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COUNTEREXAMPLES CONCERNING A WEIGHTED L^2 PROJECTION

JINCHAO XU

ABSTRACT. Counterexamples are given to show that some results concerning a weighted L^2 projection presented earlier by Bramble and the author are sharp, i.e., that certain error and stability estimates are impossible in some cases.

1. INTRODUCTION

Motivated by the numerical solution of second-order elliptic boundary value problems with discontinuous coefficients, certain weighted L^2 projections were studied in [1]. Owing to some technical difficulties, the error and stability estimates obtained in [1] are contingent upon some additional assumptions. In this paper, we study the problem further. The results we obtain are negative and demonstrate that the main results in [1] cannot be improved.

Let $\Omega \subset \mathbb{R}^d$ ($1 \le d \le 3$) be a bounded domain. For simplicity, we assume that Ω is a polyhedral domain, i.e., an interval for $d = 1$, a polygon for $d = 2$ and a polyhedron for $d = 3$. Assume the domain Ω admits the following decomposition:

$$\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i, \tag{1.1}$$

where Ω_i are mutually disjoint polyhedrons.

Given a set of positive constants $\{\omega_i\}_{i=1}^{J}$, we introduce two weighted inner products,

$$(u, v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} uv\,dx \tag{1.2}$$

and

$$(u, v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v\,dx, \tag{1.3}$$

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with the induced norms denoted by $\|\cdot\|_{L^2_\omega(\Omega)}$ and $|\cdot|_{H^1_\omega(\Omega)}$, respectively. Moreover, we define a full weighted H^1 norm by

$$\|\cdot\|^2_{H^1_\omega(\Omega)} = \|\cdot\|^2_{L^2_\omega(\Omega)} + |\cdot|^2_{H^1_\omega(\Omega)}.$$

If $\omega_i = 1$ for each i, we have the usual Sobolev space and the symbol ω will then be dropped.

Next we introduce a finite element space. For $0 < h < 1$, let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$ with simplices K of diameter less than or equal to h. An additional assumption is that this triangulation be lined up with each subdomain Ω_i. Namely, Ω_i is the union of a set of elements of $\mathcal{T}_h$. We assume that the family $\{\mathcal{T}_h\}$ is quasi-uniform, i.e., there exist positive constants c_0 and c_1 such that

$$\max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \le c_0, \qquad \frac{\max_{K\in\mathcal{T}_h} h_K}{\min_{K\in\mathcal{T}_h} h_K} \le c_1, \quad \forall h.$$

Here, h_K is the diameter of K and ρ_K the diameter of the largest ball contained in K. Corresponding to each triangulation $\mathcal{T}_h$, we define a finite element subspace $S_h \subset H^1_0(\Omega)$ that consists of continuous piecewise (with respect to the elements in $\mathcal{T}_h$) linear polynomials vanishing on $\partial\Omega$. For $G \subset \Omega$, $S_h(G)$ denotes the space of functions in S_h restricted to G.

The weighted L^2 projection $Q_h^\omega : L^2(\Omega) \mapsto S_h$ is defined by

$$(1.4) \qquad (Q_h^\omega u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)}, \qquad \forall u \in L^2(\Omega), v \in S_h.$$

If $\omega_i = 1$ for all i, we get the usual L^2 projection, denoted by Q_h. The following estimate is known (cf. [1, 3] and the reference cited therein):

$$\|u - Q_h u\|_{L^2(\Omega)} + h|Q_h u|_{H^1(\Omega)} \le Ch|u|_{H^1(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

We are interested in similar estimates for the weighted L^2 projections with the regular norms replaced by the weighted norms, and with the constant C independent of the weights ω_i's. This problem has been carefully studied in [1].

Before we review the results of [1], we introduce the following notation:

$$x \lesssim y, \quad f \gtrsim g \quad \text{and} \quad u \asymp v$$

meaning, respectively,

$$x \le Cy, \quad f \ge cg \quad \text{and} \quad cv \le u \le Cv,$$

where C and c are positive constants independent of the variables appearing in the inequalities and the other parameters related to meshes, spaces and especially the weights ω_i's. We shall use the term "interface" to denote the union of the boundaries of all Ω_i inside of Ω.

The first result shows that optimal estimates can be obtained in a special case.

Theorem 1.1 [1]. *Assume that $d = 1$, or that the decomposition* (1.1) *has no internal cross points, i.e., there is no point on the interface that belongs to more than two $\overline{\Omega}_i$'s. Then, for all $u \in H_0^1(\Omega)$,*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} + h|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim h|u|_{H^1_\omega(\Omega)}. \tag{1.5}$$

If there are internal cross points, nearly optimal estimates can be obtained under additional conditions.

Theorem 1.2 [1]. *If for all i, the $(d-1)$-dimensional Lebesgue measure of $\partial\Omega_i \cap \partial\Omega$ is positive, then for all $u \in H_0^1(\Omega)$*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} + h|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim h|\log h|^{\frac{1}{2}}|u|_{H^1_\omega(\Omega)}. \tag{1.6}$$

In order to obtain estimates without the restriction on the measure of $\partial\Omega_i \cap \partial\Omega$, as in the above theorem, we consider a special class of functions instead of all of H^1. For a given triangulation $\mathcal{T}_h$, we consider a finer quasi-uniform mesh $\mathcal{T}_{\underline{h}}$ with $\underline{h} < h$ which is obtained by refining $\mathcal{T}_h$ in such a way that

$$S_h \subset S_{\underline{h}}.$$

Here, $S_{\underline{h}} \subset H_0^1(\Omega)$ is the finite element space corresponding to $\mathcal{T}_{\underline{h}}$.

We have shown previously:

Theorem 1.3 [1]. *For any $u \in S_{\underline{h}}$,*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} + h|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim \begin{cases} h\left(\log \dfrac{h}{\underline{h}}\right)^{\frac{1}{2}} |u|_{H^1_\omega(\Omega)}, & \text{if } d = 2; \\ h\left(\dfrac{h}{\underline{h}}\right)^{\frac{1}{2}} |u|_{H^1_\omega(\Omega)}, & \text{if } d = 3. \end{cases}$$

The purpose of this paper is to show that the assumption in Theorem 1.2 concerning the measure of $\partial\Omega_i \cap \partial\Omega$ is necessary and that the estimate for $d = 3$ in Theorem 1.3 is sharp.

2. Counterexamples

In Theorem 1.2, the estimate (1.6) is established only under the condition that all the subregions meet the boundary of the original region on a subset of a positive $(d-1)$-dimensional measure. A natural question is then if this constraint is essential. The following two theorems show that this is the case.

Theorem 2.1. *Assume that there is an i_0 such that the $(d-1)$-dimensional Lebesgue measure of $\partial\Omega_{i_0} \cap \partial\Omega$ is zero. Then, there is no constant C independent of the ω_i's such that*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \le C|u|_{H^1_\omega(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{2.1}$$

Proof. For convenience, we shall use $\text{meas}_k(G)$ to denote the k-dimensional Lebesgue measure of G.

Case 1: $d = 3$ and $\text{meas}_1(\partial\Omega_{i_0} \cap \partial\Omega) = 0$. Without loss of generality, we assume that $i_0 = 1$ and that Ω_1 is the unit cube $(0, 1)^3$. It suffices to consider two cases. In the first, there is another subdomain, Ω_2, that touches Ω_1 only at the origin O. In the second case, $O \in \partial\Omega$. Because of the similarity in the proofs, we only present the proof for the first case.

Assume that there is a constant C independent of the ω_i's such that (2.1) holds. By letting $\omega_i = \omega$ for $i > 2$, we then would have

$$\|(I - Q_h^\omega)u\|_{L^2(\Omega_1\cup\Omega_2)} \le C(|u|_{H^1(\Omega_1\cup\Omega_2)} + \omega|u|_{H^1(\Omega\setminus(\Omega_1\cup\Omega_2))}).$$

In particular, the above inequality implies that $\|Q_h^\omega u\|_{L^2(\Omega_1\cup\Omega_2)}$ is bounded with respect to ω, hence it has a subsequence that converges to a function $\overline{Q}_h^\omega u \in S_h(\Omega_1 \cup \Omega_2)$. Consequently, letting $\omega \to 0$ yields

$$\|u - \overline{Q}_h^\omega u\|_{L^2(\Omega_1\cup\Omega_2)} \le C|u|_{H^1(\Omega_1\cup\Omega_2)}, \quad \forall u \in H_0^1(\Omega). \tag{2.2}$$

Take a function $\phi \in C^\infty(\mathbb{R}^1)$ such that $\phi = 0$ for $x \le \frac{1}{2}$, $\phi = 1$ for $x \ge 1$ and $|\phi'(x)| \le 4$ for any x. It is easy to see, for any $\varepsilon > 0$, that there exists a function $u_\varepsilon \in H_0^1(\Omega)$ such that

$$u_\varepsilon = \begin{cases} \phi(\frac{|x|}{\varepsilon}) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2. \end{cases}$$

For example, in the rest of Ω, u_ε can be defined by solving $-\Delta u_\varepsilon = 0$ with some properly prescribed continuous boundary data. A direct calculation shows that

$$|u_\varepsilon|_{H^1(\Omega_1\cup\Omega_2)} = |u_\varepsilon|_{H^1(\Omega_1)} \lesssim \sqrt{\varepsilon} \tag{2.3}$$

and

$$\|u_\varepsilon - \overline{u}\|_{L^2(\Omega_1\cup\Omega_2)} \lesssim \varepsilon^{\frac{3}{2}},$$

where $\overline{u}$ equals 1 in Ω_1 and 0 in Ω_2. We first observe that $\|\overline{Q}_h^\omega u_\varepsilon\|_{L^2(\Omega_1\cup\Omega_2)}$ is bounded with respect to ε. Hence, there exists a function $w_h \in S_h(\Omega_1 \cup \Omega_2)$ and a sequence $\{\varepsilon_m \to 0\}$ such that

$$\lim_{m\to\infty} \|\overline{Q}_h^\omega u_{\varepsilon_m} - w_h\|_{L^2(\Omega_1\cup\Omega_2)} = 0.$$

Consequently, we conclude from (2.2), with $u_\varepsilon = u$, and (2.3) that

$$\|\overline{u} - w_h\|_{L^2(\Omega_1\cup\Omega_2)} = 0.$$

This implies $\overline{u} = w_h$, which is a contradiction, since w_h is continuous at O but $\overline{u}$ is not.

Case 2: $d = 2$. Let Ω_1 and Ω_2 be similar as before, but $\Omega_1 = (0, 1)^2$. In this case, the construction of an appropriate u_ε is more difficult. Using the fact that

C_0^∞ is dense in $H^{\frac{1}{2}}$, we can find a sequence of smooth functions ϕ_ε on $\partial\Omega_1$ that vanish in a neighborhood of $(0, 0)$ and satisfy

$$\lim_{\varepsilon\to 0} \|\phi_\varepsilon - 1\|_{H^{\frac{1}{2}}(\partial\Omega_1)} = 0. \tag{2.4}$$

As we did above, it is easy to find a $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_1, \\ u_\varepsilon = \phi_\varepsilon & \text{on } \partial\Omega_1 \end{cases} \tag{2.5}$$

and

$$u_\varepsilon = 0 \quad \text{in } \Omega_2.$$

Notice that $u_\varepsilon - 1$ is harmonic in Ω_1, and therefore, as $\varepsilon \to 0$,

$$|u_\varepsilon|_{H^1(\Omega_1\cup\Omega_2)} = |u_\varepsilon - 1|_{H^1(\Omega_1)} \lesssim \|\phi_\varepsilon - 1\|_{H^{\frac{1}{2}}(\partial\Omega_1)} \to 0. \tag{2.6}$$

Here we have used (2.4).

The rest of the proof is the same as in the first case.

Case 3: $d = 3$ and $\operatorname{meas}_2(\partial\Omega_{i_0} \cap \partial\Omega) = 0$. In this case, we may assume that $\Omega_{i_0} = \Omega_1 = (0, 1)^3$ and

$$\partial\Omega_1 \cap \partial\Omega = \{(0, 0, x_3) : 0 < x_3 < 1\}.$$

We can construct a function $v_\varepsilon \in H_0^1(\Omega)$ satisfying

$$v_\varepsilon(x_1, x_2, x_3) = u_\varepsilon(x_1, x_2), \qquad 0 \le x_i \le 1, \ i = 1, 2, 3,$$

where u_ε satisfies (2.5).

By (2.6), we have, as $\varepsilon \to 0$,

$$|v_\varepsilon|_{H^1(\Omega_1)} = |u_\varepsilon|_{H^1((0,1)^2)} \to 0.$$

The rest of the proof is similar as above.

The following result concerns the sharpness of the estimate in Theorem 1.3 for $d = 3$.

Theorem 2.2. *Assume that* $d = 3$ *and that there is an index* i_0 *such that* $\operatorname{meas}_1(\partial\Omega_{i_0} \cap \partial\Omega) = 0$. *Then, if* $C_{\underline{h}}$ *is a constant satisfying*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \le C_{\underline{h}} |u|_{H^1_\omega(\Omega)}, \qquad \forall u \in S_{\underline{h}}, \tag{2.7}$$

there holds

$$C_{\underline{h}} \gtrsim \underline{h}^{-\frac{1}{2}}.$$

Proof. As in the proof of Theorem 2.1, there exists, for any $u \in S_{\underline{h}}$, a function $\overline{Q}_h^\omega u \in S_{\underline{h}}(\Omega_1 \cup \Omega_2)$ such that

$$\|u - \overline{Q}_h^\omega u\|_{L^2(\Omega_1\cup\Omega_2)} \le C_{\underline{h}} |u|_{H^1(\Omega_1\cup\Omega_2)}. \tag{2.8}$$

We now take $u_{\underline{h}} \in S_{\underline{h}}$ such that $u_{\underline{h}} = 1$ at all the nodes except O on $\overline{\Omega}_1$, and $u_{\underline{h}} = 0$ at all the nodes on $\overline{\Omega}_2$. A direct computation shows that

$$|u_{\underline{h}}|_{H^1(\Omega_1 \cup \Omega_2)} \lesssim \sqrt{\underline{h}}.$$

Using an argument similar to that in the proof of Theorem 2.1, we can find a $w_h \in S_h$ such that

$$\lim_{\underline{h} \to 0} \|u_{\underline{h}} - \overline{Q}_h^{\omega} u_{\underline{h}}\|_{L^2(\Omega_1 \cup \Omega_2)} = \|\overline{u} - w_h\|_{L^2(\Omega_1 \cup \Omega_2)} = \alpha_h > 0.$$

Consequently, for sufficiently small $\underline{h}$, we have

$$C_{\underline{h}} \gtrsim \underline{h}^{-\frac{1}{2}} \|(I - \overline{Q}_h^{\omega})u\|_{L^2(\Omega_1 \cup \Omega_2)} \gtrsim \tfrac{1}{2}\alpha_h \underline{h}^{-\frac{1}{2}}.$$

This completes the proof.

Remark. The questions concerning logrithmic factors appearing in the estimates of Theorems 1.2 and 1.3 are more subtle. The author does not know whether they are necessary.

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Department of Mathematics, Penn State University, University Park, Pennsylvania 16802

E-mail address: xu@math.psu.edu