

CONVERGENCE RESULTS FOR PSEUDOSPECTRAL APPROXIMATIONS OF HYPERBOLIC SYSTEMS BY A PENALTY-TYPE BOUNDARY TREATMENT

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ABSTRACT. In a previous paper we have presented a new method of imposing boundary conditions in the pseudospectral Chebyshev approximation of a *scalar* hyperbolic equation. The novel idea of the new method is to collocate the equation at the boundary points as well as in the inner grid points, using the boundary conditions as penalty terms. In this paper we extend the above boundary treatment to the case of pseudospectral approximations to general constant-coefficient hyperbolic *systems* of equations, and we provide error estimates for the pseudospectral Legendre method. The same scheme can be implemented also in the general (even nonlinear) case.

1. INTRODUCTION

The importance of the treatment of boundary conditions, in the approximation of hyperbolic systems of equations, is widely recognized. The pioneering works of Gustafsson, Kreiss, and Sundström [6] and Osher [7] provide the stability theory for the implementation of boundary conditions in the framework of the finite difference method. In fact, the question of checking the stability of a difference scheme for initial-boundary value hyperbolic problems is reduced to an algebraic problem.

The role of boundary conditions in spectral (or pseudospectral) methods is even more crucial. Spectral methods, being *global* in nature, are more sensitive to local treatments, such as boundary conditions, and therefore greater care must be taken in handling the boundaries.

In [3] the authors introduced a new method of applying the boundary conditions in the pseudospectral Chebyshev approximation of a *scalar* hyperbolic equation. The novel idea is to collocate the differential equation not only at the inner grid points, but also at the boundary itself, making use of the fact that spectral derivatives can be obtained at all grid points including the boundaries. Thus, the boundary conditions are introduced as penalty terms. In [3]

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the stability of the scheme is shown for the *scalar* case, and the merits of the new treatment are discussed.

The purpose of this paper is to extend the ideas presented in [3] to the case of hyperbolic *systems* of equations. We show how to implement the scheme for a general collocation method based on orthogonal polynomials. We also prove stability and show error estimates for the pseudospectral Legendre method applied to constant-coefficient *systems*. The extension of this idea to the Chebyshev method is trivial; however, it is more difficult to obtain similar convergence estimates.

The paper is organized as follows. In §2 we describe the method in the scalar case and prove error estimates for the pseudospectral Legendre method. In §3 we treat the case of the Legendre method applied to a *system* in diagonal form, where the unknowns are coupled via the boundary conditions. We derive error estimates for the constant-coefficient problem. In §4, we show how to implement the method for *systems* in nondiagonal form.

2. ESTIMATES FOR THE SCALAR EQUATION

In this section we derive error estimates for the new pseudospectral Legendre approximation to a scalar hyperbolic constant-coefficient problem. Let $U \equiv U(x, t)$ be the solution of

$$(2.1) \quad \begin{cases} \text{(a) } U_t = aU_x, & x \in [-1, 1], \quad 0 < t < T, \\ \text{(b) } U(1, t) = h(t), & 0 < t < T, \\ \text{(c) } U(x, 0) = f(x), & x \in [-1, 1], \end{cases}$$

where h and f are given boundary and initial functions and $a \in \mathbf{R}$, $a > 0$.

Let \mathbf{P}_N denote the space of algebraic polynomials of degree less than or equal to N . In the pseudospectral Legendre method we approximate U by $v \equiv v(x, t)$, which is a \mathbf{P}_N -polynomial in the variable x for any $0 \leq t < T$. This is done by demanding that v satisfies equation (2.1)(a) at the grid points x_j ($j = 1, \dots, N$). The points $x_0 = 1 > x_1 > \dots > x_{N-1} > x_N = -1$, taken here in decreasing order, are the nodes of the Gauss-Lobatto quadrature formula, i.e., x_j ($j = 0, \dots, N$) are the extrema in $[-1, 1]$ of the N th-degree Legendre polynomial P_N .

The choice of this particular grid allows an accurate evaluation of integrals by summing over the grid values. Namely, let ω_j ($j = 0, \dots, N$) be the weights of the Gauss-Lobatto formula; then for any polynomial p of degree at most $2N - 1$ the following equality holds (see, for instance, [2]):

$$(2.2) \quad \int_{-1}^1 p \, dx = \sum_{j=0}^N p(x_j) \omega_j.$$

In the following, we will set $\omega = \omega_0 = \omega_N = 2/N(N + 1)$.

Our new method differs from the classical one by the treatment of the boundary condition. The approximating polynomial v does not satisfy exactly the

boundary condition (2.1)(b); instead, it satisfies the differential equation (2.1)(a) at all the grid points with a penalty-type correction at the boundary. That is, v is a solution of

$$(2.3) \quad \begin{cases} \text{(a)} & v_t = av_x \quad \text{at } x = x_j, \quad j = 1, \dots, N, \\ \text{(b)} & v_t(1, t) = av_x(1, t) - \gamma(v(1, t) - h(t)), \\ \text{(c)} & v(x_j, 0) = f(x_j), \quad j = 0, \dots, N. \end{cases}$$

The coefficient $\gamma \equiv \gamma_N > 0$ in (2.3)(b) can be viewed as the amplitude of the boundary penalty term and will be specified later so that the stability of (2.3) is guaranteed.

It is convenient to compare $v(\cdot, t)$ with some \mathbf{P}_N -projection of the solution $U(\cdot, t)$. To this end, we introduce two projection operators, denoted by I_N and Π_N .

Definition 2.1. $I_N U$ is the polynomial in \mathbf{P}_N that interpolates U at the points x_j ($j = 0, \dots, N$), i.e., $I_N U(x_j) = U(x_j)$, $j = 0, \dots, N$.

Definition 2.2. $\Pi_N U$ is the polynomial in \mathbf{P}_N that is the best approximation of U in the $H^1(-1, 1)$ norm satisfying the condition $\Pi_N U(\pm 1) = U(\pm 1)$.

Here, $H^\sigma(-1, 1)$, $\sigma \in \mathbf{R}$, is the usual Sobolev space with $H^0(-1, 1) = L^2(-1, 1)$.

In the next theorem we estimate the error $E = \Pi_N U - v$. We note that, as in the Chebyshev case (see [3]), γ has to be proportional to N^2 for stability.

Theorem 2.1. Consider the scalar equation (2.1), and let v denote its pseudospectral approximation (2.3) depending on the boundary penalty amplitude γ . Then, if $\gamma \geq a/\omega = \frac{1}{2}N(N+1)a$, we have the error estimate

$$(2.4) \quad \sum_{j=0}^N E^2(x_j, T)\omega_j + a \int_0^T (E^2(1, t) + E^2(-1, t)) dt \leq \exp(1) \left(\sum_{j=0}^N (I_N f - \Pi_N f)^2(x_j)\omega_j + T \int_0^T \sum_{j=0}^N Q^2(x_j, t)\omega_j dt \right),$$

where $Q = a[\Pi_N U_x - (\Pi_N U)_x]$.

Proof. Applying Π_N to (2.1), one gets

$$(2.5) \quad \begin{cases} (\Pi_N U)_t = a(\Pi_N U)_x + Q & \text{at } x = x_j, \quad j = 0, \dots, N, \\ (\Pi_N U)(1, t) = h(t), & 0 < t < T, \\ (\Pi_N U)(x_j, 0) = (\Pi_N f)(x_j), & j = 0, \dots, N. \end{cases}$$

Therefore, upon subtracting (2.3) from (2.5), we have the following error equation:

$$(2.6) \quad \begin{cases} \text{(a)} & E_t = aE_x + Q \quad \text{at } x = x_j, \quad j = 1, \dots, N, \\ \text{(b)} & E_t(1, t) = aE_x(1, t) + Q(1, t) - \gamma E(1, t), \\ \text{(c)} & E(x_j, 0) = (\Pi_N f - I_N f)(x_j), \quad j = 0, \dots, N. \end{cases}$$

Note that E is a polynomial of degree at most N . Multiplying (2.6)(a) by $E(x_j, t)\omega_j$ ($j = 1, \dots, N$), (2.6)(b) by $E(1, t)\omega_0$, and summing, we get (using (2.2)):

$$(2.7) \quad \sum_{j=0}^N (E_t E)(x_j, t)\omega_j = a \int_{-1}^1 E_x E dx + \sum_{j=0}^N (QE)(x_j, t)\omega_j - \gamma \omega E^2(1, t), \quad 0 \leq t < T.$$

Integration by parts yields

$$(2.8) \quad \begin{aligned} \frac{d}{dt} \sum_{j=0}^N E^2(x_j, t)\omega_j &= (a - 2\gamma\omega)E^2(1, t) - aE^2(-1, t) \\ &\quad + 2 \sum_{j=0}^N (QE)(x_j, t)\omega_j \\ &\leq -a(E^2(1, t) + E^2(-1, t)) + \frac{1}{T} \sum_{j=0}^N E^2(x_j, t)\omega_j \\ &\quad + T \sum_{j=0}^N Q^2(x_j, t)\omega_j, \quad 0 \leq t < T, \end{aligned}$$

where we used the assumption $a - 2\gamma\omega \leq -a$. Finally the Gronwall lemma yields (2.4). \square

Recall now that there exist two positive constants C_1 and C_2 such that one has, for any $p \in \mathbf{P}_N$ (see [1, p. 286]):

$$(2.9) \quad C_1 \int_{-1}^1 p^2 dx \leq \sum_{j=0}^N p^2(x_j)\omega_j \leq C_2 \int_{-1}^1 p^2 dx.$$

In addition, the following error estimates concerning the projectors I_N and Π_N can be found in [1, pp. 293 and 291]:

$$(2.10) \quad \begin{aligned} \|U - I_N U\|_{H^\mu(-1, 1)} &\leq CN^{2\mu-\sigma} \|U\|_{H^\sigma(-1, 1)}, \\ &\forall U \in H^\sigma(-1, 1), \quad \sigma > \frac{1}{2}, \quad 0 \leq \mu \leq \sigma; \end{aligned}$$

$$(2.11) \quad \begin{aligned} \|U - \Pi_N U\|_{H^\mu(-1, 1)} &\leq CN^{\mu-\sigma} \|U\|_{H^\sigma(-1, 1)}, \\ &\forall U \in H^\sigma(-1, 1), \quad \sigma \geq 1, \quad 0 \leq \mu \leq \sigma. \end{aligned}$$

Using the above inequalities, we can get an estimate of the error between the solution U and the new approximate v .

Theorem 2.2. *Let $U \in L^2(0, T; H^\sigma(-1, 1)) \cap L^\infty(0, T; H^{\sigma-1}(-1, 1))$, $\sigma \geq 1$, be the solution of (2.1). Let v be the solution of the pseudospectral scheme (2.3)*

with $\gamma \geq a/\omega$. Then we have the error bounds

$$(2.12) \quad \begin{aligned} & \| (U - v)(\cdot, T) \|_{L^2(-1, 1)} \\ & \leq CN^{1-\sigma} (\sqrt{T} \|U\|_{L^2(0, T; H^\sigma(-1, 1))} + \|U(\cdot, T)\|_{H^{\sigma-1}(-1, 1)} \\ & \quad + \|f\|_{H^{\sigma-1}(-1, 1)}), \end{aligned}$$

$$(2.13) \quad \begin{aligned} & \left(\int_0^T (U - v)^2(\pm 1, t) dt \right)^{1/2} \\ & \leq CN^{1-\sigma} (\sqrt{T} \|U\|_{L^2(0, T; H^\sigma(-1, 1))} + \|f\|_{H^{\sigma-1}(-1, 1)}). \end{aligned}$$

Proof. We begin by noting that (2.11) implies for $Q = a[\Pi_N U_x - (\Pi_N U)_x]$

$$(2.14) \quad \begin{aligned} \|Q\|_{L^2(-1, 1)} & \leq a \|\Pi_N U - U\|_{H^1(-1, 1)} + a \|\Pi_N U_x - U_x\|_{L^2(-1, 1)} \\ & \leq CN^{1-\sigma} \|U\|_{H^\sigma(-1, 1)}, \quad \sigma \geq 1. \end{aligned}$$

Then, (2.12) is easily obtained by the previous theorem, by (2.9), (2.10), and by the triangle inequality:

$$(2.15) \quad \|U - v\|_{L^2(-1, 1)} \leq \|U - \Pi_N U\|_{L^2(-1, 1)} + \|\Pi_N U - v\|_{L^2(-1, 1)}.$$

To show (2.13), it is sufficient to observe that, by definition, $\Pi_N U$ coincides with U at $x = \pm 1$ for any t , so that

$$(2.16) \quad \int_0^T (\Pi_N U - U)^2(\pm 1, t) dt = 0, \quad 0 < t < T,$$

and then use the estimates shown above. \square

Note that the same results hold when $a < 0$ and the role of the boundary points $x = 1$ and $x = -1$ is interchanged.

3. ERROR ANALYSIS FOR DIAGONAL SYSTEMS OF EQUATIONS

Consider the hyperbolic system

$$(3.1) \quad \begin{cases} U_t^{(1)} = -A^{(1)} U_x^{(1)}, \\ U_t^{(2)} = A^{(2)} U_x^{(2)}, \end{cases} \quad x \in [-1, 1], \quad 0 < t < T,$$

where $U^{(1)}(x, t)$ is a vector function of $k_1 \in \mathbf{N}$ components and $U^{(2)}(x, t)$ is a vector function of $k_2 \in \mathbf{N}$ components, $\forall x \in [-1, 1], \forall t \in [0, T], T > 0$. In (3.1), $A^{(1)}$ is a $k_1 \times k_1$ constant matrix and $A^{(2)}$ is a $k_2 \times k_2$ constant matrix. Since (3.1) is a hyperbolic system, we can assume that $A^{(1)}$ and $A^{(2)}$ are positive diagonal matrices.

System (3.1) is subject to initial conditions

$$(3.2) \quad \begin{cases} U^{(1)}(x, 0) = f_1(x), \\ U^{(2)}(x, 0) = f_2(x), \end{cases} \quad x \in [-1, 1],$$

and it is coupled through the boundary conditions

$$(3.3) \quad \begin{cases} U^{(1)}(-1, t) = LU^{(2)}(-1, t) + g_1(t), \\ U^{(2)}(1, t) = RU^{(1)}(1, t) + g_2(t), \end{cases} \quad 0 < t < T,$$

where f_i, g_i ($i = 1, 2$) are given functions. Here, R and L are respectively $k_2 \times k_1$ and $k_1 \times k_2$ matrices, which determine the reflection of the ingoing characteristic variables in terms of the outgoing ones at the boundaries.

Let r and l be the norms of the operators R and L respectively, i.e.,

$$(3.4) \quad r = \|R\|_{\mathcal{L}(\mathbf{R}^{k_1}, \mathbf{R}^{k_2})}, \quad l = \|L\|_{\mathcal{L}(\mathbf{R}^{k_2}, \mathbf{R}^{k_1})}.$$

We discuss the case in which the solution to (3.1), (3.2), (3.3) does not grow in time. This is assured by demanding that

$$(3.5) \quad 0 < rl < 1.$$

In the new collocation method we seek $u^{(i)}(x, t) \in \mathbf{R}^{k_i}$, $x \in [-1, 1]$, $t \in [0, T]$, $i = 1, 2$. The vectors $u^{(i)}$, $i = 1, 2$, whose components are \mathbf{P}_N -polynomials in the variable x , are determined by the following collocation scheme:

$$(3.6) \quad \begin{cases} u_t^{(1)} = -A^{(1)}u_x^{(1)} & \text{at } x = x_j, \quad j = 0, \dots, N-1, \\ u_t^{(2)} = A^{(2)}u_x^{(2)} & \text{at } x = x_j, \quad j = 1, \dots, N, \end{cases}$$

$$(3.7) \quad \begin{cases} u_t^{(1)}(-1, t) = -A^{(1)}u_x^{(1)}(-1, t) \\ \quad - \Gamma^{(1)}\{u^{(1)}(-1, t) - Lu^{(2)}(-1, t) - g_1(t)\}, \\ u_t^{(2)}(1, t) = A^{(2)}u_x^{(2)}(1, t) - \Gamma^{(2)}\{u^{(2)}(1, t) - Ru^{(1)}(1, t) - g_2(t)\}, \end{cases}$$

$$(3.8) \quad \begin{cases} u^{(1)}(x_j, 0) = f_1(x_j), & j = 0, \dots, N, \\ u^{(2)}(x_j, 0) = f_2(x_j), & j = 0, \dots, N. \end{cases}$$

Here $\Gamma^{(i)}$, $i = 1, 2$, are positive-definite $k_i \times k_i$ diagonal matrices, which can be viewed as amplitudes of the boundary penalty terms, analogous to the role of γ in the scalar case. These matrices will be specified later. Note that we took into account at the points $x_0 = -1$ and $x_N = 1$ both the equations and the boundary conditions.

In order to study the convergence of the \mathbf{P}_N -approximations $u^{(1)}$ and $u^{(2)}$ to the exact solution $U^{(1)}$ and $U^{(2)}$ when N tends to $+\infty$, we introduce an auxiliary collocation scheme. Namely, we look for $v^{(1)}$ and $v^{(2)}$, such that

$$(3.9) \quad \begin{cases} v_t^{(1)} = -A^{(1)}v_x^{(1)} & \text{at } x = x_j, \quad j = 0, \dots, N-1, \\ v_t^{(2)} = A^{(2)}v_x^{(2)} & \text{at } x = x_j, \quad j = 1, \dots, N, \end{cases}$$

$$(3.10) \quad \begin{cases} v_t^{(1)}(-1, t) = -A^{(1)}v_x^{(1)}(-1, t) - \Gamma^{(1)}\{v^{(1)}(-1, t) - U^{(1)}(-1, t)\}, \\ v_t^{(2)}(1, t) = A^{(2)}v_x^{(2)}(1, t) - \Gamma^{(2)}\{v^{(2)}(1, t) - U^{(2)}(1, t)\}, \end{cases}$$

$$(3.11) \quad \begin{cases} v^{(1)}(x_j, 0) = f_1(x_j), & j = 0, \dots, N, \\ v^{(2)}(x_j, 0) = f_2(x_j), & j = 0, \dots, N. \end{cases}$$

The advantage of considering (3.9), (3.10), (3.11) is that the new problem is fully decoupled into $k_1 + k_2$ independent scalar equations. Therefore, the analysis of §2 will enable us to estimate the difference $U^{(i)} - v^{(i)}$, $i = 1, 2$. Thus, it remains to bound the difference $u^{(i)} - v^{(i)}$, $i = 1, 2$, due to the coupling at the boundaries.

Theorem 3.1. *Let $d^{(i)} = u^{(i)} - v^{(i)}$, $i = 1, 2$, where $u^{(i)}$ and $v^{(i)}$ are the solutions of (3.6)–(3.8) and (3.9)–(3.11), respectively. Let $\Gamma^{(i)} = A^{(i)}/\omega\sqrt{rl}$, where $\omega = \omega_0 = \omega_N$. Then we have the estimate*

$$(3.12) \quad \begin{aligned} & \sum_{j=0}^N (r\|B^{(1)}d^{(1)}(x_j, T)\|^2 + l\|B^{(2)}d^{(2)}(x_j, T)\|^2)\omega_j \\ & \leq \frac{rl}{2(1 - \sqrt{rl})} \int_0^T \left(\sqrt{\frac{r}{l}}\|(U^{(1)} - v^{(1)})(1, t)\|^2 \right. \\ & \quad \left. + \sqrt{\frac{l}{r}}\|(U^{(2)} - v^{(2)})(-1, t)\|^2 \right) dt, \end{aligned}$$

where $B^{(i)} = \sqrt{[A^{(i)}]^{-1}}$, $i = 1, 2$.

Proof. From (3.6)–(3.11) it is clear that the $d^{(i)}$ satisfy the equations

$$(3.13) \quad \begin{cases} d_t^{(1)} = -A^{(1)}d_x^{(1)} & \text{at } x = x_j, \quad j = 0, \dots, N - 1, \\ d_t^{(2)} = A^{(2)}d_x^{(2)} & \text{at } x = x_j, \quad j = 1, \dots, N, \end{cases}$$

$$(3.14) \quad \begin{cases} d_t^{(1)}(-1, t) = -A^{(1)}d_x^{(1)}(-1, t) \\ \quad - \Gamma^{(1)}\{d^{(1)}(-1, t) - Ld^{(2)}(-1, t) + LE^{(2)}(-1, t)\}, \\ d_t^{(2)}(1, t) = A^{(2)}d_x^{(2)}(1, t) \\ \quad - \Gamma^{(2)}\{d^{(2)}(1, t) - Rd^{(1)}(1, t) + RE^{(1)}(1, t)\}, \end{cases}$$

$$(3.15) \quad \begin{cases} d^{(1)}(x_j, 0) = 0, & j = 0, \dots, N, \\ d^{(2)}(x_j, 0) = 0, & j = 0, \dots, N, \end{cases}$$

where $E^{(i)} = U^{(i)} - v^{(i)}$, $i = 1, 2$. The scalar analysis provides us with estimates of the errors $E^{(i)}$, $i = 1, 2$. Note that (3.3) has been used to eliminate $U^{(1)}(-1, t)$ and $U^{(2)}(1, t)$.

Now, let $D^{(i)}$, $i = 1, 2$, be two diagonal positive-definite $k_i \times k_i$ matrices to be specified later. Multiplying the first set of equations in (3.13) and (3.14)

by $(D^{(1)}d^{(1)})(x_j, t)\omega_j$ and summing over $j = 0, \dots, N$, we get by (2.2)

$$\begin{aligned}
 & \sum_{j=0}^N \langle d_t^{(1)}, D^{(1)}d^{(1)} \rangle(x_j, t)\omega_j \\
 &= - \int_{-1}^1 \langle A^{(1)}d_x^{(1)}, D^{(1)}d^{(1)} \rangle(x, t) dx \\
 (3.16) \quad & - \omega \langle \Gamma^{(1)}d^{(1)} - \Gamma^{(1)}Ld^{(2)}, D^{(1)}d^{(1)} \rangle(-1, t) \\
 & - \omega \langle \Gamma^{(1)}LE^{(2)}, D^{(1)}d^{(1)} \rangle(-1, t) \\
 & \leq - \frac{1}{2} \langle A^{(1)}d^{(1)}, D^{(1)}d^{(1)} \rangle(1, t) + \frac{1}{2} \langle A^{(1)}d^{(1)}, D^{(1)}d^{(1)} \rangle(-1, t) \\
 & - \omega \langle \Gamma^{(1)}d^{(1)} - \Gamma^{(1)}Ld^{(2)}, D^{(1)}d^{(1)} \rangle(-1, t) \\
 & + \frac{\eta}{2} \omega \langle \Gamma^{(1)}d^{(1)}, D^{(1)}d^{(1)} \rangle(-1, t) + \frac{\omega}{2\eta} \|\sqrt{D^{(1)}\Gamma^{(1)}LE^{(2)}}(-1, t)\|^2,
 \end{aligned}$$

for any $0 < \eta < 2$. Similarly, we have

$$\begin{aligned}
 (3.17) \quad & \sum_{j=0}^N \langle d_t^{(2)}, D^{(2)}d^{(2)} \rangle(x_j, t)\omega_j \\
 & \leq \frac{1}{2} \langle A^{(2)}d^{(2)}, D^{(2)}d^{(2)} \rangle(1, t) - \frac{1}{2} \langle A^{(2)}d^{(2)}, D^{(2)}d^{(2)} \rangle(-1, t) \\
 & - \omega \langle \Gamma^{(2)}d^{(2)} - \Gamma^{(2)}Rd^{(1)}, D^{(2)}d^{(2)} \rangle(1, t) \\
 & + \frac{\eta}{2} \omega \langle \Gamma^{(2)}d^{(2)}, D^{(2)}d^{(2)} \rangle(1, t) + \frac{\omega}{2\eta} \|\sqrt{D^{(2)}\Gamma^{(2)}RE^{(1)}}(1, t)\|^2.
 \end{aligned}$$

Note that η , $D^{(1)}$, and $D^{(2)}$ are yet to be specified. We start by setting $\eta = 2(1 - \sqrt{rl})$. Recalling the hypotheses on $\Gamma^{(i)}$, by adding (3.16) and (3.17), one obtains

$$\begin{aligned}
 & \frac{d}{dt} \left(\sum_{j=0}^N \left[\|\sqrt{D^{(1)}}d^{(1)}\|^2 + \|\sqrt{D^{(2)}}d^{(2)}\|^2 \right] (x_j, t)\omega_j \right) \\
 & \leq - \left[\|\sqrt{A^{(1)}D^{(1)}}d^{(1)}\|^2 - \frac{2}{\sqrt{rl}} \langle A^{(2)}Rd^{(1)}, D^{(2)}d^{(2)} \rangle \right. \\
 & \qquad \qquad \qquad \left. + \|\sqrt{A^{(2)}D^{(2)}}d^{(2)}\|^2 \right] (1, t) \\
 (3.18) \quad & - \left[\|\sqrt{A^{(1)}D^{(1)}}d^{(1)}\|^2 - \frac{2}{\sqrt{rl}} \langle A^{(1)}Ld^{(2)}, D^{(1)}d^{(1)} \rangle \right. \\
 & \qquad \qquad \qquad \left. + \|\sqrt{A^{(2)}D^{(2)}}d^{(2)}\|^2 \right] (-1, t) \\
 & + \frac{1}{2(1 - \sqrt{rl})\sqrt{rl}} \left(\|\sqrt{D^{(2)}A^{(2)}}RE^{(1)}(1, t)\|^2 \right. \\
 & \qquad \qquad \qquad \left. + \|\sqrt{D^{(1)}A^{(1)}}LE^{(2)}(-1, t)\|^2 \right).
 \end{aligned}$$

At this point we want the two first terms in brackets on the right-hand side of (3.18) to be positive. This holds if we choose

$$D^{(1)} = (A^{(1)})^{-1}r^2, \quad D^{(2)} = (A^{(2)})^{-1}rl.$$

Indeed, in this case the quantity

$$r^2\|d^{(1)}\|^2 - 2\sqrt{rl}\langle Rd^{(1)}, d^{(2)}\rangle + rl\|d^{(2)}\|^2$$

is positive in view of the straightforward inequality

$$|2\sqrt{rl}\langle Rd^{(1)}, d^{(2)}\rangle| \leq \|Rd^{(1)}\|^2 + rl\|d^{(2)}\|^2 \leq r^2\|d^{(1)}\|^2 + rl\|d^{(2)}\|^2.$$

Similar arguments hold for the second term. Thus, we are left with the last term, which allows us to write the following inequality:

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{j=0}^N \left[r\|B^{(1)}d^{(1)}(x_j, t)\|^2 + l\|B^{(2)}d^{(2)}(x_j, t)\|^2 \right] \omega_j \right) \\ (3.19) \quad & \leq \frac{1}{2(1-\sqrt{rl})\sqrt{rl}} (l\|RE^{(1)}(1, t)\|^2 + r\|LE^{(2)}(-1, t)\|^2) \\ & \leq \frac{rl}{2(1-\sqrt{rl})} \left(\sqrt{\frac{r}{l}}\|E^{(1)}(1, t)\|^2 + \sqrt{\frac{l}{r}}\|E^{(2)}(-1, t)\|^2 \right). \end{aligned}$$

Finally, by integrating in time, we obtain (3.12). \square

Using the results of the previous section, we can finally prove our main convergence theorem.

Theorem 3.2. *Let $U^{(i)}$, $i = 1, 2$, be the solutions of (3.1), (3.2), (3.3), and let $u^{(i)}$, $i = 1, 2$, be the solutions of (3.6), (3.7), (3.8) with*

$$\Gamma^{(i)} = A^{(i)}N(N+1)/2\sqrt{rl}.$$

Then we have the following error estimate:

$$\begin{aligned} (3.20) \quad & \left\| \left(\sum_{i=1}^2 \|U^{(i)} - u^{(i)}\|^2 \right)^{1/2} (\cdot, T) \right\|_{L^2(-1, 1)} \\ & \leq CN^{1-\sigma}T \left\| \left(\sum_{i=1}^2 \|U^{(i)}\|^2 \right)^{1/2} \right\|_{C^0([0, T]; H^\sigma(-1, t))}, \end{aligned}$$

where C depends only on l, r, σ , and $A^{(i)}$, $i = 1, 2$.

Proof. We first write $U^{(i)} - u^{(i)} = U^{(i)} - v^{(i)} - d^{(i)}$. Then a bound for each component $U^{(i)} - v^{(i)}$, $i = 1, 2$, is given by (2.12), while for $d^{(i)}$ we have

$$\begin{aligned}
 & \left\| \left(\sum_{i=1}^2 \|d^{(i)}\|^2 \right)^{1/2} (\cdot, T) \right\|_{L^2(-1, 1)} \\
 &= \left\| \left(\sum_{i=1}^2 \|[B^{(i)}]^{-1} B^{(i)} d^{(i)}\|^2 \right)^{1/2} (\cdot, T) \right\|_{L^2(-1, 1)}^2 \\
 (3.21) \quad &\leq \max_{i=1,2} \{ \|[B^{(i)}]^{-1}\|_{\mathcal{L}(\mathbf{R}^{k_i}, \mathbf{R}^{k_i})}^2 \} \left\| \left(\sum_{i=1}^2 \|B^{(i)} d^{(i)}\|^2 \right)^{1/2} (\cdot, T) \right\|_{L^2(-1, 1)}^2 \\
 &\leq C(r, l) \max_{i=1,2} \max_{1 \leq j \leq k_i} \{ (a_j^{(i)})^2 \} \left(\int_0^T \|U^{(1)} - v^{(1)}\|^2(1, t) dt \right. \\
 &\qquad \qquad \qquad \left. + \int_0^T \|U^{(2)} - v^{(2)}\|^2(-1, t) dt \right),
 \end{aligned}$$

where $a_j^{(i)}$ are the diagonal entries of $A^{(i)}$, $i = 1, 2$, and we used Theorem 3.1. Finally, the last term in (3.21) is estimated as in (2.13). \square

4. SUGGESTIONS FOR THE IMPLEMENTATION OF NONDIAGONAL SYSTEMS

In this section we discuss the implementation of our new approach in the case of the general hyperbolic system

$$(4.1) \qquad \qquad \qquad \tilde{U}_t = \tilde{A} \tilde{U}_x,$$

where $\tilde{U} \equiv \tilde{U}(x, t)$ is a k -component vector and \tilde{A} is a constant $k \times k$ matrix with k_1 negative eigenvalues and k_2 positive eigenvalues ($k = k_1 + k_2$). The following boundary conditions are imposed at $x = -1$:

$$(4.2) \qquad \qquad \qquad \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix} \tilde{U}(-1, t) = \begin{pmatrix} h_1(t) \\ 0 \end{pmatrix},$$

where B_{11} is a $k_1 \times k_1$ matrix, B_{12} is a $k_1 \times k_2$ matrix, and h_1 is a given k_1 -component vector. In addition, the following boundary conditions are imposed at $x = 1$:

$$(4.3) \qquad \qquad \qquad \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix} \tilde{U}(1, t) = \begin{pmatrix} 0 \\ h_2(t) \end{pmatrix},$$

where B_{21} is a $k_2 \times k_1$ matrix, B_{22} is a $k_2 \times k_2$ matrix, and h_2 is a given k_2 -component vector.

By hyperbolicity, there exists a nonsingular matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where T_{ij} is a $k_i \times k_j$ matrix, such that the change of variables $\tilde{U} = TU$ diagonalizes the system (4.1). Thus, we get

$$(4.4) \quad \frac{\partial U}{\partial t} = T^{-1} \tilde{A} T \frac{\partial U}{\partial x} = \begin{pmatrix} -A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \frac{\partial U}{\partial x},$$

where $U \equiv (U^{(1)}, U^{(2)})$ and the $A^{(i)}$, $i = 1, 2$, have been defined in §3.

The boundary conditions are respectively transformed as follows:

$$(4.5) \quad (B_{11}T_{11} + B_{12}T_{21})U^{(1)}(-1, t) + (B_{11}T_{12} + B_{12}T_{22})U^{(2)}(-1, t) = h_1(t),$$

$$(4.6) \quad (B_{21}T_{11} + B_{22}T_{21})U^{(1)}(1, t) + (B_{21}T_{12} + B_{22}T_{22})U^{(2)}(1, t) = h_2(t).$$

Therefore, (4.2) and (4.3) are equivalent to (3.3) if and only if the matrices $B_{11}T_{11} + B_{12}T_{21}$ and $B_{21}T_{12} + B_{22}T_{22}$ are invertible. In this case we have:

$$\begin{aligned} L &= -(B_{11}T_{11} + B_{12}T_{21})^{-1}(B_{11}T_{12} + B_{12}T_{22}), \\ R &= -(B_{21}T_{12} + B_{22}T_{22})^{-1}(B_{21}T_{11} + B_{22}T_{21}), \\ g_1 &= (B_{11}T_{11} + B_{12}T_{21})^{-1}h_1, \quad g_2 = (B_{21}T_{12} + B_{22}T_{22})^{-1}h_2. \end{aligned}$$

Thus, assumption (3.5) is transferred to the B_{ij} .

We would like to show how to apply the scheme (3.9)–(3.11) directly to the system (4.1)–(4.3). According to Theorem 3.2 we set $\Gamma^{(i)} = \beta A^{(i)}$, $i = 1, 2$, where $\beta = N(N + 1)/2\sqrt{rl}$. We also set (denoting by $u \equiv (u^{(1)}, u^{(2)})$ the approximation to U)

$$\begin{aligned} \Xi(x, t) &= \begin{pmatrix} u^{(1)}(x, t) - Lu^{(2)}(x, t) - g_1(t) \\ u^{(2)}(x, t) - Ru^{(1)}(x, t) - g_2(t) \end{pmatrix} \\ &= \begin{pmatrix} I & -L \\ -R & I \end{pmatrix} u(x, t) - \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t). \end{aligned}$$

Thus, (3.1)–(3.3) can be written in the form

$$(4.7) \quad \begin{aligned} \frac{\partial u}{\partial t}(x_j, t) &= A \frac{\partial u}{\partial x}(x_j, t) + \beta \delta_{Nj} A^{(-)} \Xi(-1, t) \\ &\quad - \beta \delta_{0j} A^{(+)} \Xi(1, t), \quad j = 0, \dots, N, \end{aligned}$$

where

$$A^{(-)} = \begin{pmatrix} -A^{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \quad A^{(+)} = \begin{pmatrix} 0 & 0 \\ 0 & A^{(2)} \end{pmatrix}, \quad A = A^{(-)} + A^{(+)},$$

and δ_{ij} is the Kronecker delta.

Let us now define $\tilde{u} = Tu$. The function \tilde{u} satisfies

$$(4.8) \quad \begin{aligned} \frac{\partial \tilde{u}}{\partial t}(x_j, t) = & \tilde{A} \frac{\partial \tilde{u}}{\partial x}(x_j, t) + \beta \delta_{Nj} (TA^{(-)}T^{-1})T\Xi(-1, t) \\ & - \beta \delta_{0j} (TA^{(+)}T^{-1})T\Xi(1, t), \quad j = 0, \dots, N. \end{aligned}$$

Defining $B = T \begin{pmatrix} I & -L \\ -R & I \end{pmatrix} T^{-1}$, we get

$$(4.9) \quad T\Xi = B\tilde{u} - T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Finally, taking $\tilde{A}^{(\pm)} = TA^{(\pm)}T^{-1}$ and substituting in (4.8), we obtain the pseudospectral scheme for approximating (4.1)–(4.3), namely

$$(4.10) \quad \begin{aligned} \frac{\partial \tilde{u}}{\partial t}(x_j, t) = & \tilde{A} \frac{\partial \tilde{u}}{\partial x}(x_j, t) + \beta \delta_{Nj} \tilde{A}^{(-)} \left[B\tilde{u}(-1, t) - T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t) \right] \\ & - \beta \delta_{0j} \tilde{A}^{(+)} \left[B\tilde{u}(1, t) - T \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t) \right], \quad j = 0, \dots, N. \end{aligned}$$

This is equivalent to collocating the equation (4.1) at all the points with some suitable penalty terms, derived from the boundary conditions, added at the points $x = \pm 1$. It is clear that the same convergence estimates of Theorem 3.2 also apply for the error $\tilde{U} - \tilde{u} = T(U - u)$.

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