_h) \\
&\quad + \tilde{a}_h(\hat{u} - \tilde{u}_h, J_h E\chi_h; \gamma_h) \\
&= \int_{\Omega_0} \sum_{i,j=1}^2 (A_{ij}(\xi, \eta; \gamma) - A_{ij}^h(\xi, \eta; \gamma_h)) \, \partial_i \hat{u} \, \partial_j (J_h E\chi_h) \, d\xi d\eta \\
&\quad + \tilde{a}_h(\hat{u} - \tilde{u}_h, J_h E\chi_h; \gamma_h) \\
&= \int_{\Omega_0} \sum_{i,j=1}^2 (A_{ij}(\xi, \eta; \gamma) - A_{ij}^h(\xi, \eta; \gamma)) \, \partial_i \hat{u} \, \partial_j (J_h E\chi_h) \, d\xi d\eta \\
&\quad + \int_{\Omega_0} \sum_{i,j=1}^2 (A_{ij}^h(\xi, \eta; \gamma) - A_{ij}^h(\xi, \eta; \gamma_h)) \, \partial_i \hat{u} \, \partial_j (J_h E\chi_h) \, d\xi d\eta \\
&\quad + \tilde{a}_h(\hat{u} - \tilde{u}_h, J_h E\chi_h; \gamma_h).
\end{aligned}
$$

The representations for A and A^h (see (2.13) and (3.18), respectively) imply that

(3.23)
$$
\begin{aligned}
|A_{ij}(\xi, \eta; \gamma) - A_{ij}^h(\xi, \eta; \gamma)| &\le Ch\|\gamma\|_{W_\infty^2(0,1)}, \\
|A_{ij}^h(\xi, \eta; \gamma) - A_{ij}^h(\xi, \eta; \gamma_h)| &\le C\|\gamma - \gamma_h\|_{W_\infty^1(0,1)}.
\end{aligned}
$$

Combining the estimates (3.23) with the identity (3.22) yields

$$
\begin{aligned}
|b(\gamma - \gamma_h, \chi_h)| &\le C(h\|\gamma\|_{W_\infty^2(0,1)} + \|\gamma - \gamma_h\|_{W_\infty^1(0,1)}) \|\hat{u}\|_{W_p^1(\Omega_0)} \|J_h E\chi_h\|_{W_q^1(\Omega_0)} \\
&\quad + |a_h(\hat{u} - \tilde{u}_h, J_h E\chi_h; \gamma_h)| \\
&\le C((h\|\gamma\|_{W_\infty^2(0,1)} + \|\gamma - \gamma_h\|_{W_\infty^1(0,1)}) \|\hat{u}\|_{W_p^1(\Omega_0)} \\
&\qquad + \|\hat{u} - \tilde{u}_h\|_{W_p^1(\Omega_0)} \|\gamma_h\|_{W_\infty^1(0,1)}) \|J_h E\chi_h\|_{W_q^1(\Omega_0)} \\
&\le C((h\|\gamma\|_{W_\infty^2(0,1)} + \|\gamma - \gamma_h\|_{W_\infty^1(0,1)}) \|\hat{u}\|_{W_p^1(\Omega_0)} \\
&\qquad + \|\hat{u} - \tilde{u}_h\|_{W_p^1(\Omega_0)} \|\gamma_h\|_{W_\infty^1(0,1)}) \|\chi_h\|_{W_1^1(0,1)}.
\end{aligned}
$$

Let $\psi \in S_h$. Since

$$
b(\psi - \gamma_h, \chi_h) = b(\gamma - \gamma_h, \chi_h) - b(\gamma - \psi, \chi_h),
$$

Proposition 3.2 and Hölder's inequality imply that

$$
\begin{aligned}
\|\psi - \gamma_h\|_{W_\infty^1(0,1)} &\le C((h\|\gamma\|_{W_\infty^2(0,1)} + \|\gamma - \gamma_h\|_{W_\infty^1(0,1)}) \|\hat{u}\|_{W_p^1(\Omega_0)} \\
&\qquad + \|\hat{u} - \tilde{u}_h\|_{W_p^1(\Omega_0)} \|\gamma_h\|_{W_\infty^1(0,1)} + \|\gamma - \psi\|_{W_\infty^1(0,1)}).
\end{aligned}
$$

Subtracting and using the triangle inequality, we find

$$\|\gamma - \gamma_h\|_{W^1_\infty(0,1)}(1 - C\|u\|_{W^1_p(\Omega_0)}) \le C(h\|\gamma\|_{W^2_\infty(0,1)}\|u\|_{W^1_p(\Omega_0)}$$
$$+ \|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)}\|\gamma_h\|_{W^1_\infty(0,1)} + \|\gamma - \psi\|_{W^1_\infty(0,1)}).$$

Choosing ε appropriately, we find

$$\|\gamma - \gamma_h\|_{W^1_\infty(0,1)} \le C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)} + \|\gamma - \psi\|_{W^1_\infty(0,1)}).$$

Choosing $\psi = I_h\gamma$ and using (3.2), we have

$$\|\gamma - \gamma_h\|_{W^1_\infty(0,1)} \le C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)}).$$

Next we bound $\|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)}$. Using Proposition 3.5, or equivalently Proposition 3.3, we find for arbitrary $\tilde{v}_h \in g_h^{\gamma_h} \oplus \overset{\circ}{V}{}^0_h$ that

$$\|\tilde{v}_h - \tilde{u}_h\|_{W^1_p(\Omega_0)} \le C \sup_{0 \ne w_h \in \overset{\circ}{V}{}^0_h} \frac{\tilde{a}_h(\tilde{v}_h - \tilde{u}_h, w_h; \gamma_h)}{\|w_h\|_{W^1_q(\Omega_0)}}$$

$$= C \sup_{0 \ne w_h \in \overset{\circ}{V}{}^0_h} \frac{\tilde{a}_h(\tilde{v}_h, w_h; \gamma_h)}{\|w_h\|_{W^1_q(\Omega_0)}}$$

$$= C \sup_{0 \ne w_h \in \overset{\circ}{V}{}^0_h} \frac{\tilde{a}_h(\tilde{v}_h - \hat{u}, w_h; \gamma_h) + \tilde{a}_h(\hat{u}, w_h; \gamma_h)}{\|w_h\|_{W^1_q(\Omega_0)}}$$

$$= C \sup_{0 \ne w_h \in \overset{\circ}{V}{}^0_h} \frac{\tilde{a}_h(\tilde{v}_h - \hat{u}, w_h; \gamma_h) + \tilde{a}_h(\hat{u}, w_h; \gamma_h) - \hat{a}(\hat{u}, w_h; \gamma)}{\|w_h\|_{W^1_q(\Omega_0)}}.$$

As above, we have

$$|\tilde{a}_h(\hat{u}, w_h; \gamma_h) - \hat{a}(\hat{u}, w_h; \gamma)|$$

$$= \left| \int_{\Omega_0} \sum_{i,j=1}^2 (A^h_{ij}(\xi, \eta; \gamma_h) - A_{ij}(\xi, \eta; \gamma)) \, \partial_i\hat{u} \, \partial_j w_h \, d\xi d\eta \right|$$

$$\le C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\gamma - \gamma_h\|_{W^1_\infty(0,1)}) \|\hat{u}\|_{W^1_p(\Omega_0)} \|w_h\|_{W^1_q(\Omega_0)}.$$

By Hölder's inequality, we thus find

$$\|\tilde{v}_h - \tilde{u}_h\|_{W^1_p(\Omega_0)} \le C\|\tilde{v}_h - \hat{u}\|_{W^1_p(\Omega_0)}$$
$$+ C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\gamma - \gamma_h\|_{W^1_\infty(0,1)}) \|\hat{u}\|_{W^1_p(\Omega_0)}.$$

From the triangle inequality, we have

$$\|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)} \le (1 + C)\|\tilde{v}_h - \hat{u}\|_{W^1_p(\Omega_0)}$$
$$+ C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\gamma - \gamma_h\|_{W^1_\infty(0,1)}) \|\hat{u}\|_{W^1_p(\Omega_0)}.$$

Applying the previous estimate for $\gamma - \gamma_h$ and choosing ε sufficiently small, we find

$$\|\hat{u} - \tilde{u}_h\|_{W^1_p(\Omega_0)} \le C(h\|\gamma\|_{W^2_\infty(0,1)} + \|\tilde{v}_h - \hat{u}\|_{W^1_p(\Omega_0)}).$$

Combining this with the previous estimate for $\gamma - \gamma_h$ completes the proof. $\square$

Corollary 3.2. *If the small-norm solution* $(\gamma, u) \in V_\varepsilon$ *of problem* (2.9) *satisfies* $(\gamma, u) \in W_\infty^2(0, 1) \times W_p^2(\Omega_\gamma)$ *for* $p > 2$ *and* ε *is sufficiently small, there is a constant* $C < \infty$ *and a strictly positive* h_0 *such that for all* $0 < h \leq h_0$

$$\|\hat{u} - \tilde{u}_h\|_{W_p^1(\Omega_0)} + \|\gamma - \gamma_h\|_{W_\infty^1(0, 1)} \leq Ch(\|\gamma\|_{W_\infty^2(0, 1)} + \|\hat{u}\|_{W_p^2(\Omega_0)}).$$

Proof. We have $I_h^0 \hat{u} \in \tilde{g}_h^\gamma \oplus \overset{\circ}{V}_h^0$, so the result follows from (3.5). $\square$

4. CONCLUSIONS AND EXTENSIONS

We have demonstrated optimal-order convergence of a finite element approximation for a model free-boundary problem using piecewise linear approximation for both the field variable, u, and the free-surface representation function, γ. It would therefore appear appropriate to choose the order of approximation to be the same in this case.

We can extend this work to models involving the curvature of the free surface, instead of the expression given in the third line of (1.1). This requires only the introduction of a nonlinearity in the bilinear form, b. The definition of b depends on the chosen parameterization of the free surface, but one particular case would be

$$b(\gamma, \chi) := s \int_0^1 \frac{\gamma'(x)\chi'(x)}{[1 + \gamma'(x)^2]^{1/2}} \, dx.$$

The model in differential form corresponds to the following equation for the free surface:

$$-s \left(\frac{\gamma'(x)}{[1 + \gamma'(x)^2]^{1/2}} \right)' = \frac{\partial u(x, 1 + \gamma(x))}{\partial \vec{n}} [1 + \gamma'(x)^2]^{1/2} \quad \forall x \in [0, 1].$$

With this replacement, the results in the previous sections would remain the same. If the equation for the free surface were

$$-s \left(\frac{\gamma'(x)}{[1 + \gamma'(x)^2]^{1/2}} \right)' = \frac{\partial u(x, 1 + \gamma(x))}{\partial \vec{n}} \quad \forall x \in [0, 1],$$

the extension operator E_γ would be more complicated, namely, $E_\gamma \chi$ would be an extension of $\chi/[1 + \gamma'(x)^2]^{1/2}$ instead of just χ, but the results would be similar. Other models (cf. [16]) could be treated similarly.

Models for which the derivative of γ are specified at the endpoints of the free surface are more complicated. Although Lemma 2.0 implies that the solution to both the continuous and discrete models will exist in this case, it is not clear that convergence will be optimal-order. The corresponding difficulty arises at the first step of the proof of Theorem 3.2 in that we can no longer assure that $E\chi_h - J_h E\chi_h \in \overset{\circ}{W}_q^1(\Omega_0)$. The discrete extension as defined here would necessarily be nonzero on one edge on the side boundary (where $E\chi_h$ vanishes),

assuming the test function, χ_h, were not to vanish there. It may be possible to avoid this difficulty in some way, either by a different analysis or by a different choice of discrete extension. However, at the moment this is an open problem.

We wish to develop a theory for more general models of viscous, incompressible flow. Formally, the techniques used to convert the free-boundary problem (1.1) in differential form can be used to develop a variational form for the full Navier-Stokes equations. However, a key result needed to prove the stability of the Galerkin method for the problem (1.1) is the bound for the maximum norm of the gradient for the Galerkin method applied to a scalar elliptic prob-, lem [21]. Such bounds are not yet known for the Stokes equations (cf. [12]), so such estimates need to be derived or an alternative technique needs to be developed. Another extension we wish to make is by proving local (quadratic) convergence of the Newton iteration (cf. [24]) and other efficient iterative procedures for solving the discrete system corresponding to models such as these. Finally, variational formulations need to be developed for problems involving boundary data that are allowed to be nonzero at the point of attachment of the free surface to the boundary on which Dirichlet conditions are imposed. If care is not taken, such boundary conditions can be incompatible, leading to singular solutions.

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