MIXED FINITE ELEMENT METHODS FOR COMPRESSIBLE MISCIBLE DISPLACEMENT IN POROUS MEDIA

SO-HSIANG CHOU AND QIAN LI

ABSTRACT. A differential system describing compressible miscible displacement in a porous medium is given. The concentration equation is treated by a Galerkin method and the pressure equation is treated by a parabolic mixed finite element method. Optimal-order estimates in L^2 and almost optimal-order estimates in L^∞ are obtained for the errors in the approximate solutions under the condition that $h_p^{2k+2}(\log h_c^{-1})^{1/2} \to 0$. This condition is much weaker than one given earlier by Douglas and Roberts for the same model. Furthermore, we obtain the $L^\infty(L^2(\Omega))$-estimates for the time-derivatives of the concentration and the pressure, which were not given by the above authors. In addition, we also consider newer mixed spaces in two or three dimensions.

1. INTRODUCTION

We shall consider a two-component model for the single-phase, miscible displacement of one compressible fluid by another in a reservoir $\Omega \subset R^2$ of unit thickness. Let c_i denote the concentration of the ith component of the fluid mixture, $i = 1, 2$. Assume that the density ρ_i of the ith component and the pressure p satisfy the equation of state in the form $d\rho_i/\rho_i = z_i\, dp$, where z_i is the constant compressibility factor for the ith component. The Darcy velocity of the fluid is given by $u = -\frac{k}{\mu}\nabla p$, where $k = k(x)$ is the permeability of the medium and $\mu = \mu(c_1, c_2)$ is the concentration-dependent viscosity. Let D be a 2×2 matrix, $D = \varphi\, d_m I$, where $\varphi = \varphi(x)$ is the porosity of the rock, I is the identity matrix, and d_m is the coefficient of molecular diffusion. The model we consider is governed by the following differential system:

(1.1)
$$\text{(a)} \quad d(c)\frac{\partial p}{\partial t} + \nabla \cdot u = d(c)\frac{\partial p}{\partial t} - \nabla \cdot (a(c)\nabla p) = q,$$

$$\text{(b)} \quad \varphi\frac{\partial c}{\partial t} + b_1(c)\frac{\partial p}{\partial t} + u \cdot \nabla c - \nabla \cdot (D\nabla c) = (\hat{c} - c)q.$$

Here,

$$c = c_1 = 1 - c_2, \qquad a(c) = a(x, c) = k(x)\mu(c)^{-1},$$

$$b_1(c) = b_1(x, c) = \varphi(x)c_1 \left\{ z_1 - \sum_{j=1}^{2} z_j c_j \right\},$$

$$d(c) = d(x, c) = \varphi(x) \sum_{j=1}^{2} z_j c_j,$$

$$q = \text{the external volumetric flow rate},$$

and

$$\hat{c} = \text{the concentration of the 1st component in the external flow}.$$

We impose the no-flow conditions on the boundary:

(1.2)
    (a) $u \cdot v = 0$ on $\partial\Omega$,
    (b) $(D\nabla c - cu) \cdot v = 0$ on $\partial\Omega$,

where v is the outer normal to $\partial\Omega$. In addition, the initial conditions are

(1.3)
    (a) $p(x, 0) = p_0(x)$, $x \in \Omega$,
    (b) $c(x, 0) = c_0(x)$, $x \in \Omega$.

The differential system (1.1) under the conditions (1.2) and (1.3) has been carefully derived by Douglas and Roberts [4], using sound physical reasoning. In [4], two numerical schemes for approximating the solution of the system (1.1)–(1.3) were given. In both procedures the concentration equation (1.1b) was treated by a parabolic Galerkin procedure. In their second scheme the pressure equation was treated by a parabolic mixed finite element technique. The error analysis of the procedures was carried out under the assumptions that the solution is smooth, i.e., q is smoothly distributed, the coefficients are smooth, and the domain has at least the regularity required for a standard elliptic Neumann problem to have $H^2(\Omega)$-regularity, and more, if the piecewise-polynomial spaces used in the finite element procedures have degree greater than one. Furthermore, the coefficients a, d, φ are assumed to be bounded below positively.

The mixed space used in [4] is that of Raviart and Thomas. In this paper we also consider the newer spaces such as the BDM space [3] in two dimensions, the BDFM space [2], and the Nedelec space [8] in three dimensions.

Optimal L^2-estimates and quasi-optimal L^∞-estimates for the errors have been given in [4] under the conditions that the maximum diameters h_c and h_p for the "concentration finite element space" and the "pressure finite element space," respectively, satisfy certain restrictions [4, relations (4.18) and (4.28)]. One of the purposes of this paper is to show that one can replace these restrictions by a much weaker condition (see (4.31) below) and at the same time obtain optimal L^2-estimates and quasi-optimal L^∞-estimates. In addition, those optimal results hold for the newer spaces of [2, 3, 8]. Our analysis differs from that of [4] in one fundamental aspect. The mixed elliptic projection [4, equation

(4.1)] is replaced by a nonlinear version of the elliptic projection presented in [6].

The rest of this paper is organized as follows. In §2, we give the weak formulations of the system (1.1)–(1.3) and discuss the associated finite element spaces. Section 3 is devoted to the error estimates for the concentration equation. The error estimates for the pressure equation are derived in §4, and the main results of the paper are contained in Theorem 4.1. In §5 we extend our results to three dimensions. A relation (5.1) relating discretization parameters is found, and optimal results are proved under this relation.

2. WEAK FORMULATIONS AND FINITE ELEMENT APPROXIMATIONS

We begin by putting (1.1b) into the following equivalent form:

$$(2.1) \qquad \varphi \frac{\partial c}{\partial t} - \nabla \cdot (D \nabla c) + u \cdot \nabla c - b(c) \nabla \cdot u = g(c),$$

where $b(c) = b_1(c) d(c)^{-1}$. Let $(\cdot, \cdot)$ denote the inner product in $L^2(\Omega)$ or in $L^2(\Omega)^2$. Using Green's formula and (1.2), we have

$$-(b(c) \nabla \cdot u, z) = (b(c)u, \nabla z) + \left(\frac{\partial b}{\partial c} u \cdot \nabla c, z \right) + \left(\frac{\partial b}{\partial x} \cdot u, z \right).$$

Thus, the weak form of (2.1) is to find a map $c: J \to H^1(\Omega)$ such that

$$(2.2) \qquad \left(\varphi \frac{\partial c}{\partial t}, z \right) + (D \nabla c, \nabla z) + (b(c)u, \nabla z) + (e(c)u \cdot \nabla c, z)$$
$$+ (B(c) \cdot u, z) = (g(c), z), \qquad z \in H^1(\Omega),$$

where $e(c) = \frac{\partial b}{\partial c}(c) + 1$, $B(c) = \frac{\partial b}{\partial x} = 0$, and $J = [0, T]$.

Let $H(\mathrm{div}, \Omega) = \{v : v \in L^2(\Omega)^2, \nabla \cdot v \in L^2(\Omega)\}$, $V = \{v : v \in H(\mathrm{div}, \Omega), v \cdot v = 0 \text{ on } \partial\Omega\}$, and $W = L^2(\Omega)$. Then the pressure equation (1.1a) is equivalent to the saddle point problem [4]: find a map $\{u, p\}: J \to V \times W$ such that

$$(2.3) \qquad \begin{array}{ll} \text{(a)} & \left(d(c) \frac{\partial p}{\partial t}, w \right) + (\nabla \cdot u, w) = (q, w), \qquad w \in W, \\ \text{(b)} & (\alpha(c)u, v) - (\nabla \cdot v, p) = 0, \qquad\qquad v \in V, \end{array}$$

where $\alpha(c) = a(c)^{-1}$.

To handle the nonzero initial conditions imposed on p and c, we introduce the following transformations:

$$c(x, t) = c^*(x, t) + c_0(x), \qquad p(x, t) = p^*(x, t) + p_0(x),$$
$$(2.4) \qquad u(x, t) = [-a(c^* + c_0)\nabla(p^* + p_0) + a(c_0)\nabla p_0] - a(c_0)\nabla p_0$$
$$= u^*(x, t) + u_0(x).$$

The reason for introducing these transformations is to validate equations (3.16) and (4.22) below. Now (2.2) can be written as

$$
\left(\varphi\frac{\partial c^*}{\partial t}, z\right) + (D\nabla c^*, \nabla z) + (b^*(c^*)u^*, \nabla z) + (b^*(c^*)u_0, \nabla z)
$$

(2.5)
$$
+ (e^*(c^*)u^* \cdot \nabla c^*, z) + (e^*(c^*)u_0 \cdot \nabla c^*, z) + (A^*(c^*) \cdot u^*, z)
$$
$$
= (f^*(c^*), z) - (D\nabla c_0, \nabla z), \qquad z \in H^1(\Omega),
$$

where

$$
b^*(c^*) = b(c^* + c_0), \qquad e^*(c^*) = e(c^* + c_0),
$$
$$
A^*(c^*) = B(c^* + c_0) + e(c^* + c_0)\nabla c_0,
$$
$$
f^*(c^*) = g(c^* + c_0) - [e(c^* + c_0)\nabla c_0 + B(c^* + c_0)] \cdot u_0.
$$

Similarly, (2.3) can be expressed as

(a) $\left(d^*(c^*)\dfrac{\partial p^*}{\partial t}, w\right) + (\nabla \cdot u^*, w) = (r, w), \qquad w \in W,$

(2.6)

(b) $(\alpha^*(c^*)u^*, v) - (\nabla \cdot v, p^*)$
$\qquad = (\nabla \cdot v, p_0) - (\alpha^*(c^*)u_0, v), \qquad v \in V,$

where

$$
d^*(c^*) = d(c^* + c_0), \quad \alpha^*(c^*) = \alpha(c^* + c_0), \quad r = q - \nabla \cdot u_0.
$$

The initial conditions for c^*, p^*, and u^* can easily be seen to be

(a) $c^*(x, 0) = 0,$

(2.7)   (b) $p^*(x, 0) = 0,$

(c) $u^*(x, 0) = 0.$

Consequently, solving (1.1)–(1.3) is equivalent to solving (2.5)–(2.7). *For ease of notation, we shall drop the superscript * in (2.5)–(2.7) for the time being.*

Let $h = (h_c, h_p)$, where h_c and h_p are positive. Let $M_h = M_{h_c} \subset W^{1,\infty}(\Omega)$ be a standard finite element space associated with a quasi-regular polygonalization of Ω and piecewise polynomials of some fixed degree not exceeding l. Thus, all standard inverse relations hold on M_h and the approximation property

$$
(2.8) \qquad \inf_{z_h \in M_h} \|z - z_h\|_{1,q} \le M h_c^l \|z\|_{l+1,q}, \qquad z \in W^{l+1,q}(\Omega), \ 1 \le q \le \infty,
$$

holds, where $\|z\|_{k,q}$ is the norm in the Sobolev space $W^{k,q}(\Omega)$. In the following we also use $\|z\|_k$ for $\|z\|_{k,2}$ and $\|z\|$ for $\|z\|_{0,2}$.

Suppose that Ω is a polygonal domain. Given a quasi-uniform decomposition of Ω into triangles or rectangles of diameter less than h, there exist $V_h = V_h^k \subset V$ and $W_h = W_h^k \subset W$ for $k \ge 0$ such that the elements have diameters bounded by h_p. The boundary condition $v \cdot v = 0$ on $\partial\Omega$ is imposed

on V_h. Assume that the following approximation properties hold:

(2.9)

(a) $\displaystyle\inf_{v_h \in V_h} \|v - v_h\| = \inf_{v_h \in V_h} \|v - v_h\|_{L^2(\Omega)^2}$

$$\leq M h_p^m \|v\|_m, \qquad 1 \leq m \leq k+1,$$

(b) $\displaystyle\inf_{v_h \in V_h} \|\nabla \cdot (v - v_h)\| \leq M h_p^m (\|v\|_m + \|\nabla \cdot v\|_m), \qquad 1 \leq m \leq k,$

(c) $\displaystyle\inf_{w_h \in W_h} \|w - w_h\| \leq M h_p^m \|w\|_m, \qquad 1 \leq m \leq k.$

The spaces V_h^k and W_h^k can be the spaces of Raviart and Thomas [10] for $k \geq 0$, or the spaces of Brezzi, Douglas, and Marini [3] for $k \geq 1$. If the Raviart-Thomas spaces are used, (2.9b) and (2.9c) are also valid for $m = k+1$. The discrepancy in the range of the index m makes any attempt at handling the two spaces of the same index k simultaneously rather awkward. Hence, we will carry out the error analysis only for the Raviart-Thomas space of index k. From now on we shall assume $V_h \times W_h$ to be the space of [10]. To obtain results for the BDM space of [3], one needs only to replace every occurrence of k by $k-1$ in Theorem 4.1. This is so because the error analysis below depends only on (2.9) and (4.6) (with obvious modifications on the range of the index when the BDM space is used). Hence, any result in Theorem 4.1 obtained for the Raviart-Thomas space of index k is also valid for the BDM space of index $k - 1$. In other words, the error analysis for these two spaces are identical.

The semidiscrete finite element approximation for problem (2.5)–(2.7) amounts to finding a map $\{C, U, P\}: J \to M_h \times V_h \times W_h$ such that

(2.10)

(a) $\displaystyle\left(\varphi \frac{\partial C}{\partial t}, z\right) + (D\nabla C, \nabla z) + (b(C)U, \nabla z)$

$\qquad + (b(C)u_0, \nabla z) + (e(C)U \cdot \nabla C, z)$

$\qquad + (e(C)u_0 \cdot \nabla C, z) + (A(C) \cdot U, z)$

$\qquad = (f(C), z) - (D\nabla c_0, \nabla z), \qquad z \in M_h,$

(bi) $\displaystyle\left(d(C)\frac{\partial P}{\partial t}, w\right) + (\nabla \cdot U, w) = (r, w), \qquad w \in W_h,$

(bii) $(\alpha(C)U, v) - (\nabla \cdot v, P)$

$\qquad = (\nabla \cdot v, p_0) - (\alpha(C)u_0, v), \qquad v \in V_h,$

(c) $C(0) = 0, \qquad P(0) = 0.$

From (2.6) and (2.7), we have

(2.11) $\qquad (\nabla \cdot v, p_0) - (\alpha(0)u_0, v) = 0, \qquad v \in V.$

Then, setting $t = 0$ in (2.10bii) and using (2.10c) and (2.11), we obtain

(2.12) $\qquad\qquad\qquad U(0) = 0.$

3. ERROR ESTIMATES FOR THE CONCENTRATION EQUATION

In the analysis below, all functions of c are assumed to be extended outside $[0, 1]$ in some smooth fashion.

Let us introduce an elliptic projection of the solution of (2.5) onto M_h, which is a map $\bar{c}: J \rightarrow M_h$ defined by

(3.1)
$$\begin{aligned}(D\nabla(\bar{c} - c), \nabla z) &+ ((e(c)\nabla(\bar{c} - c)) \cdot (u + u_0), z) \\ &+ \lambda(\bar{c} - c, z) = 0, \qquad z \in M_h,\end{aligned}$$

where λ will be selected large enough to ensure the coerciveness of the bilinear form on $H^1(\Omega)$. By standard results on the finite element method for elliptic problems, and by [5, 7, 12, 13], we know that

(3.2)
$$\begin{aligned}\text{(a)} \quad &\|\bar{c} - c\| + \left\|\frac{\partial(\bar{c} - c)}{\partial t}\right\| + \left\|\frac{\partial^2}{\partial t^2}(\bar{c} - c)\right\| + h_c\|\nabla(\bar{c} - c)\| \\ &\leq Mh_c^{l+1}\left(\|c\|_{l+1} + \left\|\frac{\partial c}{\partial t}\right\|_{l+1} + \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{l+1}\right), \\ \text{(b)} \quad &\|\bar{c} - c\|_{0,\infty} \leq Mh_c^{l+1}(\log h_c^{-1})^r\|c\|_{l+1,\infty},\end{aligned}$$

where

$$r = \begin{cases} 0 & \text{when } l > 1, \\ 1 & \text{when } l = 1. \end{cases}$$

Using the inverse relations on M_h, we know for $z \in M_h$ that

$$\begin{aligned}\left\|\nabla\frac{\partial}{\partial t}\bar{c}\right\|_{0,\infty} &\leq \left\|\nabla\frac{\partial}{\partial t}(\bar{c} - z)\right\|_{0,\infty} + \left\|\nabla\frac{\partial}{\partial t}(c - z)\right\|_{0,\infty} + \left\|\nabla\frac{\partial}{\partial t}c\right\|_{0,\infty} \\ &\leq Kh_c^{-2}\left\|\frac{\partial}{\partial t}(\bar{c} - z)\right\| + \left\|\nabla\frac{\partial}{\partial t}(c - z)\right\|_{0,\infty} + \left\|\nabla\frac{\partial}{\partial t}c\right\|_{0,\infty} \\ &\leq Kh_c^{-2}\left(\left\|\frac{\partial}{\partial t}(\bar{c} - c)\right\| + \left\|\frac{\partial}{\partial t}(c - z)\right\|\right) + \left\|\nabla\frac{\partial}{\partial t}(c - z)\right\|_{0,\infty} \\ &\quad + \left\|\nabla\frac{\partial}{\partial t}c\right\|_{0,\infty}.\end{aligned}$$

Choosing z as the interpolant of c in M_h, we see from the above inequality that

$$\left\|\nabla\frac{\partial}{\partial t}\bar{c}\right\|_{0,\infty} \leq Kh_c^{l-1}\left(\|c\|_{l+1} + \left\|\frac{\partial c}{\partial t}\right\|_{l+1} + h_c^l\left\|\frac{\partial}{\partial t}c\right\|_{l+1,\infty}\right) + \left\|\nabla\frac{\partial}{\partial t}c\right\|_{0,\infty}.$$

This shows that $\|\nabla\frac{\partial}{\partial t}\bar{c}\|_{0,\infty}$ is bounded. In a similar fashion one can show that $\|\bar{c}\|_{1,\infty}$ and $\|\frac{\partial}{\partial t}\bar{c}\|_{0,\infty}$ are bounded.

Write $\xi = C - \bar{c}$, $\zeta = \bar{c} - c$, subtract (2.5) from (2.10a), and apply (3.1) to obtain

$$\left(\varphi \frac{\partial \xi}{\partial t}, z \right) + (D \nabla \xi, \nabla z)$$

(3.3)
$$= (f(C) - f(c), z) - \left(\varphi \frac{\partial \zeta}{\partial t}, z \right) - (b(C)U - b(c)u, \nabla z)$$
$$- ((b(C) - b(c))u_0, \nabla z) - (e(C)U \cdot \nabla C - e(c)u \cdot \nabla \bar{c}, z)$$
$$- ((e(C)\nabla C - e(c)\nabla \bar{c}) \cdot u_0, z)$$
$$- (A(C) \cdot U - A(c) \cdot u, z) + \lambda(\zeta, z), \qquad z \in M_h.$$

Differentiate equation (3.3) to show that

$$\left(\varphi \frac{\partial^2 \xi}{\partial t^2}, z \right) + \left(D \nabla \frac{\partial \xi}{\partial t}, \nabla z \right)$$

$$= \left(\frac{\partial}{\partial t}(f(C) - f(c)), z \right) - \left(\varphi \frac{\partial^2 \zeta}{\partial t^2}, z \right)$$

$$- \left(\frac{\partial}{\partial t}(b(C)U - b(c)u), \nabla z \right)$$

(3.4)
$$- \left(\frac{\partial}{\partial t}(b(C) - b(c))u_0, \nabla z \right)$$

$$- \left(\frac{\partial}{\partial t}(e(C)U \cdot \nabla C - e(c)u \cdot \nabla \bar{c}), z \right)$$

$$- \left(\frac{\partial}{\partial t}(e(C)\nabla C - e(c)\nabla \bar{c}) \cdot u_0, z \right)$$

$$- \left(\frac{\partial}{\partial t}(A(C) \cdot U - A(c) \cdot u), z \right) + \lambda \left(\frac{\partial \zeta}{\partial t}, z \right), \qquad z \in M_h.$$

By selecting the test function $z = \frac{\partial \xi}{\partial t}$, and using the boundedness of $\|\bar{c}\|_{0, \infty}$ and $\|\frac{\partial \bar{c}}{\partial t}\|_{1, \infty}$, it follows from (3.2) that

$$\frac{1}{2} \frac{d}{dt} \left(\varphi \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right) + \left(D \nabla \frac{\partial \xi}{\partial t}, \nabla \frac{\partial \xi}{\partial t} \right)$$

$$\leq K \left\{ \|\xi\|^2 + \left\| \frac{\partial \xi}{\partial t} \right\|^2 + h_c^{2l+2} \right\} - \left(\frac{\partial}{\partial t}(b(C)U - b(c)u), \nabla \frac{\partial \xi}{\partial t} \right)$$

$$- \left(\frac{\partial}{\partial t}(b(C) - b(c))u_0, \nabla \frac{\partial \xi}{\partial t} \right)$$

(3.5)
$$- \left(\frac{\partial}{\partial t}(e(C)U \cdot \nabla C - e(c)u \cdot \nabla \bar{c}), \frac{\partial \xi}{\partial t} \right)$$

$$- \left(\frac{\partial}{\partial t}(e(C)\nabla C - e(c)\nabla \bar{c}) \cdot u_0, \frac{\partial \xi}{\partial t} \right)$$

$$- \left(\frac{\partial}{\partial t}(A(C) \cdot U - A(c) \cdot u), \frac{\partial \xi}{\partial t} \right).$$

Next, observe from (3.2) and the inequality $ab \leq \varepsilon a^2 + b^2/4\varepsilon$ that

$$
\left| \left(\frac{\partial}{\partial t} (b(C)U - b(c)u), \nabla \frac{\partial \xi}{\partial t} \right) \right|
$$

$$
= \left| \left(\frac{\partial b}{\partial c}(C) \frac{\partial \xi}{\partial t}(U - u), \nabla \frac{\partial \xi}{\partial t} \right) \right.
$$

$$
+ \left(\frac{\partial b}{\partial c}(C) \frac{\partial \bar{c}}{\partial t}(U - u) + \frac{\partial b}{\partial c}(C) \frac{\partial(\xi + \zeta)}{\partial t} u \right.
$$

$$
+ \left(\frac{\partial b}{\partial c}(C) - \frac{\partial b}{\partial c}(c) \right) \frac{\partial c}{\partial t} u + b(C) \frac{\partial(U - u)}{\partial t}
$$

$$
\left. + (b(C) - b(c)) \frac{\partial u}{\partial t}, \nabla \frac{\partial \xi}{\partial t} \right) \right|
$$

(3.6)

$$
\leq K \left\| \nabla \frac{\partial \xi}{\partial t} \right\| \left\{ \left(\left\| \frac{\partial \xi}{\partial t} \right\|_{0,\infty} + 1 \right) \| U - u \| \right.
$$

$$
\left. + \| \xi \| + \left\| \frac{\partial \xi}{\partial t} \right\| + \left\| \frac{\partial(U - u)}{\partial t} \right\| + h_c^{l+1} \right\}
$$

$$
\leq \varepsilon \left\| \nabla \frac{\partial \xi}{\partial t} \right\|^2 + K \left\{ \| \xi \|^2 + \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \left(\left\| \frac{\partial \xi}{\partial t} \right\|_{0,\infty}^2 + 1 \right) \| U - u \|^2 \right.
$$

$$
\left. + \left\| \frac{\partial(U - u)}{\partial t} \right\|^2 + h_c^{2l+2} \right\}.
$$

Similarly,

$$
\left(\frac{\partial}{\partial t} (e(C)U \cdot \nabla C - e(c)u \cdot \nabla \bar{c}), \frac{\partial \xi}{\partial t} \right)
$$

$$
= \left(\frac{\partial e}{\partial c}(C) \frac{\partial C}{\partial t}(U - u) \cdot \nabla C + \frac{\partial e}{\partial c}(C) \frac{\partial C}{\partial t} u \cdot \nabla \xi \right.
$$

$$
+ \frac{\partial e}{\partial c}(C) \frac{\partial(\xi + \zeta)}{\partial t} u \cdot \nabla \bar{c} + \left(\frac{\partial e}{\partial c}(C) - \frac{\partial e}{\partial c}(c) \right) \frac{\partial c}{\partial t} u \cdot \nabla \bar{c}, \frac{\partial \xi}{\partial t} \right)
$$

(3.7)

$$
+ \left(e(C)U \cdot \nabla \frac{\partial \xi}{\partial t} + e(C)(U - u) \cdot \nabla \frac{\partial \bar{c}}{\partial t} \right.
$$

$$
\left. + (e(C) - e(c))u \cdot \nabla \frac{\partial \bar{c}}{\partial t}, \frac{\partial \xi}{\partial t} \right)
$$

$$
+ \left(e(C) \frac{\partial(U - u)}{\partial t} \cdot \nabla C + e(C) \frac{\partial u}{\partial t} \cdot \nabla \xi \right.
$$

$$
\left. + (e(C) - e(c)) \frac{\partial u}{\partial t} \cdot \nabla \bar{c}, \frac{\partial \xi}{\partial t} \right),
$$

where

$$\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial C}{\partial t}(U-u)\cdot\nabla C,\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$\leq\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial\xi}{\partial t}(U-u)\cdot\nabla\xi,\,\frac{\partial\xi}{\partial t}\right)\right|+\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial\bar{c}}{\partial t}(U-u)\cdot\nabla\xi,\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$+\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial\xi}{\partial t}(U-u)\cdot\nabla\bar{c},\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$(3.8)\qquad+\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial\bar{c}}{\partial t}(U-u)\cdot\nabla\bar{c},\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$\leq K\|U-u\|\left\{\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}\|\nabla\xi\|+\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}\left(\|\nabla\xi\|+\left\|\frac{\partial\xi}{\partial t}\right\|\right)+\left\|\frac{\partial\xi}{\partial t}\right\|\right\}$$

$$\leq K\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}+1\right)\left\{\|\nabla\xi\|^{2}+\left\|\frac{\partial\xi}{\partial t}\right\|^{2}+\|U-u\|^{2}\right\},$$

$$(3.9)\quad\left|\left(\frac{\partial e}{\partial c}(C)\frac{\partial C}{\partial t}u\cdot\nabla\xi,\,\frac{\partial\xi}{\partial t}\right)\right|\leq K\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}+1\right)\left\{\|\nabla\xi\|^{2}+\left\|\frac{\partial\xi}{\partial t}\right\|^{2}\right\},$$

$$\left|\left(e(C)U\cdot\nabla\frac{\partial\xi}{\partial t},\,\frac{\partial\xi}{\partial t}\right)\right|\leq K\left\|\nabla\frac{\partial\xi}{\partial t}\right\|\left\{\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}\cdot\|U-u\|+\left\|\frac{\partial\xi}{\partial t}\right\|\right\}$$

$$(3.10)\qquad\leq\varepsilon\left\|\nabla\frac{\partial\xi}{\partial t}\right\|^{2}+K\left\{\left\|\frac{\partial\xi}{\partial t}\right\|^{2}+\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}\|U-u\|^{2}\right\},$$

$$\left|\left(e(C)\frac{\partial(U-u)}{\partial t}\cdot\nabla C,\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$(3.11)\qquad\leq K\left\|\frac{\partial(U-u)}{\partial t}\right\|\left\{\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}\|\nabla\xi\|+\left\|\frac{\partial\xi}{\partial t}\right\|\right\}$$

$$\leq K\left\{\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}\|\nabla\xi\|^{2}+\left\|\frac{\partial\xi}{\partial t}\right\|^{2}+\left\|\frac{\partial(U-u)}{\partial t}\right\|^{2}\right\}.$$

The remaining terms of (3.7) may be treated the same way as above. Hence,

$$\left|\left(\frac{\partial}{\partial t}(e(C)U\cdot\nabla C-e(c)u\cdot\nabla\bar{c}),\,\frac{\partial\xi}{\partial t}\right)\right|$$

$$\leq\varepsilon\left\|\nabla\frac{\partial\xi}{\partial t}\right\|^{2}+K\left\{\|\xi\|^{2}+\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^{2}+1\right)\right.$$

$$(3.12)\qquad\qquad\cdot\left(\|\nabla\xi\|^{2}+\left\|\frac{\partial\xi}{\partial t}\right\|^{2}+\|U-u\|^{2}\right)$$

$$\left.+\left\|\frac{\partial(U-u)}{\partial t}\right\|^{2}\right\}Kh_{c}^{2l+2}.$$

An argument similar to that used above can be applied to the remaining terms in (3.5) to show that

$$
\frac{1}{2}\frac{d}{dt}\left(\varphi\frac{\partial\xi}{\partial t},\frac{\partial\xi}{\partial t}\right)+\frac{1}{2}D_*\left\|\nabla\frac{\partial\xi}{\partial t}\right\|^2
$$

(3.13)
$$
\leq K\left\{\|\xi\|^2+\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^2+1\right)\left(\|\nabla\xi\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2+\|U-u\|^2\right)\right.
$$
$$
\left.+\left\|\frac{\partial}{\partial t}(U-u)\right\|^2+h_c^{2l+2}\right\},
$$

where D_* is a lower bound of D. Integrate the inequality (3.13) with respect to time t to show that

$$
\left\|\frac{\partial\xi}{\partial t}\right\|^2+\int_0^t\left\|\nabla\frac{\partial\xi}{\partial t}\right\|^2\,ds
$$
$$
\leq K\left\{\left\|\frac{\partial\xi}{\partial t}(0)\right\|^2+\int_0^t\left[\|\xi\|^2+\left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^2+1\right)\right.\right.
$$

(3.14)
$$
\cdot\left(\|\nabla\xi\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2+\|U-u\|^2\right)
$$
$$
\left.\left.+\left\|\frac{\partial}{\partial t}(U-u)\right\|^2\right]\,ds+h_c^{2l+2}\right\}.
$$

Set $t=0$ in (2.5) and (2.10a) and use (2.7), (2.10c), and (2.12) to obtain

(3.15)
$$
\left(\varphi\frac{\partial}{\partial t}(C-c)(0),z\right)=0,\qquad z\in M_h.
$$

Let $z=\frac{\partial\xi}{\partial t}(0)$ in (3.15); then

(3.16)
$$
\left\|\frac{\partial\xi}{\partial t}(0)\right\|\leq K\left\|\frac{\partial\zeta}{\partial t}(0)\right\|\leq Kh_c^{l+1}.
$$

Adding the following inequalities

(3.17)
$$
\|\xi\|^2=\int_0^t\frac{d}{dt}\|\xi\|^2\,ds\leq\int_0^t\left[\|\xi\|^2+\left\|\frac{\partial\xi}{\partial t}\right\|^2\right]\,ds,
$$

(3.18)
$$
\|\nabla\xi\|^2=\int_0^t\frac{d}{dt}\|\nabla\xi\|^2\,ds\leq\varepsilon\int_0^t\left\|\nabla\frac{\partial\xi}{\partial t}\right\|^2\,ds+K\int_0^t\|\nabla\xi\|^2\,ds
$$

to (3.14), we obtain

$$
\|\xi\|_1^2 + \left\|\frac{\partial\xi}{\partial t}\right\|^2 + \int_0^t \left\|\nabla\frac{\partial\xi}{\partial t}\right\|^2 ds
$$

$$
\leq K_1 \left\{ \int_0^t \left[\|\xi\|^2 + \left(\left\|\frac{\partial\xi}{\partial t}\right\|_{0,\infty}^2 + 1 \right) \right.\right.
$$

(3.19)

$$
\cdot \left(\|\nabla\xi\|^2 + \left\|\frac{\partial\xi}{\partial t}\right\|^2 + \|U - u\|^2 \right)
$$

$$
\left.\left. + \left\|\frac{\partial}{\partial t}(U - u)\right\|^2 \right] ds + h_c^{2l+2} \right\},
$$

where

$$
K_1 = K_1 \left(\|\bar{c}\|_{1,\infty}, \left\|\frac{\partial\bar{c}}{\partial t}\right\|_{1,\infty}, \|c\|_{l+1}, \left\|\frac{\partial c}{\partial t}\right\|_{l+1}, \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{l+1} \right).
$$

4. ERROR ESTIMATES FOR THE PRESSURE EQUATION

In this section we first give error estimates for the pressure equation. These estimates are then combined with the estimates obtained in §3 to yield the main theorem of the paper.

To estimate the errors in the approximation of the pressure equation, we again introduce an elliptic projection in the sense of Johnson and Thomée [6]. Let $(\bar{u}, \bar{p}) \in V_h \times W_h$ be the solution of the system

(4.1)
    (a) $(\nabla \cdot (\bar{u} - u), w) + (\bar{p} - p, w) = 0,$ $w \in W_h,$
    (b) $(\alpha(c)(\bar{u} - u), v) - (\nabla \cdot v, \bar{p} - p) = 0,$ $v \in V_h,$
    (c) $(\bar{p}, 1) = (p, 1),$

where (c, p, u) is the solution of (2.5)–(2.7).

To show the uniqueness and existence of the solution of (4.1), it suffices to show that the associated homogeneous system has only the trivial solution. The associated homogeneous system is

(4.2)
    (a) $(\nabla \cdot \bar{u}, w) + (\bar{p}, w) = 0,$ $w \in W_h,$
    (b) $(\alpha(c)\bar{u}, v) - (\nabla \cdot v, \bar{p}) = 0,$ $v \in V_h.$

Choosing $w = \bar{p}$ and $v = \bar{u}$ in (4.2) leads to

(4.3) $0 \leq \alpha_*(\bar{u}, \bar{u}) \leq (\alpha(c)\bar{u}, \bar{u}) = (\nabla \cdot \bar{u}, \bar{p}) = -(\bar{p}, \bar{p}) \leq 0,$

where α_* is a positive number such that $\alpha_* \leq \alpha(c)$. Hence, $\bar{u} = 0$ and $\bar{p} = 0$.

In the analysis to follow we shall employ three linear operators: the L^2-orthogonal projections $r_{h_c}: H^1(\Omega) \to M_h$ and $r_{h_p}: W \to W_h$, respectively defined by

(4.4)
    (a) $(\varphi - r_{h_c}\varphi, m) = 0,$ $m \in M_h,$
    (b) $(\varphi - r_{h_p}\varphi, w) = 0,$ $w \in W_h,$

and the linear operator $T_h : V \to V_h$ having the property

$$(4.5) \qquad (\nabla \cdot (v - T_h v), w) = 0, \qquad w \in W_h.$$

From [6] the following results are standard:

$$
\begin{array}{lll}
(4.6) & \text{(a)} \quad \|\varphi - r_{h_c}\varphi\| \le M h_c^{l+1}\|\varphi\|_{l+1}, & l \ge 1, \\[4pt]
& \text{(b)} \quad \|\varphi - r_{h_p}\varphi\| \le M h_p^{k+1}\|\varphi\|_{k+1}, & k \ge 0, \\[4pt]
& \text{(c)} \quad \|\varphi - r_{h_p}\varphi\|_{0,\infty} \le M h_p^{k+1}(\log h_p^{-1})^{1/2}\|\varphi\|_{k+2}, & k \ge 0, \\[4pt]
& \text{(d)} \quad \|v - T_h v\| \le M h_p^{k+1}\|v\|_{k+1}, & k \ge 0.
\end{array}
$$

Choosing $w = \bar{p} - r_{h_p}p$ and $v = \bar{u} - T_h u$ in (4.1) and adding the two resulting equations, we have

$$(4.7) \qquad
\begin{aligned}
&(\alpha(c)(\bar{u} - u), \bar{u} - T_h u) - (\nabla \cdot (\bar{u} - T_h u), \bar{p} - p) \\
&\quad + (\nabla \cdot (\bar{u} - u), \bar{p} - r_{h_p}p) + (\bar{p} - p, \bar{p} - r_{h_p}p) = 0.
\end{aligned}
$$

Use the properties of r_{h_p} and T_h to show that

$$
\begin{aligned}
&(\nabla \cdot (\bar{u} - T_h u), \bar{p} - p) - (\nabla \cdot (\bar{u} - u), \bar{p} - r_h p) \\
&\quad = (\nabla \cdot (\bar{u} - T_h u), \bar{p} - r_{h_p}p) - (\nabla \cdot (\bar{u} - T_h u), \bar{p} - r_h p) = 0.
\end{aligned}
$$

Then from (4.7),

$$
\begin{aligned}
&(\alpha(c)(\bar{u} - u), \bar{u} - u) + (\bar{p} - p, \bar{p} - p) \\
&\quad = (\alpha(c)(\bar{u} - u), T_h u - u) + (\bar{p} - p, r_{h_p}p - p) \\
&\quad \le \frac{\alpha_*}{2}\|\bar{u} - u\|^2 + K\|T_h u - u\|^2 + \frac{1}{2}\|\bar{p} - p\|^2 + \frac{1}{2}\|r_{h_p}p - p\|^2,
\end{aligned}
$$

and from (4.6),

$$(4.8) \qquad \|\bar{u} - u\| + \|\bar{p} - p\| \le K h_p^{k+1}\{\|u\|_{k+1} + \|p\|_{k+1}\}, \qquad k \ge 0.$$

Now we derive the L^∞-error estimates. By (4.1b) and (4.4),

$$(4.9) \qquad (\alpha(c)(\bar{u} - u), v) + (r_{h_p}p - \bar{p}, \nabla \cdot v) = 0, \qquad v \in V_h.$$

Thus, from [6, Lemma 1.2] and (4.8) we have

$$(4.10) \qquad
\begin{aligned}
\|r_{h_p}p - \bar{p}\|_{0,\infty} &\le K \log h_p^{-1}\|\alpha(c)(\bar{u} - u)\| \\
&\le K h_p^{k+1}\log h_p^{-1}\{\|u\|_{k+1} + \|p\|_{k+1}\}.
\end{aligned}
$$

Consequently, from (4.6c)

$$
(4.11) \quad
\begin{aligned}
\|\bar{p} - p\|_{0,\infty} &\leq \|\bar{p} - r_{h_p} p\|_{0,\infty} + \|r_{h_p} p - p\|_{0,\infty} \\
&\leq K h_p^{k+1} \log h_p^{-1} \{\|u\|_{k+1} + \|p\|_{k+2}\}, \qquad k \geq 0.
\end{aligned}
$$

As for the L^2-estimates for $\frac{\partial}{\partial t}(\bar{u} - u)$ and $\frac{\partial}{\partial t}(\bar{p} - p)$, we differentiate (4.1) in time to show that

$$
(4.12) \quad
\begin{aligned}
&\text{(a)} \quad \left(\nabla \cdot \frac{\partial}{\partial t}(\bar{u} - u), w\right) + \left(\frac{\partial}{\partial t}(\bar{p} - p), w\right) = 0, \qquad w \in W_h, \\
&\text{(b)} \quad \left(\alpha(c)\frac{\partial}{\partial t}(\bar{u} - u), v\right) + \left(\frac{\partial \alpha}{\partial c}(c)\frac{\partial c}{\partial t}(\bar{u} - u), v\right) \\
&\qquad\qquad - \left(\nabla \cdot v, \frac{\partial}{\partial t}(\bar{p} - p)\right) = 0, \qquad v \in V_h.
\end{aligned}
$$

Select $w = \frac{\partial \bar{p}}{\partial t} - r_{h_p}\frac{\partial p}{\partial t}$ and $v = \frac{\partial \bar{u}}{\partial t} - T_h\frac{\partial u}{\partial t}$ in (4.12) and proceed as in deriving (4.8) to obtain

$$
(4.13) \quad
\begin{aligned}
&\left(\alpha(c)\frac{\partial}{\partial t}(\bar{u} - u), \frac{\partial}{\partial t}(\bar{u} - u)\right) + \left(\frac{\partial}{\partial t}(\bar{p} - p), \frac{\partial}{\partial t}(\bar{p} - p)\right) \\
&= \left(\alpha(c)\frac{\partial}{\partial t}(\bar{u} - u), T_h\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t}\right) - \left(\frac{\partial \alpha}{\partial c}(c)\frac{\partial c}{\partial t}(\bar{u} - u), \frac{\partial}{\partial t}(\bar{u} - u)\right) \\
&\quad - \left(\frac{\partial \alpha}{\partial c}(c)\frac{\partial c}{\partial t}(\bar{u} - u), \frac{\partial u}{\partial t} - T_h\frac{\partial u}{\partial t}\right) + \left(\frac{\partial}{\partial t}(\bar{p} - p), r_{h_p}\frac{\partial p}{\partial t} - \frac{\partial p}{\partial t}\right) \\
&\leq \frac{1}{2}\alpha_* \left\|\frac{\partial}{\partial t}(\bar{u} - u)\right\|^2 + \frac{1}{2}\left\|\frac{\partial}{\partial t}(\bar{p} - p)\right\|^2 \\
&\quad + K\left\{\left\|\frac{\partial u}{\partial t} - T_h\frac{\partial u}{\partial t}\right\|^2 + \|\bar{u} - u\|^2 + \left\|r_{h_p}\frac{\partial p}{\partial t} - \frac{\partial p}{\partial t}\right\|^2\right\}.
\end{aligned}
$$

From (4.6) and (4.8), we have

$$
(4.14) \quad
\begin{aligned}
&\left\|\frac{\partial}{\partial t}(\bar{u} - u)\right\| + \left\|\frac{\partial}{\partial t}(\bar{p} - p)\right\| \\
&\leq K h_p^{k+1}\left\{\|u\|_{k+1} + \|p\|_{k+1} + \left\|\frac{\partial u}{\partial t}\right\|_{k+1} + \left\|\frac{\partial p}{\partial t}\right\|_{k+1}\right\}, \qquad k \geq 0.
\end{aligned}
$$

Similarly,

$$
(4.15) \quad
\begin{aligned}
\left\|\frac{\partial^2}{\partial t^2}(\bar{p} - p)\right\| &\leq K h_p^{k+1}\left\{\|u\|_{k+1} + \|p\|_{k+1} + \left\|\frac{\partial u}{\partial t}\right\|_{k+1} \right. \\
&\left. \quad + \left\|\frac{\partial p}{\partial t}\right\|_{k+1} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{k+1} + \left\|\frac{\partial^2 p}{\partial t^2}\right\|_{k+1}\right\}.
\end{aligned}
$$

Now we can turn to the estimate for the pressure equation. Let

$$\pi = P - \bar{p}, \quad \eta = \bar{p} - p, \quad \sigma = U - \bar{u}, \quad \rho = \bar{u} - u.$$

Combine (2.6), (2.10b), and (4.1) to show that

(4.16)

(a) $\left(d(C)\dfrac{\partial \pi}{\partial t}, w\right) + (\nabla \cdot \sigma, w)$
$$= (\eta, w) - \left(d(C)\frac{\partial \eta}{\partial t}, w\right)$$
$$- \left((d(C) - d(c))\frac{\partial p}{\partial t}, w\right), \quad w \in W_h,$$

(b) $(\alpha(C)\sigma, v) - (\nabla \cdot v, \pi)$
$$= ((\alpha(c) - \alpha(C))\bar{u}, v) + ((\alpha(c) - \alpha(C))u_0, v), \quad v \in V_h.$$

Differentiate (4.16) with respect to time to show that

(4.17)

(a) $\left(d(C)\dfrac{\partial^2 \pi}{\partial t^2}, w\right) + \left(\nabla \cdot \dfrac{\partial \sigma}{\partial t}, w\right)$
$$= \left(\frac{\partial \eta}{\partial t}, w\right) - \left(\frac{\partial d}{\partial c}(C)\frac{\partial C}{\partial t}\frac{\partial \pi}{\partial t}, w\right)$$
$$- \left(d(C)\frac{\partial^2 \eta}{\partial t^2}, w\right) - \left(\frac{\partial d}{\partial c}(C)\frac{\partial C}{\partial t}\frac{\partial \eta}{\partial t}, w\right)$$
$$- \left(\frac{\partial}{\partial t}[d(C) - d(c)]\frac{\partial p}{\partial t}, w\right)$$
$$- \left([d(C) - d(c)]\frac{\partial^2 p}{\partial t^2}, w\right), \quad w \in W_h,$$

(b) $\left(\alpha(C)\dfrac{\partial \sigma}{\partial t}, v\right) - \left(\nabla \cdot v, \dfrac{\partial \pi}{\partial t}\right)$
$$= \left(\frac{\partial}{\partial t}[\alpha(c) - \alpha(C)]\bar{u}, v\right) + \left([\alpha(c) - \alpha(C)]\frac{\partial \bar{u}}{\partial t}, v\right)$$
$$+ \left(\frac{\partial}{\partial t}[\alpha(c) - \alpha(C)]u_0, v\right)$$
$$- \left(\frac{\partial \alpha}{\partial c}(C)\frac{\partial C}{\partial t}\sigma, v\right), \quad v \in V_h.$$

Select the test functions $w = \frac{\partial \pi}{\partial t}$ and $v = \frac{\partial \sigma}{\partial t}$ in (4.17), add the two equations, and use the equality

$$\frac{d}{dt}\left(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\right) = 2\left(d(C)\frac{\partial^2 \pi}{\partial t^2}, \frac{\partial \pi}{\partial t}\right) + \left(\frac{\partial d}{\partial c}(C)\frac{\partial C \partial \pi}{\partial t \partial t}, \frac{\partial \pi}{\partial t}\right)$$

to see that

$$
\frac{1}{2}\frac{d}{dt}\left(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\right) + \left(\alpha(C)\frac{\partial \sigma}{\partial t}, \frac{\partial \sigma}{\partial t}\right)
$$

$$
= \left(\frac{\partial \eta}{\partial t}, \frac{\partial \pi}{\partial t}\right) - \frac{1}{2}\left(\frac{\partial d}{\partial c}(C)\frac{\partial C\partial \pi}{\partial t\partial t}, \frac{\partial \pi}{\partial t}\right) - \left(d(C)\frac{\partial^2 \eta}{\partial t^2}, \frac{\partial \pi}{\partial t}\right)
$$

$$
- \left(\frac{\partial d}{\partial c}(C)\frac{\partial C\partial \eta}{\partial t\partial t}, \frac{\partial \pi}{\partial t}\right) - \left(\frac{\partial}{\partial t}[d(C)-d(c)]\frac{\partial p}{\partial t}, \frac{\partial \pi}{\partial t}\right)
$$

$$
- \left([d(C)-d(c)]\frac{\partial^2 p}{\partial t^2}, \frac{\partial \pi}{\partial t}\right) + \left(\frac{\partial}{\partial t}[\alpha(c)-\alpha(C)]\bar{u}, \frac{\partial \sigma}{\partial t}\right)
$$

$$
+ \left([\alpha(c)-\alpha(C)]\frac{\partial \bar{u}}{\partial t}, \frac{\partial \sigma}{\partial t}\right) + \left(\frac{\partial}{\partial t}[\alpha(c)-\alpha(C)]u_0, \frac{\partial \sigma}{\partial t}\right)
$$

$$
\text{(4.18)} \qquad - \left(\frac{\partial \alpha}{\partial c}(C)\frac{\partial C}{\partial t}\sigma, \frac{\partial \sigma}{\partial t}\right)
$$

$$
\leq \varepsilon \left\|\frac{\partial \sigma}{\partial t}\right\|^2 + K\left\{\left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left\|\frac{\partial \eta}{\partial t}\right\|^2\right.
$$

$$
\left. + \left\|\frac{\partial^2 \eta}{\partial t^2}\right\|^2 + \|\zeta\|^2 + \left\|\frac{\partial \zeta}{\partial t}\right\|^2\right\}
$$

$$
- \frac{1}{2}\left(\frac{\partial d}{\partial c}(C)\frac{\partial C\partial \pi}{\partial t\partial t}, \frac{\partial \pi}{\partial t}\right) - \left(\frac{\partial d}{\partial c}(C)\frac{\partial C\partial \eta}{\partial t\partial t}, \frac{\partial \pi}{\partial t}\right)
$$

$$
- \left(\frac{\partial \alpha}{\partial c}(C)\frac{\partial C}{\partial t}\sigma, \frac{\partial \sigma}{\partial t}\right).
$$

Note that by the triangle inequality

$$
\left\|\frac{\partial C}{\partial t}\right\|_{0,\infty} \leq \left\|\frac{\partial \xi}{\partial t}\right\|_{0,\infty} + \left\|\frac{\partial \bar{c}}{\partial t}\right\|_{0,\infty}.
$$

By the boundedness of $\left\|\frac{\partial \zeta}{\partial t}\right\|_{0,\infty}$ and the known estimates of η and ξ, it follows from (4.18) that

$$
\frac{1}{2}\frac{d}{dt}\left(d(C)\frac{\partial \pi}{\partial t}, \frac{\partial \pi}{\partial t}\right) + \alpha_* \left\|\frac{\partial \sigma}{\partial t}\right\|^2
$$

$$
\text{(4.19)} \qquad \leq \varepsilon \left\|\frac{\partial \sigma}{\partial t}\right\|^2 + K\left\{\|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left(\left\|\frac{\partial \xi}{\partial t}\right\|_{0,\infty}^2 + 1\right)\right.
$$

$$
\left. \cdot \left(\left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2\right) + h_c^{2l+2} + h_p^{2k+2}\right\},
$$

which upon integration gives

$$
\left\|\frac{\partial \pi}{\partial t}\right\|^2 + \int_0^t \left\|\frac{\partial \sigma}{\partial t}\right\|^2 ds
$$

$$
(4.20) \quad
\begin{aligned}
&\leq K \left\{ \left\|\frac{\partial \pi}{\partial t}(0)\right\|^2 + \int_0^t \left[\|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left(\left\|\frac{\partial \xi}{\partial t}\right\|_{0,\infty} + 1 \right) \right.\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.\cdot \left(\left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2 \right) \right] ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + h_c^{2l+2} + h_p^{2k+2} \right\}.
\end{aligned}
$$

Choose $t = 0$ in (2.6a) and (2.10bi), and subtract the resulting equations, to show that

$$
(4.21) \qquad \left(\alpha(0) \left(\frac{\partial P}{\partial t} - \frac{\partial p}{\partial t} \right)(0), w \right) = 0, \qquad w \in W_h,
$$

and

$$
(4.22) \qquad \left\| \frac{\partial \pi}{\partial t}(0) \right\| \leq K \left\| \frac{\partial \eta}{\partial t}(0) \right\| \leq K h_p^{k+1}.
$$

Using (2.7), we see from (4.1) that $\bar{p}(0) = 0$ and $\bar{u}(0) = 0$. Thus, $\pi(0) = 0$ and $\sigma(0) = 0$. The techniques used in deriving (3.19) from (3.17) and (3.18) can be applied to (4.20) to show that

$$
\|\pi\|^2 + \left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2 + \int_0^t \left\|\frac{\partial \sigma}{\partial t}\right\|^2 ds
$$

$$
(4.23) \quad
\begin{aligned}
&\leq K \left\{ \int_0^t \left[\|\xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left(\left\|\frac{\partial \xi}{\partial t}\right\|_{0,\infty} + 1 \right) \left(\left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2 \right) \right] ds \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + h_c^{2l+2} + h_p^{2k+2} \right\}.
\end{aligned}
$$

Take a $(1 + K_1)$-multiple of (4.23), add it to (3.19), and use (4.8) and (4.14) to have

$$
\|\xi\|_1^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \|\pi\|^2 + \left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2 + \int_0^t \left\|\nabla \frac{\partial \xi}{\partial t}\right\|^2 ds + \int_0^t \left\|\frac{\partial \sigma}{\partial t}\right\|^2 ds
$$

$$
(4.24) \quad
\begin{aligned}
&\leq K \left\{ \int_0^t \left[\|\xi\|^2 + \left(\left\|\frac{\partial \xi}{\partial t}\right\|_{0,\infty}^2 + 1 \right) \right.\right.\\
&\qquad\qquad \left.\cdot \left(\|\nabla \xi\|^2 + \left\|\frac{\partial \xi}{\partial t}\right\|^2 + \left\|\frac{\partial \pi}{\partial t}\right\|^2 + \|\sigma\|^2 + h_c^{2k+2} \right) \right] ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + h_c^{2l+2} + h_p^{2k+2} \right\}.
\end{aligned}
$$

To estimate (4.24), we make the induction hypothesis that

$$
\begin{aligned}
(4.25) \quad
& \left\| \frac{\partial \xi}{\partial t} \right\|_{L^2(J;L^\infty(\Omega))}
\left(\|\nabla \xi\|_{L^\infty(J;L^2(\Omega))} + \left\| \frac{\partial \xi}{\partial t} \right\|_{L^\infty(J;L^2(\Omega))} \right. \\
& \left. \qquad + \left\| \frac{\partial \pi}{\partial t} \right\|_{L^\infty(J;L^2(\Omega))} + \|\sigma\|_{L^\infty(J;L^2(\Omega))} + h_p^{k+1} \right) \\
& \qquad \leq K_2(h_c^{l+1} + h_p^{k+1}), \qquad l \geq 1, \ k \geq 0.
\end{aligned}
$$

Thus, from Gronwall's lemma we can derive that

$$
\begin{aligned}
(4.26) \quad
& \|\xi\|_{L^\infty(J;H^1(\Omega))} + \left\| \frac{\partial \xi}{\partial t} \right\|_{L^\infty(J;L^2(\Omega))} + \|\pi\|_{L^\infty(J;L^2(\Omega))} + \left\| \frac{\partial \pi}{\partial t} \right\|_{L^\infty(J;L^2(\Omega))} \\
& \qquad + \|\sigma\|_{L^\infty(J;L^2(\Omega)^2)} + \left\| \nabla \frac{\partial \xi}{\partial t} \right\|_{L^2(J;L^2(\Omega))} + \left\| \frac{\partial \sigma}{\partial t} \right\|_{L^2(J;L^2(\Omega)^2)} \\
& \qquad \leq K_3(h_c^{l+1} + h_p^{k+1}), \qquad l \geq 1, \ k \geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
K_3 = K_3 \left(K_1, K_2, \|u\|_{k+1}, \|p\|_{k+1}, \left\| \frac{\partial u}{\partial t} \right\|_{k+1}, \right. \\
\left. \left\| \frac{\partial p}{\partial t} \right\|_{k+1}, \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{k+1}, \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{k+1} \right).
\end{aligned}
$$

Now we turn to the justification of the induction hypothesis (4.25). Obviously, (4.25) holds for $t = 0$. Since

$$
\begin{aligned}
(4.27) \quad
F(t) = & \left\| \frac{\partial \xi}{\partial t} \right\|_{L^2(0,t;L^\infty(\Omega))}
\left(\|\nabla \xi\|_{L^\infty(0,t;L^2(\Omega))} + \left\| \frac{\partial \xi}{\partial t} \right\|_{L^\infty 0,t;L^2(\Omega)} \right. \\
& \left. + \left\| \frac{\partial \pi}{\partial t} \right\|_{L^\infty(0,t;L^2(\Omega))} + \|\sigma\|_{L^\infty(0,t;L^2(\Omega))} + h_p^{k+1} \right)
\end{aligned}
$$

is a continuous function in t, there exists some $t^* > 0$ such that

$$
(4.28) \quad
\begin{aligned}
&\text{(a)} \quad F(t) < K_2(h_c^{l+1} + h_p^{k+1}), \qquad 0 < t < t^*, \\
&\text{(b)} \quad F(t) = K_2(h_c^{l+1} + h_p^{k+1}), \qquad t = t^*.
\end{aligned}
$$

We want to show that $t^* = T$. By the well-known imbedding inequality [11] for two space dimensions,

$$
(4.29) \quad \|v\|_{0,\infty} \leq K_4 (\log h_c^{-1})^{1/2} \|v\|_1, \qquad v \in M_h,
$$

and using (4.26) and (4.27), we can derive

$$
\text{(4.30)} \quad F(t) \le K_3 K_4 (\log h_c^{-1})^{1/2} \{ 4K_3 h_c^{2l+2} + (8K_3 + 1) h_c^{l+1} h_p^{k+1} \\
+ (4K_3 + 1) h_p^{2k+2} \}, \qquad 0 \le t \le t^*.
$$

Suppose that the relation

$$
\text{(4.31)} \qquad h_p^{2k+2} (\log h_c^{-1})^{1/2} \to 0 \quad \text{as } h \to 0
$$

holds. Then h can be selected to be small enough so that

$$
F(t^*) \le \frac{K_2}{2} (h_c^{l+1} + h_p^{k+1}).
$$

Hence, $t^* = T$, and the proof is completed.

The desired L^2-optimal error estimates now result from the combination of (4.26), (3.2a), (4.8), and (4.14):

$$
\text{(4.32)} \quad \begin{aligned}
&\| C - c \|_{L^\infty(J; L^2(\Omega))} + \left\| \frac{\partial}{\partial t}(C - c) \right\|_{L^\infty(J; L^2(\Omega))} + \| P - p \|_{L^\infty(J; L^2(\Omega))} \\
&\quad + \left\| \frac{\partial}{\partial t}(P - p) \right\|_{L^\infty(J; L^2(\Omega))} + \| U - u \|_{L^\infty(J; L^2(\Omega)^2)} \\
&\quad + \left\| \frac{\partial}{\partial t}(U - u) \right\|_{L^2(J; L^2(\Omega)^2)} \\
&\quad \le K_5 (h_c^{l+1} + h_p^{k+1}), \qquad l \ge 1, \ k \ge 0,
\end{aligned}
$$

where

$$
\begin{aligned}
K_5 = K_5 \Bigg(& \| \bar{c} \|_{L^\infty(J; W^{1,\infty}(\Omega))}, \left\| \frac{\partial \bar{c}}{\partial t} \right\|_{L^\infty(J; W^{1,\infty}(\Omega))}, \| c \|_{L^\infty(J; H^{l+1}(\Omega))}, \\
& \left\| \frac{\partial c}{\partial t} \right\|_{L^\infty(J; H^{l+1}(\Omega))}, \left\| \frac{\partial^2 c}{\partial t^2} \right\|_{L^2(J; H^{l+1}(\Omega))}, \| p \|_{L^\infty(J; H^{k+1}(\Omega))}, \\
& \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(J; H^{k+1}(\Omega))}, \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^2(J; H^{k+1}(\Omega))}, \| u \|_{L^\infty(J; H^{k+1}(\Omega))}, \\
& \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty(J; H^{k+1}(\Omega))}, \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(J; H^{k+1}(\Omega))} \Bigg).
\end{aligned}
$$

Combining (3.2b), (4.26), and (4.29) results in

$$
\text{(4.33)} \quad \begin{aligned}
\| C - c \|_{L^\infty(J; L^\infty(\Omega))} &\le K_6 (h_c^{l+1} + h_p^{k+1})(\log h_c^{-1})^\beta, \\
& l \ge 1, \ k \ge 0, \ \beta = \begin{cases} 1, & l = 1, \\ \frac{1}{2}, & l \ge 1, \end{cases}
\end{aligned}
$$

where

$$
K_6 = K_6 (K_5, \| c \|_{L^\infty(J; W^{l+1,\infty}(\Omega))}).
$$

Applying a similar argument as in deriving (4.10), we see from (4.16b) that

$$\|\pi\|_{0,\infty} \le K\|\alpha(C)\sigma + (\alpha(C) - \alpha(c))(\bar{u} + u_0)\| \log h_p^{-1}$$
$$\le K(h_c^{l+1} + h_p^{k+1}) \log h_p^{-1},$$

by (4.26) and (4.32). Then the L^∞-estimates for the pressure follows by this estimate and (4.11):

$$(4.34) \quad \|P - p\|_{L^\infty(J;L^\infty(\Omega))} \le K_7(h_c^{l+1} + h_p^{k+1}) \log h_p^{-1}, \quad l \ge 1, \ k \ge 0,$$

where $K_7 = K_7(K_5)$.

Finally, we can return to the error estimates for the original dependent variables c, p, and u, which are the solution of the system (1.1)–(1.3). The reader is being reminded that the error estimates obtained before pertain to the transformed system whose variables have "*" superscripts. Now define

$$(4.35) \quad C = C^* + r_{h_c}c_0, \quad P = P^* + r_{h_p}p_0, \quad U = U^* + T_h u_0,$$

where (C^*, P^*, U^*) is the solution of the semidiscrete system (2.5)–(2.7). By (2.4), the triangle inequality

$$\|C - c\| \le \|C^* - c^*\| + \|r_{h_c}c_0 - c_0\|$$

and (4.32) can be combined with (4.6) to show that

$$\|C - c\| \le K(h_c^{l+1} + h_p^{k+1}), \quad l \ge 1, \ k \ge 0.$$

By repeated use of (4.6) and (4.32) in the way shown above, we obtain the main theorem of this paper. For the case of using the Raviart-Thomas space of index k, $k \ge 0$, we thus have

Theorem 4.1. *Let* (c, p, u) *be the solution of the continuous problem* (1.1)–(1.3) *and* (C, P, U) *be defined by* (4.35). *Suppose that the relation* (4.31) *on the diameters* h_p *and* h_c *holds. Then the following estimates hold:*
  (i)

$$(4.36) \quad \begin{aligned} &\|C - c\|_{L^\infty(J;L^2(\Omega))} + \left\|\frac{\partial}{\partial t}(C - c)\right\|_{L^\infty(J;L^2(\Omega))} + \|P - p\|_{L^\infty(J;L^2(\Omega))} \\ &\quad + \left\|\frac{\partial}{\partial t}(P - p)\right\|_{L^\infty(J;L^2(\Omega))} + \|U - u\|_{L^\infty(J;L^2(\Omega))^2} \\ &\quad + \left\|\frac{\partial}{\partial t}(U - u)\right\|_{L^2(J;L^2(\Omega)^2)} \\ &\le K_8(h_c^{l+1} + h_p^{k+1}), \quad l \ge 1, \ k \ge 0, \end{aligned}$$

where the constant K_8 depends on

$$\|\bar{c}\|_{L^\infty(J;W^{1,\infty}(\Omega))}, \quad \left\|\frac{\partial\bar{c}}{\partial t}\right\|_{L^\infty(J;W^{1,\infty}(\Omega))}, \quad \|c\|_{L^\infty(J;H^{l+1}(\Omega))},$$

$$\left\|\frac{\partial c}{\partial t}\right\|_{L^\infty(J;H^{l+1}(\Omega))}, \quad \left\|\frac{\partial^2 c}{\partial t^2}\right\|_{L^2(J;H^{l+1}(\Omega))}, \quad \|p\|_{L^\infty(J;H^{k+1}(\Omega))},$$

$$\left\|\frac{\partial p}{\partial t}\right\|_{L^\infty(J;H^{k+1}(\Omega))}, \quad \left\|\frac{\partial^2 p}{\partial t^2}\right\|_{L^2(J;H^{k+1}(\Omega))}, \quad \|u\|_{L^\infty(J;H^{k+1}(\Omega))},$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J;H^{k+1}(\Omega))}, \quad and \quad \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(J;H^{k+1}(\Omega))}.$$

(ii)

(4.37)
$$\|C - c\|_{L^\infty(J;L^\infty(\Omega))} \le K_9(h_c^{l+1} + h_p^{k+1})(\log h_c^{-1})^\beta,$$
$$l \ge 1, \quad k \ge 0, \quad \beta = \begin{cases} 1, & l = 1, \\ \frac{1}{2}, & l > 1, \end{cases}$$

where K_9 depends on K_8 and $\|c\|_{L^\infty(J;W^{l+1,\infty}(\Omega))}$.
   (iii)

(4.38) $\|P - p\|_{L^\infty(J;L^\infty(\Omega))} \le K_{10}(h_c^{l+1} + h_p^{k+1})\log h_p^{-1}, \quad l \ge 1, \ k \ge 0,$

where K_{10} depends on K_8.

As for the case of the BDM space of index k, $k \ge 1$, replace every occurrence of k by $k - 1$ in (4.36)–(4.38).

5. EXTENSIONS TO SPACE IN THREE DIMENSIONS

In this section we indicate possible extensions of the analysis in the previous sections to three dimensions. The pressure equation can be approximated using the spaces of [2, 8]. The error analysis can be carried out in the same way, assuming different relations between the discretization parameters.

Theorem 5.1. *Suppose that the mesh parameters satisfy the relations*

(5.1) $h_p^{2k+2}h_c^{-1/2} \to 0 \quad as \ h \to 0.$

Then the estimate (4.36) holds.

Proof. We follow the proof of Theorem 4.1, making changes when necessary. It suffices to prove that the induction hypothesis (4.25) holds.
   Using the imbedding theorem for Sobolev spaces in R^3, we get

(5.2) $\|v\|_{0,6} \le K\|v\|_1, \quad v \in W^{1,2}(\Omega),$

and then, by the inverse inequality in the space M_h,

(5.3) $\|v\|_{0,\infty} \le Kh^{-1/2}\|v\|_{0,6}, \quad v \in M_h.$

Hence,

$$(5.4) \qquad \|v\|_{0,\infty} \leq K_9 h^{-1/2} \|v\|_1, \qquad v \in M_h.$$

It follows from (5.4), (4.26), and (4.27) that

$$(5.5) \quad F(t) \leq K_3 K_9 h_c^{-1/2} \{ 4K_3 h_c^{2l+2} + (8K_3 + 1) h_c^{l+1} h_p^{k+1} + (4K_3 + 1) h_p^{2k+2} \}.$$

By (5.1) and (5.5), the induction hypothesis (4.25) is verified. □

As a final remark, we point out that the relation (4.31) or (5.1) in the paper is much weaker than the condition

$$\min(h_p^{k+1} h_c^{-1}, h_c^{l+1} h_p^{-1}) \to 0 \quad \text{as } h \to 0 \text{ for all } k,$$

and

$$\gamma_1 \leq h_p h_c^{-2} \leq \gamma_2 \quad \text{for } k = 0$$

given in [4].

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DEPARTMENT OF MATHEMATICS AND STATISTICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403-0221