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Supplement to

NUMERICAL APPROXIMATIONS OF ALGEBRAIC RICCATI EQUATIONS FOR ABSTRACT SYSTEMS MODELLED BY ANALYTIC SEMIGROUPS, AND APPLICATIONS

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<u>Section 3</u>

<u>Proof of Lemma 3.1</u>. (i) We compute, with Π_h the orthogonal projection H onto V_h, after adding and subtracting

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(H;U)} \le \|B_h^*(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t})\|_{\mathcal{L}(H;U)} + \|B^* e^{A^* t} - B_h^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \tag{S.3.1}$$

(using (A.3) = (1.16), the rough data estimate (1.20) for $\Theta = 1$ and $\Pi_h^2 = \Pi_h$ in the first term of (S.3.1); and (A.5) = (1.18) on the second term of (S.3.1))

$$\le C\,h^{-\gamma s}\,\frac{h^s e^{(\omega_0+\varepsilon)t}}{t} + C\,h^{s(1-\gamma)}\|A^* e^{A^* t}\|_{\mathcal{L}(H)}$$

$$\le C\,h^{s(1-\gamma)}\,\frac{e^{(\omega_0+\varepsilon)t}}{t}, \tag{S.3.2}$$

where in the last step we have used the analyticity of $e^{A^* t}$. Thus (3.1) is proved.

(ii) Similarly,

$$\|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \le \|B_h^*(e^{A_h^* t}\Pi_h - \Pi_h e^{A^* t})\|_{\mathcal{L}(\mathcal{D}(A^*);U)}$$

$$+ \|B^* e^{A^* t} - B_h^*\Pi_h e^{A^* t}\|_{\mathcal{L}(\mathcal{D}(A^*);U)} \tag{S.3.3}$$

(using (A.3) = (1.16) and (1.22) on the first term of (S.3.3); and (A.5) = (1.18) on the second term of (S.3.3)),

$$\le \tilde{C}\,h^{-\gamma s}h^s + C\,h^{s(1-\gamma)}\,e^{(\omega_0+\varepsilon)t}, \tag{S.3.4}$$

and *a fortiori* (3.2) follows from (S.3.4).

(iii) Eq. (3.3) follows from (3.1) and (3.2) by use of the interpolation (moment) inequality [17, p. 19].

(iv) First, from the full assumption (A.3) = (1.16) and uniform analyticity (A.1) = (1.14a), we obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\,h^{-\gamma s}\,e^{(\omega_0+\varepsilon)t}. \tag{S.3.5}$$

Next, we shall obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\,\frac{h^{s(1-\gamma)}}{t}\,e^{(\omega_0+\varepsilon)t} \quad t > 0, \tag{S.3.6}$$

through a computation similar to the ones above. Indeed, adding and subtracting

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le \|B_h^* e^{A_h^* t}\Pi_h - B^* e^{A^* t}\|_{\mathcal{L}(H;U)} + \|B^*(I-\Pi_h)e^{A^* t}\|_{\mathcal{L}(H;U)} \tag{S.3.7}$$

(using (3.1) of part (i) on the first term of (S.3.7) and (A.4) = (1.17) on the second term of (S.3.7)),

$$\le C\,\frac{h^{s(1-\gamma)}}{t}\left[e^{(\omega_0+\varepsilon)t} + \|A^* e^{A^* t}\|_{\mathcal{L}(H)}\right], \tag{S.3.8}$$

and (S.3.6) follows from (S.3.8) by analyticity of $e^{A^* t}$. Next, we raise (S.3.5) to the power $(1-\gamma)$, we raise (S.3.6) to the power γ, and we multiply the resulting expressions together. This way we obtain

$$\|B_h^* e^{A_h^* t}\Pi_h - B^*\Pi_h e^{A^* t}\|_{\mathcal{L}(H;U)} \le C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^{\gamma}}, \quad t > 0. \tag{S.3.9}$$

On the other hand, by assumption (A.6) = (1.19) and analyticity of $e^{A^* t}$, we obtain, recalling the notation in the standing assumption (ii) below (1.2):

$$\|B^*\Pi_h e^{A^*t}\|_{\mathcal{L}(H;U)} \le C\|(\hat{A}^*)^{\gamma} e^{A^*t}\|_{\mathcal{L}(H)}$$

$$= C\, e^{\omega t}\|(\hat{A}^*)^{\gamma} e^{-\hat{A}t}\|_{\mathcal{L}(H)} \le C\,\frac{e^{(\omega-\hat{\omega})t}}{t^{\gamma}} = C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^{\gamma}}. \qquad \text{(S.3.10)}$$

joining (S.3 9) with (S.3.10), we obtain (3.4) as desired.

(v) First, from assumption (1.3) and analyticity of e^{A^*t} we have

$$\|B^* e^{A^*t}\|_{\mathcal{L}(H,U)} \le C\|(\hat{A}^*)^{\gamma} e^{A^*t}\|_{\mathcal{L}(H)} \le C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^{\gamma}}, \quad t > 0, \qquad \text{(S.3.11)}$$

calling the computations leading to (S 3 10) Then (S.3 11) and (3 4) of part (iv)

lly

$$\|B_h^* e^{A_h^*t}\Pi_h - B^* e^{A^*t}\|_{\mathcal{L}(H;U)} \le C\,\frac{e^{(\omega_0+\varepsilon)t}}{t^{\gamma}}, \quad t > 0. \qquad \text{(S.3.12)}$$

Finally, we raise (3.1) of part (i) to the power Θ, we raise (S 3.12) to power Θ), and multiply the resulting expressions together. This way we obtain (3 5) .

(vi) Eq (3 6) follows from (3.5) and (3 2) via the interpolation (moment) equality.

Lemma 3.1 is completely proved ■

<u>proof of Theorem 3.2</u>. (i) We compute from (2.3), (2.12) after recalling the estimate 5) of Lemma 3.1(v), with $\beta = \Theta + (1-\Theta)\gamma$, where $\gamma \le \beta < 1$, for any $\Theta < 1$ and $\gamma < 1$:

$$\|L_h^* v - L^* v\|^2_{L_2(0,T;U)} = \left\|\int_t^T [B_h^* e^{A_h^*(\tau-t)}\Pi_h - B^* e^{A^*(\tau-t)}]v(\tau)d\tau\right\|^2_{L_2(0,T;U)}$$

$$\le C_T^2\, h^{2s(1-\gamma)\Theta}\int_0^T\left[\int_t^T \frac{1}{(\tau-t)^{\beta}}\|v(\tau)\|_H d\tau\right]^2 dt$$

$$\le C_T^2\, h^{2s(1-\gamma)\Theta}\int_0^T \left[\int_t^T \frac{1}{(\tau-t)^{\beta}}d\tau\right]\left[\int_t^T \frac{\|v(\tau)\|_H^2}{(\tau-t)^{\beta}}d\tau\right]dt$$

$$\le C_T^2\, T^{1-\beta}\, h^{2s(1-\gamma)\Theta}\int_0^T \|v(\tau)\|_H^2\left[\int_\tau^T \frac{1}{(\tau-t)^{\beta}}dt\right]d\tau$$

$$\le C_{T,\beta} h^{2s(1-\gamma)\Theta}\|v\|^2_{L_2(0,T;H)} \qquad \text{(S.3.13)}$$

after using the Schwarz inequality and changing the order of integration. Then (S.3.13) proves (3.7).

(ii) Similarly, from (2.1) and (2.11), again by use of estimate (3.5) with $\gamma \le \beta = \Theta + (1-\Theta)\gamma < 1$:

$$\|L_h u - Lu\|_{C([0,T];H)} = \left\|\int_0^t (e^{A_h(t-\tau)}B_h - e^{A(t-\tau)}B)u(\tau)d\tau\right\|_{C([0,T];H)}$$

$$\le C\, h^{s(1-\gamma)\Theta} \sup_{0\le t\le T}\left[\int_0^T \frac{1}{(t-\tau)^{\beta}}\|u(\tau)\|_U d\tau\right]$$

$$\le C_{T,\beta} h^{s(1-\gamma)\Theta}\|u\|_{L_\infty(0,T;U)}, \qquad \text{(S 3 14)}$$

and (S.3.14) proves (3.8). ■

<u>Proof of Theorem 3.3</u> (i) From (2.5) and (2 13), we compute with $v \in L_2(0,\infty;H)$

$$\|\hat{L}_h^* v - \hat{L}^* v\|^2_{L_2(0,\infty;U)} = \int_0^\infty \left\|\int_t^\infty [B_h^* e^{-\hat{A}_h^*(\tau-t)}\Pi_h - B^* e^{-\hat{A}^*(\tau-t)}]v(\tau)d\tau\right\|_U^2 dt$$

$$\le C^2 h^{2s(1-\gamma)\Theta}\int_0^\infty\left[\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^{\beta}}\|v(\tau)\|_H d\tau\right]^2 dt, \qquad \text{(S.3 15)}$$

after recalling (3 5) with $\gamma \le \beta = \theta+(1-\theta)\gamma < 1$, for any $\theta < 1$ and $\gamma < 1$ (as in the proof of Theorem 3 2), as well as $-\hat{A}^* = A^*-\omega I$, $\omega_0+\varepsilon-\omega = -\hat{\omega} < 0$ from (ii) below (1.2), and $-\hat{A}_h^* = A_h^*-\omega I$ from (2 15) Next, as in the proof of Theorem 3.2(i),

$$\int_0^\infty\int_t^\infty \frac{e^{\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}\|v(\tau)\|_H d\tau\, dt \le \int_0^\infty\Big(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}d\tau\Big)\Big(\int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}\|v(\tau)\|^2 d\tau\Big)dt \tag{S.3.16}$$

$$\le C_\beta\int_0^\infty\Big(\int_0^\tau \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\beta}d\tau\Big)\|v(\tau)\|^2 d\tau \tag{S 3 17}$$

$$\le C_\beta\|v\|^2_{L_2(0,\infty;H)}, \tag{S.3 18}$$

since the first internal integral in (S.3.16) is bounded by a constant, and after changing the order of integration from (S 3 16) to (S.3.17). Then (S 3.15) and (S 3.18) prove (3 9)

(ii) The proof is similar to that of part (i) ∎

<u>Section 4</u>

4.1.1. <u>Uniform analyticity</u>

<u>Proof of Lemma 4.1</u> <u>Step 1</u> We have stipulated above (4 3) that, for the sake of definiteness, we are taking the case $\Sigma(F) \cap \Sigma_{app}(A)$. Thus, for all $\lambda \in \Sigma^c_{app}(A) = \Sigma^c_{app}(A,a;\theta_a)$, the complement of $\Sigma_{app}(A)$, we have that $R(\lambda,A_F)$ and $R(\lambda,A_h)$ are well-defined, bounded operators on $L_2(\Omega)$ and V_h respectively, $\forall\, h \le h_a$.

<u>Step 2</u>. Given $\delta > 0$, there exist $r_\delta > 0$ and $0 < h_\delta \le h_a$, such that

$$\|R(\lambda,A_h)B_hF_h\|_{\mathcal{L}(V_h)} < \delta, \quad \text{for all } \lambda \text{ and } h \text{ as in (4.4).} \tag{S.4 1}$$

Indeed, (S 4 1) follows from

$$\|B_h^*R(\bar{\lambda},A_h^*)\|_{\mathcal{L}(H;U)} \le \frac{C}{|\bar{\lambda}-a|^{1-\gamma}}, \quad \lambda \in \Sigma^c_{app}(A), \tag{S.4.2}$$

which is the Laplace transform version in the λ-domain of estimate (3.4), Lemma 3 1(iv), in the t-domain (to be proved by contour integration, as usual), by taking $C\|F_h\|/|\lambda-a|^{1-\gamma} < \delta$ with $\|F_h\|$ uniformly bounded.

<u>Step 3</u>. We have

$$R(\lambda,A_F) = [I-R(\lambda,A)BF]^{-1}R(\lambda,A), \quad \forall\, \lambda \in \Sigma^c(A_F); \tag{S.4.3}$$

$$R(\lambda,A_{h,F_h}) = [I-R(\lambda,A_h)B_hF_h]^{-1}R(\lambda,A_h), \quad \forall\, \lambda \in \Sigma^c(A); \tag{S.4.4}$$

and, in both cases, $|\lambda|$ sufficiently large, and h sufficiently small in (S.4.4). In fact, (S.4.3) and (S.4.4) are the standard perturbation identities for the perturbed operators in (4.1) and (4.3) respectively, where we note, in the case of (S 4.4), that

$$\|[I-R(\lambda,A_h)B_hF_h]^{-1}\|_{\mathcal{L}(V_h)} \le \frac{1}{1-\|R(\lambda,A_h)B_hF_h\|_{\mathcal{L}(V_h)}}$$

$$\text{(by (S.4.1))} \qquad \le \frac{1}{1-\delta}, \ \lambda \text{ and } h \text{ as in (4.4).} \tag{S.4.5}$$

In the case of (S.4.3), we recall (1.3) and obtain the continuous version of (S.4.1),

$$\|R(\lambda,A)BF\| = \|R(\lambda,A)\hat{A}^\gamma\hat{A}^{-\gamma}BF\| \le \frac{C}{|\lambda-\omega_0|^{1-\gamma}} \to 0 \text{ as } |\lambda| \to \infty, \quad \lambda \in \Sigma^c_{app}(A). \tag{S.4 6}$$

Then the analog of (S.4.5) needed for (S.4.3) follows in the same way.

<u>Step 4</u>. The desired estimate (4 4) of part (i) follows from (S.4.4) and (S 4 5). Then (4.4) implies the desired estimate (4.5) of part (ii) via $\|R(\lambda,A_h)\|_{\mathcal{L}(V_h)} \le C/|\lambda-a|$, $\lambda \in \Sigma^c_{app}(A)$, see (A.1) = (1.14b).

Finally, the desired estimate (4.6) of part (iii) follows from (4.5) in the usual

ly. We write $R(\lambda,A_{h,F_h})A_{h,F_h} = \lambda R(\lambda,A_{h,F_h})-I$, and use on this (4.5), thereby obtaining

6) for $\Theta = 1$ Then the cases $\Theta = 0$ and $\Theta = 1$ imply the cases $0 < \Theta < 1$ via the

iterpolation (moment) inequality. Lemma 4 1 is proved. ■

1.2. Uniform exponential stability of A_{h,F_h} and A_{F_h}

roof of Theorem 4.2 Orientation From Lemma 4.1 we know, *a fortiori*, that $e^{A_{h,F_h}t}$ are iiformly (in h) analytic on H and that the spectrum $\sigma(A_{h,F_h})$ is uniformly (in h) intained in a common sector, which preliminarily can be taken to be $\Sigma^c_{app}(A)$ The next ep is to show that as a consequence of (4.8), in fact, $\sigma(A_{h,F_h})$ satisfies (4.11), i.e., a (4.10), $\sigma(A_{h,F_h})$ is contained on a three-sided sector on the left-hand side of the mplex plane. Finally, uniform analyticity combined with the 'correct' location of the iectrum will imply the remaining parts (ii) = (4 12) and part (iii) = (4.13) of Theorem 2 via operator calculus Details follows We begin with

mma S.4.1.

) Under the same assumptions as in Lemma 4.1, let, for the sake of definiteness, λ_0 be fixed with Re $\lambda_0 > r_\delta$, r_δ as in (4.4). Then the following convergence holds true for all $\varepsilon_0 < s(1-\gamma)$:

$$\|R(\lambda_0,A_F)-R(\lambda_0,A_{h,F_h})\Pi_h\|_{\mathcal{L}(H)} \le h^{\varepsilon_0}+\|F_h-F\|_{\mathcal{L}(U;H)} \to 0 \quad \text{as } h \downarrow 0, \tag{S 4 7}$$

for all $\varepsilon_0 < s(1-\gamma)$;

i) in (S.4.7), one may replace the chosen λ_0 with any other $\lambda \in \rho(A_F)$;

ii) for any compact set $\mathcal{K} \subset \rho(A_F)$, we have

$$\sup_{\lambda\in\mathcal{K};h} \|R(\lambda,A_{h,F_h})\|_{\mathcal{L}(H)} \le \text{const}_{\mathcal{K}}. \tag{S.4.8}$$

Proof of Lemma S.4.1. (i) Recalling (S.4.3) and (S.4.4), we compute in the $\mathcal{L}(H)$-norm:

$$\|R(\lambda_0,A_F)-R(\lambda_0,A_{h,F_h})\Pi_h\| = \|[I-R(\lambda_0,A)BF]^{-1}R(\lambda_0,A)-[I-R(\lambda_0,A_h)B_hF_h]^{-1}R(\lambda_0,A_h)\Pi_h\|$$

$$= \|(1)+(2)\|, \tag{S.4.9}$$

where, after adding and subtracting,

$$(1) = [I-R(\lambda_0,A)BF]^{-1}(R(\lambda_0,A)-R(\lambda_0,A_h)\Pi_h); \tag{S.4.10}$$

$$(2) = \{[I-R(\lambda_0,A)BF]^{-1}-[I-R(\lambda_0,A_h)B_hF_h]^{-1}\}R(\lambda_0,A_h)\Pi_h \tag{S.4.11}$$

Thus, by assumption (A.2) = (1.15),

$$\|(1)\| \le C\,h^s\|[I-R(\lambda_0,A)BF]^{-1}\| \to 0 \quad \text{as } h \downarrow 0. \tag{S.4 12}$$

As to (2), we use the identity

$$[I-T_1]^{-1}-[I-T_2]^{-1} = [I-T_1]^{-1}[T_1-T_2][I-T_2]^{-1} \tag{S.4.13}$$

in (S.4.11) with $T_1 = R(\lambda_0,A)BF$, $T_2 = R(\lambda_0,A_h)B_hF_h$. By (S.4.4), $R(\lambda_0,A_{h,F_h}) = [I-T_2]^{-1}R(\lambda_0,A_h)$. Using these and $K = \|[I-T_1]^{-1}\|$, we can write from (S.4.11),

$$\|(2)\| \le c_r K\|R(\lambda_0,A_{h,F_h})\|\;\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\|$$

$$\le \frac{c}{1-\delta}\,\frac{1}{|\lambda-a|}\,\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\|, \tag{S.4.14}$$

where in the last step we have used (4.5) of Lemma 4.1(ii), with $0 < \delta < 1$, preassigned, and $0 < h \le h_\delta$. We next compute the term in (S.4.14),

$$\|R(\lambda_0,A)BF-R(\lambda_0,A_h)B_hF_h\| \le \|[F^*-F_h^*]B^*R(\lambda_0,A^*)\|+\|F_h^*[B^*R(\lambda_0,A^*)-B_h^*R(\lambda_0,A_h^*)\Pi_h]\|. \quad (S.4.15)$$

As to the first term in (S.4.15), we invoke assumption (4.9) to see that it tends to 0 as $h \downarrow 0$. As to the second term in (S.4.15), we use the assumption $\|F_h\| \le$ const, as well as the Laplace (λ-) version of the estimate (3.5) of Lemma 3.1(v) with $\Theta < 1$, to be proved by contour integration, thereby obtaining

$$\|F_h^*[B^*R(\lambda_0,A^*)-B_h^*R(\lambda_0,A_h^*)\Pi_h]\| \le \text{const } h^{s(1-\gamma)\Theta} \to 0 \text{ as } h \downarrow 0. \quad (S.4.16)$$

Thus, the term (2) in (S.4.14) also tends to zero as $h \downarrow 0$. Then, by (S.4.9), the desired convergence (S.4.7) in part (i) is proved.

(ii) The statement for any other $\lambda \in \rho(A_F)$ follows now from standard results [13, Thm. 3.15, p. 206; also Remark 3.13, p. 211].

(iii) Part (iii), Eq. (S.4.8) is a consequence of the joint continuity of the resolvent $R(\lambda,A_F)$ in both arguments [13, Thm. 3.15, p. 212]. ■

Continuing with the proof of Theorem 4.2, we return to Lemma 4.1(i), Eq. (4.4): Given $1 > \delta > 0$, there exist r_δ, $h_\delta > 0$ such that for all $0 < h < h_\delta$,

$$[\Sigma^c_{app}(A) \cap \{|\lambda| \ge r_\delta\}] \subset \rho(A_{h,F_h}), \quad (S.4.17)$$

$\rho(\cdot)$ denoting the resolvent set.

We next complement the statement in (S.4.17) by virtue of the following:

Lemma S.4.2. For any $\varepsilon' > 0$ there exists $h_{\varepsilon'} > 0$ such that

$$\sup \text{Re } \sigma(A_{h,F_h}) \le -\omega_F+\varepsilon', \quad 0 < h \le h_{\varepsilon'}, \quad (S.4.18)$$

where ε' may be taken 0 if $-\omega_F \in \rho(A_F)$.

Proof of Lemma S.4.2. The proof of this result is the same as the proof of [18, Lemma 4.4] and is omitted here. ■

Thus, conclusion (4.10) of Theorem 4.2 has been proved. In order to complete the proof of Theorem 4.2, we combine the results of Lemma 4.1, Lemmas S.4.1 and S.4.2, and we integrate along a path in $\Sigma^c_{app}(A_F)$ in (4.10) which follows its boundary. The computations are the same as those given in [18, pp. 200-201] and will not be repeated here. ■

4.3. Uniform stability of the feedback semigroup $\exp(A_{h,P_h}t)$

Proof of Theorem 4.6. The first step is a consequence of Theorem 4.2.

Step 1. Lemma S.4.3. We have

$$\|P_h\|_{\mathcal{L}(V_h)} \le \text{const, uniformly in } h. \quad ■ \quad (S.4.19)$$

Proof of Lemma S.4.3. We note first that the assumption of uniform convergence (4.9) for Theorem 4.2 holds true for the choice $F_h = F\Pi_h$, by virtue of the compactness assumption (1.27a), or else (1.27b), see Remark 4.2. Thus, Theorem 4.2(ii), Eq. (4.12), implies that $\exp\{(A_{h,F_h})t\} = \exp\{(A_h+B_hF_h)t\}$ is uniformly stable. For the approximating optimal control problem (1.23), (1.24) on V_h with optimal pair

$$\Phi_h(t)x = y_h^0(t;x) = e^{A_{h,P_h}t}x; \quad u_h^0(t;x) = -B_h^*P_h e^{A_{h,P_h}t}x \quad (S.4.20)$$

and initial point $x \in V_h$, the feedback control $F_h e^{A_{h,F_h} t} x$ and corresponding solution $e^{A_{h,F_h} t} x$ form a competing pair. Thus, by (2.20) we get

$$J(u_h^0, y_h^0) = (P_h x, x)_H = \int_0^\infty \|u_h^0(t;x)\|_U^2 + \|R_h y_h^0(t;x)\|_Z^2 dt$$

$$\leq \int_0^\infty \left\|F_h e^{A_{h,F_h} t} x\right\|_U^2 + \left\|R_h e^{A_{h,F_h} t} x\right\|_Z^2 dt$$

$$\leq C \int_0^\infty e^{2(-\omega_F+\epsilon)t} dt \|x\|^2 \leq C\|x\|_H^2, \tag{S.4.21}$$

here in the last step we have invoked (4.12) Since P_h is non-negative, self-adjoint, (S.4.21) yields $\|P_h\|_{\mathcal{L}(V_h)} = \sup(P_h x, x) \leq C$, over all $x \in V_h$ with $\|x\| \leq 1$ ∎

<u>step 2</u>. <u>Lemma S.4.4</u> We have

$$\int_0^\infty \left\|e^{A_{h,P_h} t} x\right\|_H^2 dt \leq c\|x\|_H^2, \quad x \in H. \tag{S.4.22}$$

<u>Poof of Lemma S.4.4</u>. If $R > 0$, then (S.4.22) is a direct consequence of (S.4.21) via (S.4.20). Otherwise, we shall use, as usual, the more general detectability assumption which leads to the uniform estimate (4.19). Writing by (4.16) and (4.26),

$$A_{h,P_h} = A_{h,K} - \Pi_h K R \Pi_h - B_h B_h^* P_h, \tag{S.4.23}$$

and recalling that $y_h^0 = A_{h,P_h} y_h^0$ for the approximating problem we have by (S.4.23),

$$y_h^0(t,x) = e^{A_{h,P_h} t} x = e^{A_{h,K} t} x - \int_0^t e^{A_{h,K}(t-\tau)} \Pi_h K R \Pi_h y_h^0(\tau,x) d\tau$$

$$- \int_0^t e^{A_{h,K}(t-\tau)} B_h B_h^* P_h y_h^0(\tau,x) d\tau \tag{S.4.24}$$

$$= e^{A_{h,K} t} x - \{L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot,x)\}(t)$$

$$- \{L_{h,K} B_h B_h^* P_h y_h^0(\cdot,x)\}(t), \tag{S.4.25}$$

after recalling the operators $L_{h,K}$ in (4.22), whose regularity properties (4.23), (4.24) will now be invoked. In fact, by (4.24) we have the first step of

$$\|L_{h,K} \Pi_h K R \Pi_h y_h^0(\cdot,x)\|_{L_2(0,\infty;H)} \leq C\|R \Pi_h y_h^0(\cdot,x)\|_{L_2(0,\infty;Z)}$$

$$\leq C\|y_h^0(\cdot,x)\|_{L_2(0,\infty;H)}$$

$$\text{(by (S.4.20) and by (4.27))} \quad = C\left\|e^{A_{h,P_h} \cdot} x\right\|_{L_2(0,\infty;H)} \leq C\|x\|_H, \tag{S.4.26}$$

as desired. Similarly, by (4.23) we have the first step of

$$\|L_{h,K} B_h B_h^* P_h y_h^0(\cdot,x)\|_{L_2(0,\infty;H)} \leq C\|B_h^* P_h y_h^0(\cdot,x)\|_{L_2(0,\infty;U)}$$

$$\text{(by (S.4.20) and by (S.4.21))} \quad \leq C\|u_h^0(\cdot,x)\|_{L_2(0,\infty;U)} \leq C\|x\|_H, \tag{S.4.27}$$

as desired. Finally, using (S.4.26), (S.4.27), in (S.4.22), as well as (4.19) for the first term in (S.4.25), we obtain (S.4.22) as desired. ∎

<u>Step 3</u>. <u>Proposition S.4.5</u>. There are numbers $c > 0$ and $a > 0$, independent of h, such that

$$\left\| e^{A_{h,P_h} t} \right\|_{\mathcal{L}(V_h)} \leq c\, e^{at}, \quad t \geq 0. \tag{S.4.28}$$

<u>Proof of Proposition S.4.5</u>. We shall prove (S.4.28), in fact with $a = \omega$, by using a 'bootstrap' argument, as in [18], based on the following equations for the optimal pair:

$$e^{-\omega t} e^{A_{h,P_h} t} x_h = \hat{y}_h^0(t, x_h) = e^{-\hat{A}_h t} x_h + \{\hat{L}_h \hat{u}_h^0\}(t) \tag{S.4.29}$$

$$\hat{u}_h^0(t, x_h) = -\{\hat{L}_h^* [R^* R + 2\omega P_h] \hat{y}_h^0(\cdot, x_h)\}(t) \tag{S 4.30}$$

with $x_h \in V_h$, see (2 21), (2.22) in Section 2. The 'bootstrap' argument uses the following result, which is of interest only in the more demanding situation where $\frac{1}{2} < \gamma < 1$

<u>Lemma S.4.6</u>. For the operators $\hat{L}_h$ and $\hat{L}_h^*$ defined by (2.13), (2.14), we have

(i) $$\hat{L}_h : \text{continuous } L_2(0,\infty;U) \to L_r(0,\infty;H) \text{ uniformly in } h \downarrow 0, \tag{S.4.31}$$

$$\text{i.e., } \sup_{h>0} \|\hat{L}_h\|_{2,r} \leq \text{const},$$

where r is an arbitrary number satisfying $r < 2/(2\gamma-1)$, where $2/(2\gamma-1) > 2$ for $\frac{1}{2} < \gamma < 1$; for $0 \leq \gamma \leq \frac{1}{2}$, one can take $r = \infty$.

(ii) $$\hat{L}_h^* : \text{continuous } L_r(0,\infty;H) \to L_{r'}(0,\infty;U), \text{ uniformly in } h \downarrow 0, \tag{S.4.32}$$

$$\text{i.e., } \sup_{h>0} \|\hat{L}_h^*\|_{r,r'} \leq \text{const},$$

where r is as in (i), and r' is any number satisfying $r' < 2/(4\gamma-3)$, where $\frac{3}{4} < \gamma < 1$, $2/(4\gamma-3) > r$; for $0 < \gamma < \frac{3}{4}$, we can take $r' = \infty$.

(iii) With $p > 1(1-\gamma)$,

$$\hat{L}_h \cdot \text{continuous } L_p(0,\infty;U) \to C([0,\infty];H), \text{ uniformly in } h \downarrow 0, \tag{S.4.33}$$

$$\text{i e., } \sup_{h>0} \|\hat{L}_h\|_{p,\text{contin}} < \infty.$$

<u>Proof of Lemma S.4.6</u> As in [18], the proof is based on Young's inequality [23, p. 29]. By using Lemma 3.1, inequality (3 4), as well as $\hat{A}_h = -A_h + \omega I$, $-\hat{\omega} = \omega_0 + \epsilon - \omega$, we have preliminarily

$$\|e^{-\hat{A}_h t} B_h\|_{\mathcal{L}(U;V_h)} = \|B_h^* e^{-\hat{A}_h t} \Pi_h\|_{\mathcal{L}(V_h;U)} = \mathcal{O}\left(\frac{e^{-\hat{\omega} t}}{t^\gamma}\right). \tag{S.4.34}$$

Thus, it follows from (2 13), respectively (2.14), via (S.4.34) that

$$\|(\hat{L}_h u)(t)\|_H \leq C \int_0^t \frac{e^{-\hat{\omega}(t-\tau)}}{(t-\tau)^\gamma} \|u(\tau)\|_U d\tau; \tag{S.4.35}$$

$$\|(\hat{L}_h^* v)(t)\|_U \leq C \int_t^\infty \frac{e^{-\hat{\omega}(\tau-t)}}{(\tau-t)^\gamma} \|v(\tau)\| d\tau. \tag{S.4.36}$$

Then parts (i) and (ii) follow immediately from (S.4.35), (S.4.36) via Young's inequality. Part (iii) follows likewise with $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1 = 0$, where we have $\gamma q < 1$, and $\frac{1}{q} = 1 - \frac{1}{p} \downarrow \gamma$ as $p \downarrow \frac{1}{1-\gamma}$. The proof of Lemma S.4.6 is complete. ∎

To complete the proof of Proposition S.4.5, Eq. (S.4.28), we start with $\hat{u}_h^0 \in L_2(0,\infty;U)$ and apply a bootstrap argument on (S.4.29), (S.4.30) using Lemma S.4.6. After a finite number of iterations we obtain $(\hat{u}_h^0 \in L_\infty(0,\infty;U)$ and) $y_h^0 \in C([0,\infty];H)$ from which (S 4.28) follows from (S.4.29) with the constant $a = \omega$. ∎

Step 4. Starting from the uniform bound in (S.4.28) of Proposition S.4 5, we can complete the proof of Theorem 4.6 and obtain estimate (4.27) by simply proceeding as in the continuous case; see [22, p. 121]. Theorem 4 6 is proved ■

4. Uniform regularity of P_h

Proof of Theorem 4.7. (i) We return to identity (2 19) for P_h and obtain with $x \in V_h$,

$$(\hat{A}_h^*)^{\Theta} P_h x = \int_0^\infty (\hat{A}_h^*)^{\Theta} e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt \tag{S 4 37}$$

voking estimate (S 4 16) of Lemma S.4.3 and analyticity, we obtain

$$\|(\hat{A}_h^*)^{\Theta} P_h\|_{\mathcal{L}(H)} \le C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^{\Theta}} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(H)} dt, \quad 0 \le \Theta < 1. \tag{S.4.38}$$

then (4.28) of part (i) follows immediately from (S.4.35) via (2.16) and the uniform and (4.27) of Theorem 4 6 for $\Phi_h(t)$.

(ii) The proof of (4.29) is similar· From (2.19) with $x \in V_h$,

$$B_h^* P_h x = \int_0^\infty B_h^* e^{-\hat{A}_h^* t} \Pi_h [R^* R + 2\omega P_h] \hat{\Phi}_h(t) x \, dt, \tag{S.4.39}$$

and invoking estimates (3.4) of Lemma 3 1 and (S.4.16) of Lemma S.4 3, and $-\hat{A}_h = A_h - \omega I$, $= \omega_0 + \epsilon - \omega$, we obtain from (S.4.36),

$$\|B_h^* P_h\|_{\mathcal{L}(H;U)} \le C \int_0^\infty \frac{e^{-\hat{\omega} t}}{t^{\gamma}} \|\hat{\Phi}_h(t)\|_{\mathcal{L}(H)} dt, \tag{S.4.40}$$

and (4.29) follows likewise via (2.16) and (4.27).

(iii) Part (iii) follows from part (i) via self-adjoint calculus as in [10, Lemma 3.3]. ■

Section 5

5.1. Uniform convergence $P_h \Pi_h \to P$ of Riccati operators

Proof of Theorem 5.1.

Proof. Step 1. The following four operators will play a key role. The first and the fourth, defined by (1.12) and (4.26), refer to the optimal dynamics, continuous and discrete. The second and third are introduced here for the first time They will define competitive dynamics:

$$A_P = A - BB^* P; \quad A_{h,P} = A_h - B_h B^* P, \tag{S.5.1}$$

$$A_{P_h} = A - BB_h^* P_h; \quad A_{h,P_h} = A_h - B_h B_h^* P_h. \tag{S.5.2}$$

The semigroup generated by A_P is analytic and stable, Section 1.1. As to the other operators, we have

Proposition S.5.1. The semigroups generated by the operators $A_{h,P}$, A_{P_h}, A_{h,P_h} are both uniformly analytic (in the sense of Lemma 4.1) and uniformly stable.

<u>Proof</u>. In the case of A_{h,P_h}, uniform analyticity was established in Corollary 4 8, while uniform stability was established in Theorem 4.6, Eq (4.27). The same properties then hold true for $A_{h,P}$ as special case of the latter, where $F_h \equiv B^*P$. Next, uniform stability of $e^{A_{P_h}t}$ follows from Remark 4.4, where we already know that $e^{A_{h,P_h}t}$ is uniformly stable, and thus we take $F = F_h = B_h^*P_h$, so that the required assumption (4.9) holds true Finally, uniform analyticity of $\exp(A_{P_h}t)$ follows from Lemma 4.1 with $A_h \equiv A$, $B_h \equiv B$, and $F_h \equiv B_h^*P_h$ which is uniformly bounded by (4.29), as required. ■

<u>Step 2</u> <u>Proposition S.5.2</u>. Let ε_0 be the same number $\varepsilon_0 < s(1-\gamma)$ as in Lemma S.4.1, Eq (S 4.7) of the Supplement. Then

$$\|A_P^{-1}-A_{h,P}^{-1}\|_{\mathcal{L}(H)} \le C\,h^{\varepsilon_0} \to 0 \quad \text{as } h \to 0; \tag{S.5.3}$$

$$\|A_{P_h}^{-1}-A_{h,P_h}^{-1}\|_{\mathcal{L}(H)} \le C\,h^{\varepsilon_0} \to 0 \quad \text{as } h \to 0. \tag{S.5.4}$$

<u>Proof</u>. By use of the first resolvent equation for resolvent operators, it suffices to show (S.5 3), (S.5.4) with $R(0,\cdot)$ replaced by $R(\lambda_0,\cdot)$, for Re λ_0 sufficiently large. In this latter case then, these desired bounds follow from Lemma 4.3 with $F = F_h = B^*P$ in the case of (S 5 3), and with $F = F_h = B_h^*P_h$ in the case of (S.5.4), where we note that the fact that now F depends on h does not make any difference in the argument of Lemma S.4 1 as long as $\|F_h\| \le$ const, which is true by (4.29). ■

<u>Step 3</u> By (1 10) and (2.23), we have

$$|(([P_h\Pi_h-P]x,x)_H| = |J(u_h^0(\cdot,\Pi_h x),y_h^0(\cdot,\Pi_h x)-J(u^0(\cdot,x),y^0(\cdot,x))|. \tag{S.5.5}$$

Now, with x and h fixed, if $J(u_h^0,y_h^0)-J(u^0,y^0) > 0$ we introduce the competing pair

$$\tilde{u}_h(t,\Pi_h x) = -B^*P\,e^{A_{h,P}t}\Pi_h x;\quad \tilde{y}_h(t,\Pi_h x) = e^{A_{h,P}t}\Pi_h x, \tag{S.5.6}$$

for the approximating problem so that in this case

$$|J(u_h^0,y_h^0)-J(u^0,y^0)| \le J(\tilde{u}_h,\tilde{y}_h)-J(u^0,y^0). \tag{S.5.7}$$

Instead, if $J(u^0,y^0)-J(u_h^0,y_h^0) > 0$, we introduce the competing pair

$$\bar{u}_h(t,x) = -B_h^*P_h\,e^{A_{P_h}t}\Pi_h x,\quad \bar{y}_h(t,x) = e^{A_{P_h}t}\Pi_h x, \tag{S 5.8}$$

for the continuous problem so that in this case

$$|J(u^0,y^0)-J(u_h^0,y_h^0)| < J(\bar{u},\bar{y})-J(u_h^0,y_h^0). \tag{S.5.9}$$

Thus, in all cases, we have from (S.5.7), (S.5.9),

$$|J(u_h^0,y_h^0)-J(u^0,y^0)| \le |J(\tilde{u}_h,\tilde{y}_h)-J(u^0,y^0)| + |J(\bar{u}_h,\bar{y}_h)-J(u_h^0,y_h^0)|. \tag{S.5.10}$$

Recalling (S.5.6), (S.5.8), we rewrite the right-hand side (R.H.S.) of (S.5.10) after recalling the costs (1.2) and (1.24), as well as (S.4.17)

$$\text{R.H.S. of (S.5.10)} = \Big|\int_0^\infty \{\|R\,e^{A_Pt}x\|_Z^2-\|R\,e^{A_{h,P}t}\Pi_h x\|_Z^2$$

$$+\|B^*P\,e^{A_Pt}x\|_U^2-\|B^*P\,e^{A_{h,P}t}\Pi_h x\|_U^2\}dt\Big|+\Big|\int_0^\infty\{\|R\,e^{A_{P_h}t}\Pi_h x\|_Z^2-\|R\,e^{A_{h,P_h}t}\Pi_h x\|_Z^2$$

$$+\|B_h^*P_h e^{A_{P_h}t}x\|_U^2-\|B_h^*P_h e^{A_{h,P_h}t}\Pi_h x\|_U^2\}dt\Big| \tag{S.5.11}$$

$$\leq C\left\{\int_0^\infty \|e^{A_P t}x - e^{A_{h,P_h}t}\Pi_h x\|_H dt + \int_0^\infty \|e^{A_{P_h}t}\Pi_h x - e^{A_{h,P_h}t}\Pi_h x\|_H dt\right\}\|x\|_H, \quad (S.5.12)$$

ere in the last step we have used $|\|a\|^2 - \|b\|^2| \leq \|a-b\|(\|a\|+\|b\|)$, as well as $^*P\| + \|B_h^* P_h\| \leq$ const, and also the uniform exponential decay of the semigroups. Finally, litting the integration over $[0,h^s]$ and $[h^s,\infty]$ and using uniform bounds on $[0,h^s]$, we tain from (S.5.12),

$$\text{R.H.S. of (S.5 10)} \leq C\|x\|_H\left\{h^s\|x\|_H + \left\|e^{A_P t}x - e^{A_{h,P_h}t}\Pi_h x\right\|_{L_1(h^s,\infty)}\right.$$
$$\left. + \left\|e^{A_{P_h}t}\Pi_h x - e^{A_{h,P_h}t}\Pi_h x\right\|_{L_1(h^s,\infty)}\right\}. \quad (S.5.13)$$

w, if Γ. $\rho e^{\pm i\Theta} - k$, $\frac{\pi}{2} < \Theta < \pi$, $0 \leq \rho < \infty$, $k > 0$, is the boundary of a triangular sector ataining the spectrum of A_P, A_{h,P_h}, uniformly in h, as guaranteed by Proposition S.5.1, compute

$$\|e^{A_P t} - e^{A_{h,P_h}t}\Pi_h\|_{\mathcal{L}(H)} = \left\|\int_\Gamma e^{\lambda t}R(\lambda,A_P) - R(\lambda,A_{h,P_h})d\lambda\right\| \quad (S.5.14)$$

$$= \left\|\int_\Gamma e^{\lambda t}R(\lambda,A_P)(A_P - A_{h,P_h})R(\lambda,A_{h,P_h})d\lambda\right\| \quad (S.5.15)$$

$$= \left\|\int_\Gamma e^{\lambda t}R(\lambda,A_P)A_P(A_P^{-1} - A_{h,P_h}^{-1})A_{h,P_h}R(\lambda,A_{h,P_h})d\lambda\right\|$$

$$\leq C\left(\int_\Gamma e^{(\mathrm{Re}\,\lambda)t}|d\lambda|\right)\|A_P^{-1} - A_{h,P_h}^{-1}\|_{\mathcal{L}(H)}, \quad (S.5.16)$$

where in going from (S.5.14) to (S.5.15) we have used an identity like the one in (S.4.13), while in going from (S.5.15) to (S.5.16), we have used analyticity of $e^{A_P t}$, uniform analyticity of $e^{A_{h,P_h}t}$, see Proposition S.5.1, in the sense of (4.6) Lemma 4.1 with $\Theta = 1$. Thus from (S.5.16) we obtain, with $k > 0$ (below (S.5.13)) and $t > 0$,

$$\left\|e^{A_P t} - e^{A_{h,P_h}t}\right\|_{\mathcal{L}(H)} \leq C\frac{e^{-kt}}{t}\left\|A_P^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(H)}, \quad (S.5.17)$$

and by a similar argument,

$$\left\|e^{A_{P_h}t}\Pi_h - e^{A_{h,P_h}t}\Pi_h\right\|_{\mathcal{L}(H)} \leq C\frac{e^{-kt}}{t}\left\|A_{P_h}^{-1} - A_{h,P_h}^{-1}\right\|_{\mathcal{L}(H)}. \quad (S.5.18)$$

Then, by (S.5.17), splitting the integration over $[h^s,1]$ and $[1,\infty]$ with $h^s < 1$ as $h \downarrow 0$, we obtain:

$$\int_{h^s}^\infty \left\|e^{A_P t}x - e^{A_{h,P_h}t}\Pi_h x\right\|_H dt \leq \|A_P^{-1} - A_{h,P_h}^{-1}\| \int_{h^s}^\infty \frac{e^{-kt}}{t}dt\ \|x\|_H$$

$$\text{(by (S.5.3))} \qquad \leq C\|A_P^{-1} - A_{h,P_h}^{-1}\|(s\ \ell n\ \tfrac{1}{h})\ \|x\|_H \leq C\ h^{\varepsilon_0 - \varepsilon}\|x\|_H, \quad (S.5.19)$$

for all $\varepsilon > 0$. The same estimate can be obtained starting from (S.5.18) and using (S.5.4)

$$\int_{h^s}^\infty \left\|e^{A_{P_h}t}\Pi_h x - e^{A_{h,P_h}t}\Pi_h x\right\|_H dt \leq C\ h^{\varepsilon_0 - \varepsilon}\|x\|_H. \quad (S.5.20)$$

Using estimates (S.5.19), (S.5.20), and (S.5.13), as well as recalling (S.5.5) and (S.5.10), we obtain

$$|([P_h\Pi_h - P]x,x)_H| \leq C\ h^{\varepsilon_0}\|x\|_H^2, \quad (S.5.21)$$

with any $\varepsilon_0 < s(1-\gamma) < s$, as desired. Then (S.5.21) implies (5.1) by taking the sup over all x, $\|x\| \le 1$, since P_h and P are self-adjoint. Theorem 5.1 is proved. ■

<u>Proof of Theorem 5.2</u>. We interpolate between inequality (S.5.17) and (see (1.12) and (4.27))

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(H)} \le C\, e^{-kt}, \tag{S.5.22}$$

$k > 0$, to obtain for any $0 < \Theta < 1$,

$$\|e^{A_P t} - e^{A_{h,P_h} t}\Pi_h\|_{\mathcal{L}(H)} \le C \frac{e^{-kt}}{t^{\Theta}} \|A_P^{-1} - A_{h,P_h}^{-1}\|^{\Theta}_{\mathcal{L}(H)}. \tag{S.5.23}$$

Then (5.4) follows from (S.5.23) after recalling (S.5.3), as we can take $k = \bar{\omega}_P$, see (4.27). ■

5.2. <u>Uniform convergence $B_h^* P_h \Pi_h \to B^* P$ of gain operators</u>

<u>Proof of Theorem 5.3</u>. From (2.19) (or (S.4.39)) and (2.10) we compute

$$B_h^* P_h \Pi_h - B^* P = \int_0^\infty B_h^* e^{-\hat{A}_h^* t}\Pi_h [R^* R + 2\omega P_h]\hat{\Phi}_h(t)\Pi_h dt$$

$$- \int_0^\infty B^* e^{-\hat{A}^* t}[R^* R + 2\omega P]\hat{\Phi}(t) dt = I_{1,h} + I_{2,h} + I_{3,h}, \tag{S.5.24}$$

where, after suitable adding and subtracting,

$$I_{1,h} = \int_0^\infty [B_h^* e^{-\hat{A}_h^* t}\Pi_h - B^* e^{-\hat{A}^* t}][R^* R + 2\omega P_h]\hat{\Phi}_h(t)\Pi_h dt; \tag{S.5.25}$$

$$I_{2,h} = \int_0^\infty B^* e^{-\hat{A}^* t} 2\omega [P_h \Pi_h - P]\hat{\Phi}_h(t)\Pi_h dt; \tag{S.5.26}$$

$$I_{3,h} = \int_0^\infty B^* e^{-\hat{A}^* t}[RR^* + 2\omega P][\hat{\Phi}_h(t)\Pi_h - \hat{\Phi}(t)] dt. \tag{S.5.27}$$

To handle $I_{1,h}$, we recall that from Lemma 3.1(v), Eq. (3.5), applied with $\Theta < 1$, we have (by the definitions of $\hat{A}^*$ and $\hat{A}_h^*$ below (1.2) and (2.15)):

$$\|B_h^* e^{-\hat{A}_h^* t}\Pi_h - B^* e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} \le \frac{C\, h^{s(1-\gamma)\Theta}}{t^{\Theta + (1-\Theta)\gamma}} e^{-\hat{\omega} t}. \tag{S.5.28}$$

Thus, by (S.5.28) with $\Theta + (1-\Theta)\gamma < 1$, as well as the uniform bound (S.4.19) of Lemma S.4.3 for P_h, and (4.27) of Theorem 4.6 for $\Phi_h(t)$, we readily obtain from (S.5.25) that

$$\|I_{1,h}\|_{\mathcal{L}(H;U)} \le C\, h^{s(1-\gamma)\Theta} \downarrow 0 \quad \text{as } h \downarrow 0, \quad \Theta < 1. \tag{S.5.29}$$

To handle $I_{2,h}$, we recall Eq. (5.1) of Theorem 5.1, and the uniform bound (4.27), and note the bound

$$\|B^* e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} = \|B^* (\hat{A}^*)^{-\gamma} (\hat{A}^*)^{\gamma} e^{-\hat{A}^* t}\|_{\mathcal{L}(H;U)} \le C \frac{e^{-\hat{\omega} t}}{t^{\gamma}} \tag{S.5.30}$$

(from (1.3) and analyticity) to conclude from (S.5.26) that

$$\|I_{2,h}\|_{\mathcal{L}(H;U)} \le C\, h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0, \quad \varepsilon_0 < s(1-\gamma). \tag{S.5.31}$$

To complete the proof of Theorem 5.3 by showing that $I_{3,h}$ also goes to zero, we need (part of) Lemma 5.4.

<u>Proof of Lemma 5.4</u>.

<u>Proof (i) and (ii)</u>. <u>Step 1</u>. We return to (2.18), (2.19) rewritten here for convenience as

$$\hat{y}_h^0(\cdot,\Pi_h x) = [I+\hat{L}_h\hat{L}_h^*\Pi_h(R^*R+2\omega P_h)]^{-1}\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \tag{S.5.32}$$

$$-\hat{u}_h^0(\cdot,\Pi_h x) = [I+\hat{L}_h^*\Pi_h(R^*R+2\omega P_h)\hat{L}_h]^{-1}\hat{L}_h^*\Pi_h[R^*R+2\omega P_h]\{e^{-\hat{A}_h\cdot}\Pi_h x\}, \tag{S.5.33}$$

and take the limit as $h \downarrow 0$. Using the uniform convergence,

$$\|e^{-\hat{A}_h t}\Pi_h - e^{-\hat{A}t}\|_{\mathcal{L}(H;L_2(0,\infty;H))} \le C\,h^{s\Theta} \to 0 \text{ as } h \downarrow 0,\ \Theta < 1,$$

which follows from assumption (1.20), the uniform convergence $\|P_h\Pi_h - P\| \le C\,h^{\varepsilon_0}$ in (5.1) of Theorem 5.1 and the uniform convergence $\|\hat{L}_h - \hat{L}\| \le C\,h^{s(1-\gamma)\Theta}$, $\Theta < 1$, of (3.9) of Theorem 3.3, we conclude via (2.8), (2.9) (since the rates of convergence are preserved by the inverses by an identity as (S.4.13)), that as $h \downarrow 0$,

$$\|\hat{u}_h^0(\cdot,\Pi_h x)-\hat{u}^0(\cdot,x)\|_{\mathcal{L}(H;L_2(0,\infty;U))} \le C\,h^{\varepsilon_0}\|x\|_H \to 0 \quad \text{as } h \downarrow 0; \tag{S.5.34}$$

$$\|\hat{\Phi}_h(\cdot)\Pi_h - \hat{\Phi}(\cdot)\|_{\mathcal{L}(H;L_2(0,\infty;H))} \le C\,h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0. \tag{S.5.35}$$

<u>Step 2</u>. *A fortiori*, from (S.5.34), (S.5.35) we have for any $0 < T < \infty$:

$$\|u_h^0(\cdot,\Pi_h x)-u_0(\cdot,x)\|_{\mathcal{L}(H;L_2(0,T;U))} \le \|\Phi_h(\cdot)\Pi_h - \Phi(\cdot)\|_{\mathcal{L}(H;L_2(0,T;H))}$$
$$\le C\,h^{\varepsilon_0} \to 0 \quad \text{as } h \downarrow 0. \tag{S.5.36}$$

On the other hand, we have

$$\int_T^\infty \|u_h^0(t,\Pi_h x)-u^0(t,x)\|_U^2 dt \le 2\int_T^\infty \|u_h^0(t,\Pi_h x)\|_U^2 + \|u^0(t,x)\|_U^2 dt$$

$$(\text{by } ((S.4.20) \text{ and } (1.7))) \quad = \int_T^\infty \|B_h^* P_h e^{A_{h,P_h}t}\Pi_h x\|_U^2 dt + \int_T^\infty \|B^* P e^{A_P t} x\|_U^2 dt$$

$$\le \int_T^\infty C(e^{-2\omega_P t} + e^{-2\bar{\omega}_P t})dt\|x\|_H$$

$$\le C\,e^{-\omega/h^{\varepsilon_0}} \to 0 \quad \text{as } T = 1/h^{\varepsilon_0} \to \infty, \tag{S.5.37}$$

where in the last step we have used the bound (4.29) of Theorem 4.7 and the exponential bound (4.27) of Theorem 4.6 in the first integral and (1.11), (1.12) for the second integral. A similar upper bound and a similar convergence as in (S.5.37) holds true *a fortiori* if u_h^0, u^0 are replaced by $\Phi_h(\cdot)\Pi_h x$ and $\Phi(\cdot)x$. Thus (S.5.37), combined with (S.5.36) for any T, yields the desired conclusions, (5.6) and (5.7). Thus, parts (i) and (ii) of Lemma 5.4 are proved.

(iii) From the representation (see (S.4.20) and point (6) of Theorem 1.0)

$$R(\lambda,A_{h,P_h})\Pi_h x - R(\lambda,A_P)x = \int_0^\infty e^{-\lambda t}[\Phi_h(t)\Pi_h x - \Phi(t)x]dt \tag{S.5.38}$$

for Re $\lambda > \min\{-\omega_P, -\bar{\omega}_P\}$ (defined in (1.12) and (4.27)), and from (5.7) of part (ii), we obtain with $\varepsilon_0 < s(1-\gamma)$:

$$\|R(\lambda,A_{h,P_h})\Pi_h - R(\lambda,A_P)\|_{\mathcal{L}(H)} \le C\,h^{\varepsilon_0} \to 0 \text{ as } h \downarrow 0. \tag{S.5.39}$$

Then (S.5.39), combined with the uniform bounds (1.12) and (4.27) for $\Phi(t)$ and $\Phi_h(t)$, allows us to invoke the Trotter–Kato Theorem [22, p. 87] and obtain

$$\|[\Phi_h(\cdot)\Pi_h-\Phi(\cdot)]x\|_{C([0,T],H)} \to 0 \quad \text{as } h \downarrow 0,\ x \in H. \tag{S.5.40}$$

Then (S.5.40), combined with the exponential decay of $\Phi(t)$ and $\Phi_h(t)$ (uniformly in h) from (1.12), (4.27), implies (5.8). ∎

Continuing with the proof of Theorem 5.3, we can now handle the term $I_{3,h}$ in (S.5.27). From (S.5.27) and (1.3) we obtain

$$\|I_{3,h}\| \le C\int_0^\infty \frac{e^{-\hat{\omega}t}}{t^\gamma}\|\hat{\Phi}_h(t)\Pi_h-\hat{\Phi}(t)\|_{\mathcal{L}(H)}dt \tag{S.5.41}$$

But the norm inside the integral in (S.5.41) is dominated by a decaying exponential by (1.12) and (4.27). Thus, Lebesgue's dominated convergence theorem applies in (S.5.41) and yields

$$\|I_{3,h}\| \to 0 \quad \text{as } h \downarrow 0, \tag{S.5.42}$$

as desired. (Note that for $\gamma < \frac{1}{2}$, one still obtains $|I_{3,h}| \le C\,h^{\varepsilon_0}$.) Then, (S.5.29), (S.5.31), and (S.5.42) used in (S.5.24) produce the claimed convergence in (5.5). Theorem 5.3 is proved. ∎

5.3. <u>Uniform convergence</u> $u_h^0 \to u^0$

<u>Proof of Corollary 5.5.</u> <u>Step 1</u>. We shall first show that (5.9) holds with u replaced by $\hat{u}$. Indeed, from (2.9b) and (2.22) = (S.4.30)

$$\hat{u}_h^0(\ ,\Pi_h x)-\hat{u}^0(\ ,x) = \hat{L}_h^*(I+2\omega P_h)\hat{y}_h^0(\ ,\Pi_h x)-\hat{L}^*(I+2\omega P)\hat{y}^0(\ ,x)$$
$$= [\hat{L}_h^*(I+2\omega P_h)-\hat{L}^*(I+2\omega P)]\hat{y}_h^0(\ ,\Pi_h x)$$
$$+ \hat{L}^*(I+2\omega P)[\hat{y}_h^0(\ ,\Pi_h x)-\hat{y}^0(\ ,x)] \equiv (1)+(2). \tag{S.5.43}$$

By (3.9b), Eq. (5.1), and uniform boundedness of y_h^0 in (4.27), we obtain

$$\|(1)\|_{C([0,\infty];U)} \le C\,h^{\varepsilon_0}\|x\|_H,\ \forall\ \varepsilon_0 < s(1-\gamma) \tag{S.5.44}$$

As for the term (2), we invoke the result (5.4) of Theorem 5.2 which, together with the estimate (via (2.4), (1.3))

$$|\hat{L}^*(t^{-\varepsilon}f)|_{C([0,\infty];U)} \le \sup_{t\in[0,\infty]}\int_t^\infty \frac{\|f(\tau)\|_U d\tau}{(\tau-t)^\gamma\tau^\varepsilon} \le C\|f\|_{L_\infty(0,\infty;H)}, \quad \text{for } \gamma+\varepsilon < 1,$$

gives for $\varepsilon_0 < s(1-\gamma)$,

$$\|(2)\|_{C([0,\infty];U)} \le C\,h^{\varepsilon_0}\|x\|_H. \tag{S.5.45}$$

Thus we obtain the desired estimate by using (S.5.43)-(S.5.45),

$$\|\hat{u}_h^0(\ ,\Pi_h x)-\hat{u}^0(\ ,x)\|_{\mathcal{L}(H;C([0,\infty];U)} \le C\,h^{\varepsilon_0}. \tag{S.5.46}$$

<u>Step 2</u>. From (S.5.46), it follows that for any fixed T > 0,

$$\|u_h^0(\ ,\Pi_h x)-u^0(\ ,x)\|_{\mathcal{L}(H;C([0,2T];U)} \le C_T\,h^{\varepsilon_0}.$$

Hence, in particular,

$$\sup_{0\le t\le 2T}\left\|B_h^*P_h e^{A_{h,P_h}T}\Pi_h - B^*P e^{A_P t}\right\|_{\mathcal{L}(H;U)} \le C_T\,h^{\varepsilon_0}. \tag{S.5.47}$$

y the semigroup property, for any $t > 2T$ we compute via (S.4.20), (1.7),

$$\|u_h^0(t,\Pi_h x)-u^0(t,x)\|_{\mathcal{L}(H;U)} = \left\|B_h^* P_h e^{A_{h,P_h}T} e^{A_{h,P_h}(t-T)}\Pi_h - B^* P e^{A_P T} e^{A_P(t-T)}\right\|_{\mathcal{L}(H)}$$

$$\leq \left\|B_h^* P_h e^{A_{h,P_h}T}\Pi_h - B^* P e^{A_P T}\right\|_{\mathcal{L}(H;U)} \left\|e^{A_{h,P_h}(t-T)}\Pi_h\right\|_{\mathcal{L}(H)}$$

$$+ \|B^* P e^{A_P T}\|_{\mathcal{L}(H;U)} \left\|e^{A_{h,P_h}(t-T)}\Pi_h - e^{A_P(t-T)}\right\|_{\mathcal{L}(H)}$$

$$\leq C_T h^{\varepsilon_0} e^{-\bar{\omega}_P(t-T)} + C e^{-\omega_P T} \frac{h^{\varepsilon_0}}{t-T} e^{-\bar{\omega}_P(t-T)} \qquad \text{(S.5.48)}$$

ere in the last step we have used (S.5.47), (4.27) for the first term and (1 11), .12), and (5.4) with $\varepsilon = 1$ for the second term: this way we obtain the desired result (5 9) Corollary 5.5 is proved. ∎

4. Convergence $(\hat{A}^*)^\Theta(P_h\Pi_h - P)x \to 0$

oof of Proposition 5.6. (i) We return to (2.20) and get

$$(\hat{A}^*)^\Theta P_h \Pi_h x = \int_0^\infty (\hat{A}^*)^\Theta e^{-\hat{A}_h^* t}\Pi_h[R^* R + 2\omega P_h]\hat{\Phi}_h(t)x\,dt, \quad x \in H, \qquad \text{(S.5 49)}$$

ere $(\hat{A}^*)^\Theta e^{-\hat{A}_h^* t} = (\hat{A}^*)^\Theta(\hat{A}_h^{*-1})^\Theta(\hat{A}_h^*)^\Theta e^{-\hat{A}_h^* t} = \mathcal{O}(e^{-\hat{\omega}t}/t^\Theta)$ by (5.10) and uniform alyticity (1.14), $\Theta < 1$. Moreover, $\hat{\Phi}_h(\cdot)x \to \hat{\Phi}(\cdot)x$ in $C([0,\infty];H)$ by (5.8) of Lemma 4(iii), and $P_h \to P$ by (5.1). Thus, letting $h \downarrow 0$ in (S.5.49) and recalling (2.10), we tain the limit $(\hat{A}^*)^\Theta Px$ and (5.11) is proved.

(ii) By (4.33) of Corollary 4.8 for the discrete problem and the corresponding version for the continuous problem, we have

$$\|(\hat{A}^*)^\Theta(P_h\Pi_h - P)\hat{A}^\Theta\| \leq \text{const}_\Theta, \quad 0 \leq \Theta < ½. \qquad \text{(S.5.50)}$$

Then (5.11) of part (i), combined with density of $\mathcal{D}(\hat{A}^\Theta)$ and with the uniform bound (S.5.50) yields (5.12) as desired ∎

5.5. Completion of the proof of main Theorem 1.2

The conclusion (1.41) of Theorem 1.2 follows from Theorem 4.2 (see also Remark 4.2(ii)) and Theorem 5.3: the latter provides $\|B_h^* P_h \Pi_h - B^* P\| \to 0$ in $\mathcal{L}(H;U)$, see (5.5), and in the former we take $F = -B^* P$, $F_h = -B_h^* P_h$. Then, by virtue of the exponential decay (1.7) of $e^{A_P t}$, we obtain the counterpart of conclusion (4.12), which is precisely (1.41).

As for (1.42), we obtain from (5.2) = (S.5.1) and (5.3) = (S.5.2), using the same formula for the difference of inverses as the one in (S.4.13),

$$R(\lambda,A_P) - R(\lambda,A_{P_h}) = R(\lambda,A_P)B[B^* P - B_h^* P_h]R(\lambda,A_h), \qquad \text{(S.5.51)}$$

since the term in the bracket in (S.5 51) is $A_P - A_{P_h}$, where

$$R(\lambda,A_P)B = [I + R(\lambda,A)BB^* P]^{-1}R(\lambda,A)B. \qquad \text{(S.5.52)}$$

It follows that

$$\|R(\lambda,A_P)B\|_{\mathcal{L}(U;H)} \leq \frac{C}{|\lambda|^{1-\gamma}}, \quad \lambda \in \Gamma_P, \qquad \text{(S.5.53)}$$

where Γ_P is the path $\rho e^{\pm i\Theta_P} - \bar{\omega}_P$, $\Pi/2 < \Theta_P < \Pi$, $0 \leq \rho < \infty$. In fact, (S.5.53) is true for $|\lambda|$ sufficiently large, by (S.5.52), (1 11) and $\|R(\lambda,A)B\| \leq C/|\lambda|^{1-\gamma}$, in view of analyticity and (1.3); and hence, for $\lambda \in \Gamma_P$ by using the first resolvent equation on

$R(\lambda, A_p)$ (which we already know to be well defined on Γ_p). Thus, (S.5.53) used in (S.5 51) yields by uniform analyticity (1.14b) with $\Theta = 0$:

$$\|R(\lambda, A_p) - R(\lambda, A_{P_h})\|_{\mathcal{L}(U,H)} \leq \frac{C}{|\lambda|^{2-\gamma}} \|B^*P - B_h^*P_h\|_{\mathcal{L}(H)}. \tag{S.5.54}$$

From (S 5.54) we obtain, as usual,

$$\|e^{A_P t} - e^{A_{P_h} t}\|_{\mathcal{L}(H)} \leq C \int_{\Gamma_P} e^{\lambda t} \frac{C}{|\lambda|} d\lambda \, \|B^*P - B_h^*P_h\|_{\mathcal{L}(H;U)}$$

$$\leq C e^{-\bar{\omega}_P t} \|B^*P - B_h^*P_h\|_{\mathcal{L}(H;U)}, \tag{S.5.55}$$

and (1.42) follows Theorem 1.2 is proved ∎

Section 6: Approximation framework and verification of all required assumptions

Example 6.1: Heat equation with Dirichlet boundary control

Assumption (1.3). $(\hat{A})^{-\gamma} B \in \mathcal{L}(U,Y)$. Assumption (1.3) is satisfied in our present case with $\gamma = \frac{3}{4}+\varepsilon$, $\forall\, \varepsilon > 0$ In fact, we may take $\hat{A} = A_D$. From (6.7), we have

$$B = -AD_1: \text{ continuous } L_2(\Gamma) \to [\mathcal{D}(\hat{A}^{\frac{1}{4}+\varepsilon})]' = [\mathcal{D}(A_D^{\frac{1}{4}+\varepsilon})]', \tag{S.6.1}$$

and we then have with $\gamma = \frac{3}{4}+\varepsilon$ via (6.5) that our claim is verified.

$$\hat{A}^{-\gamma} B = -A_D^{-\gamma} AD_1 \in \mathcal{L}(L_2(\Gamma), L_2(\Omega)) = \mathcal{L}(U,H). \tag{S.6.2}$$

Stabilizability condition (1.5). The generator A has (for suitably large constant c^2 in (6.3)) only finitely many unstable eigenvalues of finite multiplicity, since its resolvent is compact and e^{At} is analytic. Thus, the stabilization theory as in [25], [21; App.], etc., applies: The problem is stabilizable on $L_2(\Omega)$ if and only if its projection onto the finite-dimensional unstable subspace is controllable. In particular, as shown in [27], one may prescribe the stabilizing feedback to be of the form

$$u(t) = \sum_{n=1}^{N} (y(t), w_n)_{L_2(\Omega)} g_n \tag{S.6.3}$$

for suitable vectors $w_k \in L_2(\Omega)$ and $g_k \in L_2(\Gamma)$ and suitable (minimal) N as described there, in order to stabilize uniformly the corresponding feedback system in the norm of $H^{\frac{1}{2}-\varepsilon}(\Omega)$ in fact. Thus, *a fortiori* the Finite Cost Condition on $L_2(\Omega)$ is satisfied

Detectability Condition (1.6). This is automatically satisfied since in our case R = I, see (6.5).

Conclusion. Theorem 1.0 applies to problem (6.1), (6.2).

6.1.2. Discrete problem

We present next the approximation framework for the problem (6.1), (6.2) [18].

Choice of V_h. We shall select the approximating space $V_h \subset H_0^1(\Omega)$ to be a space of splines (linear, quadratic, curvilinear, etc.) which comply with the usual approximation properties ([2], [24]):

$$\|\Pi_h y - y\|_{H^{\ell}(\Omega)} \leq C h^{s-\ell} \|y\|_{H^s(\Omega)}, \quad s \leq 2; \quad s-\ell \geq 0; \quad 0 \leq \ell \leq 1; \tag{S.6.4}$$

(inverse approximation properties)

$$\|y_h\|_{H^{\alpha}(\Omega)} \leq C h^{-\alpha} \|y_h\|_{L_2(\Omega)}, \quad 0 \leq \alpha \leq 1, \tag{S.6.5}$$

$$\left\|\frac{\partial}{\partial \nu} (y - \Pi_h y)\right\|_{L_2(\Gamma)} \leq C h^{s-\frac{3}{2}} \|y\|_{H^s(\Omega)}, \quad \frac{3}{2} < s \leq 2, \tag{S.6.6}$$

$$\left\|\frac{\partial y_h}{\partial \nu}\right\|_{L_2(\Gamma)} \le C\, h^{-3/2} \|y_h\|_{L_2(\Omega)}, \quad y_h \in V_h, \tag{S.6.7}$$

here Π_h is the orthogonal projection of $L_2(\Omega)$ onto V_h

<u>hoice of</u> A_h We define A_h $V_h \to V_h$ as usual, where the inner products are in L_2

$$(A_h x_h, y_h)_\Omega = (A x_h, y_h)_\Omega = \int_\Omega \nabla x_h \cdot \nabla y_h \, d\Omega - c^2 (x_h, y_h)_\Omega, \quad x_h, y_h \in V_h \tag{S.6.8}$$

<u>hoice of</u> B_h With reference to (6 5), we define B_h $U \to V_h$ by

$$B_h = -\Pi_h A D_1, \tag{S.6.9}$$

as in (6 6), (6 7), and we notice that (L_2-inner products)

$$(B_h u, y_h)_\Omega = -(A D_1 u, y_h)_\Omega = -(u, D_1^* A y_h)_\Gamma = (u, \frac{\partial y_h}{\partial \nu})_\Gamma \tag{S.6.10}$$

the

$$B_h^* y_h = \frac{\partial y_h}{\partial \nu} \tag{S.6.11}$$

<u>oproximating control problem</u> This is given by the O D E problem

$$\begin{cases} (\dot y_h, \phi_h)_\Omega + \int_\Omega \nabla y_h \cdot \nabla \phi_h \, d\Omega - c^2 \int_\Omega y_h \phi_h \, d\Omega = (u, \frac{\partial}{\partial \nu} \phi_h)_\Gamma, \quad \phi_h \in V_h, \\ (y_h(0), \phi_h)_\Omega = (y(0), \phi_h)_\Omega \end{cases} \tag{S.6.12}$$

The optimal feedback control for the approximating finite-dimensional problem is

$$u_h^0(t, y_0) = -\frac{\partial}{\partial \nu} P_h y_h^0(t, y_0),$$

The P_h satisfies the following discrete Algebraic Riccati Equation

$$-\int_\Omega \nabla P_h x_h \cdot \nabla y_h \, d\Omega - \int_\Omega \nabla x_h \cdot \nabla P_h y_h \, d\Omega + (x_h, y_h)_\Omega = (\frac{\partial}{\partial \nu} P_h x_h, \frac{\partial}{\partial \nu} P_h y_h)_\Gamma, \quad \forall\, x_h, y_h \in V_h \tag{S.6.13}$$

Eq (S 6 13) leads to a standard matrix Riccati equation, which can be effectively solved by finite-dimensional methods [12]

<u>Verification of assumptions of Theorem 1.1. Assumptions (1.26) and (1.27)</u>. These are plainly satisfied since $R = I$ and $\hat A^{-\gamma} B \in \mathcal{L}(U,H)$ with $\gamma = 3/4 + \varepsilon$ in our case Because of the compactness of A^{-1} (since Ω is bounded), this then implies in turn that $\hat A^{-1} B$ is compact $U \to H$, and thus $B^* (\hat A^*)^{-1}$ is compact $H \to U$, as desired

<u>Assumption (A.1) = (1.14)</u> (uniform analyticity). That this is satisfied follows from results on Galerkin approximations of elliptic operators, see [3] for the self-adjoint case and [14] for the general non-self-adjoint case

<u>Assumption (A.2) = (1.15)</u>. The standard elliptic approximation estimate is

$$\|\Pi_h \hat A^{-1} - \hat A_h^{-1} \Pi_h\|_{\mathcal{L}(L_2(\Omega))} \le C\, h^2,$$

so that (A 2) holds with $s = 2$

<u>Assumption (A.3) = (1.16)</u> By (S.6 11) and (S 6 7), we obtain with $U = L_2(\Gamma)$ and $H = L_2(\Omega)$,

$$\|B^* y_h\|_U = \|B_h^* y_h\|_U = \|\frac{\partial}{\partial \nu} y_h\|_{L_2(\Gamma)} \le C\, h^{-3/2} \|y_h\|_{L_2(\Omega)} \tag{S.6.14}$$

and (A 3) follows since $\gamma s = 2(3/4 + \varepsilon) > 3/2$

<u>Assumption (A.4) = (1.17)</u>. By (S.6.11) and (S 6.6) applied with $s = 2$,

$$\|B^*(\Pi_h x - x)\|_{L_2(\Gamma)} = \|\frac{\partial}{\partial \nu} (\Pi_h x - x)\|_{L_2(\Gamma)} \le C\, h^{1/2} \|x\|_{H^2(\Omega)}, \tag{S.6.15}$$

which implies (A.4) in view of the fact that $\mathcal{D}(A) \subset H^2(\Omega)$ and $s(1-\gamma) = 2(1-\frac{3}{4}-\varepsilon) = \frac{1}{2}-2\varepsilon < \frac{1}{2}$.

<u>Assumption (A.5) = (1.18)</u>. Since in our case $B_h^* \Pi_h = B^* \Pi_h$, (A.4) coincides with (A.5).

<u>Assumption (A.6) = (1.19)</u>. From (S.6.6) applied with $s = \frac{1}{2}+\varepsilon$ and from the trace theorem, we obtain

$$\|B^* \Pi_h x\|_{L_2(\Gamma)} = \|\frac{\partial}{\partial\nu} \Pi_h x\|_{L_2(\Gamma)} \le \|\frac{\partial}{\partial\nu}(\Pi_h - I)x\|_{L_2(\Gamma)} + \|\frac{\partial}{\partial\nu} x\|_{L_2(\Gamma)} \le C\, h^{\varepsilon}\|x\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)} + C\|x\|_{H^{\frac{3}{2}+\varepsilon}(\Omega)}$$

(A.6) follows now from $\mathcal{D}(A^{*\frac{3}{4}+\varepsilon}) \subset H^{\frac{3}{2}+2\varepsilon}(\Omega)$.

<u>Conclusion</u>. Thus, we have verified all the assumptions of Theorems 1.1 and 1.2 in the case of the heat equation problem with Dirichlet boundary control as in (6.1). Then, application of Theorem 1.1 yields the following convergence results (see also [18]):

(i) $\|P_h \Pi_h - P\|_{\mathcal{L}(L_2(\Omega))} \le C\, h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(ii) $\|\frac{\partial}{\partial\nu}[P_h \Pi_h - P]\|_{\mathcal{L}(L_2(\Omega);L_2(\Gamma))} \to 0$ as $h \downarrow 0$;,

(iii) $\|y_h^0 - y^0\|_{\mathcal{L}(L_2(\Omega);L_2(0,\infty;L_2(\Omega)))} + \sup_{t\ge 0} e^{\bar{\omega}_P t}\|y_h^0(t) - y^0(t)\|_{\mathcal{L}(L_2(\Omega))} \le C\, h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(iv) $\sup_{t\ge 0} e^{\bar{\omega}_P t}\|u_h^0(t) - u^0(t)\|_{\mathcal{L}(L_2(\Omega);L_2(\Gamma))} \le C\, h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{1}{2}$;

(v) since $\|\hat{A}^{*\Theta} x_h\| = \|\hat{A}_h^{*\Theta} x_h\|$ for $0 \le \Theta \le \frac{1}{2}$, (1.39) gives

$$\|(P_h \Pi_h - P)x\|_{H^1(\Omega)} \to 0, \quad x \in L_2(\Omega).$$

Application of Theorem 1.2 yields the following result: If we use the feedback law given by

$$u_h^*(t) = -\frac{\partial}{\partial\nu} P_h\, y^h(t),$$

which we insert into the original dynamics

$$\begin{cases} y_t^h = (\Delta + c^2) y^h, \\ y^h|_{\Sigma} = u_h^*, \end{cases} \tag{S.6.16}$$

then the corresponding system is exponentially stable in $L_2(\Omega)$ uniformly in the parameter h. Moreover,

$$\sup_{t\ge 0} e^{\bar{\omega}_P t}\|y^h(t) - y^0(t)\|_{\mathcal{L}(L_2(\Omega))} \to 0. \tag{S.6.17}$$

Other boundary conditions, like Neumann or Robin can be treated similarly (see [19]). In fact, the analysis here is even simpler as $\gamma = \frac{1}{4}+\varepsilon < \frac{1}{2}$, if one takes $H = L_2(\Omega)$; otherwise, $\gamma = \frac{3}{4}+\varepsilon$ if one takes $H = H^1(\Omega)$.

<u>Example 6.2: Structurally damped plates with point control</u>

<u>Assumption (1.3)</u>: $(-A)^{-\gamma}B \in \mathcal{L}(U,H)$. It is easy to verify that assumption (1.3) is satisfied with $\gamma = 1$. Indeed, from (6.9), we require that

$$(-A)^{-1}Bu = \begin{bmatrix} \rho\mathcal{A}^{-\frac{1}{2}} & \mathcal{A}^{-1} \\ -I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \delta(x-x^0)u \end{bmatrix} = \begin{bmatrix} \mathcal{A}^{-1}\delta(x-x^0)u \\ 0 \end{bmatrix} \in H, \tag{S.6.18}$$

i.e., from (6.12), we require that $\mathcal{A}^{-\frac{1}{2}}\delta(x-x^0) \in L_2(\Omega)$, or that (#): $\delta(x-x^0) \in [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, the dual of $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ with respect to $L_2(\Omega)$. Since it is true that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset H^2(\Omega)$ for the fourth-order operator $\mathcal{A}$ in (6.11) (in fact, regardless of the particular boundary conditions), and thus $[H^2(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\frac{1}{2}})]'$, then condition (#) is satisfied provided $\delta(x-x^0) \in [H^2(\Omega)]'$, i.e., provided $H^2(\Omega) \subset C(\bar{\Omega})$, which is indeed the case by Sobolev embedding provided $2 > \frac{n}{2}$, or $n < 4$, as required.

However, the above result is not sufficient for our purposes as--according to our assumption--we need to show that we can take $\gamma < 1$ in (1.3). As a matter of fact, we now show that assumption (1.3) holds true for any $\gamma > \frac{n}{4}$, which then for $n \leq 3$ yields $\gamma < 1$ as desired. To this end, we note that

$$(-A)^{-\gamma} B \in \mathcal{L}(U,H) \text{ if and only if } B \in \mathcal{L}(U,[\mathcal{D}((-A^*)^{\gamma})]') \qquad (S.6.19)$$

with duality with respect to H. But $\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma})$: this follows since A is the direct sum of two normal operators on H, with possibly an additional finite-dimensional component (if 1 is an eigenvalue of $\mathcal{A}$) [4], [5, Lemma A.1, case v(a) with $\alpha = \frac{1}{2}$]. moreover, [7, with $\alpha = \frac{1}{2}$], we have

$$\mathcal{D}((-A^*)^{\gamma}) = \mathcal{D}((-A)^{\gamma}) = \mathcal{D}(\mathcal{A}^{\frac{1}{2}+\gamma/2})\times\mathcal{D}(\mathcal{A}^{\gamma/2}),\ 0 < \gamma < 1 \qquad (S.6.20)$$

the first component does not really matter in the argument below). Thus, from (S.6.20) and B as in (6.13), it follows that (S 6.19) holds true, provided $\delta(x-x^0) \in [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$ (duality with respect to $L_2(\Omega)$), where $\mathcal{D}(\mathcal{A}^{\gamma/2}) \subset H^{2\gamma}(\Omega)$, and hence, provided $(x-x^0) \in [H^{2\gamma}(\Omega)]' \subset [\mathcal{D}(\mathcal{A}^{\gamma/2})]'$. But this in turn is the case, provided $H^{2\gamma}(\Omega) \subset C(\bar{\Omega})$; i.e., by Sobolev embedding provided $2\gamma > \frac{n}{2}$, as desired. We conclude: <u>assumption</u> (1.3) $[4\text{-}A)^{-\gamma}B \in \mathcal{L}(U,H)$ <u>holds true for problem</u> (6.9) <u>with</u> $\frac{n}{4} < \gamma < 1$, $n \leq 3$.

Also, the operator A in (6.13) generates an s.c. contraction semigroup e^{At} on H, which moreover is analytic here for $t > 0$. (This is a special case of a much more general result [4-5]). This, along with the requirement $\gamma < 1$ proved above guarantees that problem (6.9) satisfies our preliminary assumption (ii) of the Introduction, below (1.2).

<u>Stabilizability Condition (1.5)</u> With A as in (6.13), the semigroup e^{At} is uniformly (exponentially) stable in H [5], and thus the Finite Cost Condition (1.5) holds true with $u \equiv 0$.

<u>Remark 6.1</u>. Suppose that instead of Eq (6 9a), one has

$$w_{tt}+(\Delta^2+k_1)w-(\Delta+k_2)w_t = \delta(x-x^0)u(t) \quad \text{in } Q, \qquad (S\ 6\ 21)$$

along with (6.9b-c) Then, if $0 < k_1+k_2$ is sufficiently large, the generator A has finitely many unstable eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$ Since e^{At} is analytic on H, the usual theory [25] applies: The problem is stabilizable on H if [25] and only if [21] its projection onto the finite-dimensional unstable subspace is controllable

For instance, if $\lambda_1, \ldots, \lambda_K$ are the unstable eigenvalues of A, assumed for simplicity to be simple, and $\Phi_1, \ldots, \Phi_K$ are the corresponding eigenfunctions in H, then the necessary and sufficient condition for stabilization is that $\Phi_k(x^0) \neq 0$, $k = 1, \ldots, K$.

If $\lambda_1, \ldots, \lambda_K$ are not simple, then their largest multiplicity M determines the smallest number of scalar controls needed for the stabilization of (S.6 21), where now the right-hand side is replaced by $\sum_{i=1}^{M} \delta(x-x^i)u_i(t)$, along with (6.9b-c) The necessary and sufficient condition for stabilization is now a well-known full-rank condition [25].

<u>Detectability Condition (D.C.)</u>. This is satisfied since in our case $R = I$, see (6 13)

<u>Conclusion</u>. Theorem 1.0 applies to problem (6.9)-(6 10), $n \leq 3$, and provides existence and uniqueness of the solution to the ARE (1.8), with Riccati operator $P \in \mathcal{L}(H,\mathcal{D}(A))$ (since A, as remarked above (S.6.20), is the direct sum of two normal operators on H, plus possibly a finite-dimensional component, in particular, A has a Riesz basis of

eigenvectors on H), where $\mathcal{D}(A) = \mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}^{1/2})$, see (6.11), (6.12) for the characterizations of these spaces. Thus, in particular, we have $B^*P \in \mathcal{L}(H;U)$, where $B^*\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2(x^0)$.

6.2.2. Discrete problem

Choice of V_h. We shall select the approximating space $V_h \subset H^2(\Omega) \cap H_0^1(\Omega)$ to be a space of splines (e g , quadratic or cubic splines or curvilinear), which comply with the usual approximation properties

$$\|Q_h z - z\|_{H^\ell(\Omega)} \le C\, h^{s-\ell}\|z\|_{H^s(\Omega)}, \quad z \in H^s(\Omega) \cap H_0^1(\Omega), \quad 0 \le \ell \le 2;\ \ell \le s < r. \tag{S.6.22}$$

$$\|z_h\|_{H^\alpha(\Omega)} \le C\, h^{-s}\|z_h\|_{H^{\alpha-s}(\Omega)}, \quad 0 \le \alpha \le 2, \tag{S.6.23}$$

where Q_h is the orthogonal projection of $L_2(\Omega)$ onto V_h and where r is the order of approximation

Choice of $\mathcal{A}_h$ We let

$$\mathcal{A}_h = Q_h \mathcal{A} Q_h \cdot V_h \to V_h,$$

i.e ,

$$(\mathcal{A}_h\phi_h,\psi_h)_\Omega = (\Delta\phi_h,\Delta\psi_h)_\Omega = (\mathcal{A}^{1/2}\phi_h,\mathcal{A}^{1/2}\psi_h)_\Omega, \quad \phi_h,\psi_h \in V_h, \tag{S 6 24}$$

$$\|\mathcal{A}_h^{1/2}\phi_h\|_{L_2(\Omega)} = \|\mathcal{A}^{1/2}\phi_h\|_{L_2(\Omega)};\quad \|\mathcal{A}_h^{1/2}\phi_h\| \sim \|\phi_h\|_{H^2(\Omega)}, \quad \phi_h \in V_h, \tag{S 6.25}$$

where (S 6 25) is a consequence of (S.6 24) From the estimates for the biharmonic operator, see [2], we obtain

$$\begin{cases} \|(\mathcal{A}^{-1}-\mathcal{A}_h^{-1}Q_h)z\|_{H^2(\Omega)} \le C\, h^2\|z\|_{L_2(\Omega)}; \\ \|(\mathcal{A}^{-1/2}-\mathcal{A}_h^{-1/2}Q_h)z\|_{H^2(\Omega)} \le C\, h^2\|z\|_{H^2(\Omega)}, \quad z \in \mathcal{D}(\mathcal{A}^{1/2}). \end{cases} \tag{S.6.26}$$

Choice of A_h and B_h. To begin with, we let $H_h \equiv V_{h1}\times V_{h2}$, where V_{h1} consists of the elements of V_h equipped with norm $\|v_h\|_{V_h^1} = \|\mathcal{A}_h^{1/2}v_h\|_{L_2(\Omega)} = \|\mathcal{A}^{1/2}v_h\|_{L_2(\Omega)}$, and V_{h2} consists of the elements of V_h equipped with the $L_2(\Omega)$-norm. We shall write $x_h = [x_{h1},x_{h2}] \in H_h$. Next, we define

$$A_h : H_h \to H_h : A_h = \begin{bmatrix} 0 & Q_h \\ -\mathcal{A}_h & -\mathcal{A}_h^{1/2} \end{bmatrix}, \tag{S 6.27}$$

$$B_h \cdot L_2(\Gamma) \to H_h : B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix}, \quad (\mathcal{B}_h u, v_h)_\Omega = v_h(x^0)u. \tag{S.6.28}$$

Finally, we let $\Pi_h : H \to H_h$ be defined as

$$\Pi_h = \begin{bmatrix} Q_h & 0 \\ 0 & Q_h \end{bmatrix}$$

Computation of adjoints A_h^* and B_h^*. To compute the adjoints of A_h and B_h, we use the inner products generated by the topology on V_{h1} and V_{h2}. We find, as in the continuous case,

$$A_h^* = \begin{bmatrix} 0 & -Q_h \\ \mathcal{A}_h & -\mathcal{A}_h^{1/2} \end{bmatrix};\quad B_h^* x_h = x_{h2}(x^0)$$

as it follows from $(A_h x_h, y_h)_{H_h} = (x_h, A_h^* y_h)_{H_h}$ and $(B_h u, x_h)_{H_h}$ and $(u, B_h^* x_h)_U$, respectively

Approximating control problem With the above notation, the approximating version of the dynamics is

$$\begin{cases} (\ddot{y}_h,\phi_h)+(A_h y_h,\phi_h)+(A_h^{1/2} y_h,\phi_h) = \phi_{h2}(x^0)u; \\ (A_h y_h,\phi_h) = (\Delta y_n,\Delta\phi_h), \quad \text{all } \phi_h \in V_h; \\ (y_h(0),\phi_h) = (y_0,\phi_h);\ (\dot{y}_h(0),\phi_h) = (y_1,\phi_h), \end{cases} \tag{S.6.29}$$

where all inner products are in $L_2(\Omega)$.

The optimal feedback control for the finite-dimensional problem is given by $u_h^0(t) \equiv -[\bar{P}_{h2}y_h(t)][x^0]$, where

$$P_h y_h \equiv \begin{cases} P_{h1}y_{h1}+P_{h2}y_{h2} \equiv \bar{P}_{h1}y_h; \\ P_{h3}y_{h1}+P_{h4}y_{h2} \equiv \bar{P}_{h2}y_h; \end{cases} \tag{S.6.30}$$

and P_h satisfies the following algebraic equation with $L_2(\Omega)$-inner products

$$-(A_h x_{h2},\bar{P}_{h1}y_h) + (A_h x_{h1}-A_h^{1/2}x_{h2},\bar{P}_{h2}y_h) - (A_h\bar{P}_{h1}x_h,y_{h2})$$

$$+ (\bar{P}_{h2}x_h,A_h y_{h1}-A_h^{1/2}y_{h2}) + (A_h x_{h1},y_{h1}) + (x_{h2},y_{h2})$$

$$= (\bar{P}_{h2}x_h)(x^0)(\bar{P}_{h2}y_h)(x^0). \tag{S.6.31}$$

Again, (S.6.31) leads to a matrix Riccati equation, which can be effectively solved by finite-dimensional methods [12].

<u>Verification of assumptions of Theorem 1.1</u>. In order to apply Theorem 1.1, we need to verify the approximating assumptions (A.1)-(A.6), as well as assumptions (1.26), (1.27). Indeed, the last two are plainly satisfied: (1.26) since R = I, while (1.27) follows from (S.6.18) and the argument below it (in essence, $A^{-(1/2-\epsilon)}\delta \in L_2(\Omega)$), while $A^{-\epsilon}$ is compact on $L_2(\Omega)$).

<u>Assumption (A.1)</u>. This follows by applying the arguments of [5] of the continuous case o the finite-dimensional operator given by (S.6.27).

<u>Assumption (A.2)</u>. By (6.38), we have that (A.2) with s = 2 holds true:

$$\|(A_h^{-1}\Pi_h-A^{-1})x_h\|_H = \|-(A^{-1/2}-A_h^{-1/2})x_{h1}+(A^{-1}-A_h^{-1})x_{h2}\|_{H^2(\Omega)}$$

$$\le C\,h^2[\|x_{h1}\|_{H^2(\Omega)}+\|x_{h2}\|_{L_2(\Omega)}] = C\,h^2[\|x_h\|_H].$$

The same result holds for the adjoint A^*, in view of its definition.

<u>Assumption (A.3)</u>. By Sobolev embedding and the inverse approximation property (S.6.23), we have for any $\epsilon > 0$,

$$\|B_h^*x_h\|_U = \|B^*x_h\|_U = \|x_{h2}(x^0)\| \le C\|x_{h2}\|_{H^{n/2+\epsilon}(\Omega)}$$

$$\le C\,h^{-n/2-\epsilon}\|x_{h2}\|_{L_2(\Omega)} \le C\,h^{-n/2-\epsilon}\|x_h\|_H,$$

and (A.3) follows since $\gamma s = (n/4+\epsilon)2 > n/2$.

<u>Assumption (A.4)</u>. By (S.6.22) we compute

$$\|B^*(\Pi_h x-x)\|_U = |(Q_h x_2)(x^0)-x_2(x^0)|_{\mathbb{R}^1} \le C\|Q_h x_2-x_2\|_{H^{n/2+\epsilon}(\Omega)}$$

$$\le C\,h^{2-n/2-\epsilon}\|x_2\|_{H^2(\Omega)}$$

$$\le C\,h^{2-n/2-\epsilon}\|x\|_{\mathcal{D}(A^*)}$$

Since $2(1-\gamma) = 2(1-\frac{n}{4}-\epsilon) < 2-\frac{n}{2}-\epsilon$, then (A.4) is satisfied.

<u>Assumption (A.5)</u>. It coincides with (A.4).

Assumption (A.6).

$$\|B^*\Pi_h x\|_U = \|x_{h_2}(x^0)\|_{L_2(\Omega)} \le C\|x_{h_2}\|_{H^{n/2+\varepsilon}(\Omega)} \le C\|x_h\|_{\mathcal{D}(A^{*\gamma})}$$

(as in [7]), $\mathcal{D}(A^{*\gamma}) \subset H^{4\gamma}(\Omega)\times H^{2\gamma}(\Omega)$ and $2\gamma = 2(\frac{n}{4}+\varepsilon) = \frac{n}{2}+2\varepsilon > \frac{n}{2}+\varepsilon$.

Conclusion. Thus, we have verified all the assumptions of Theorem 1.1. Thus, Theorem 1.1 applies to our problem, and yields the following convergence results:

(i) $$\|P_h\Pi_h - P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{4-n}{2};$$

(ii) $$\left\|(P_h\Pi_h - P)|_{x=x^0}\right\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R^1)} \to 0 \quad \text{as } h\to 0,$$

or equivalently

$$\|B^*P_h\Pi_h - B^*P\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R)} \to 0 \quad \text{as } h \downarrow 0,$$

where P_h is computed from (S.6.31).

(iii) $$\sup_{t\ge 0} e^{\bar{\omega}_P t}\|u_h^0(t)-u^0(t)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega);R)} \le C\,h^{\varepsilon_0};$$

(iv) $$\sup_{t\ge 0} t\, e^{\bar{\omega}_P t}\|y_h^0(t)-y^0(t)\|_{\mathcal{L}(H^2(\Omega)\times L_2(\Omega))} \le C\,h^{\varepsilon_0}, \quad \varepsilon_0 < \frac{4-n}{2}.$$

Application of Theorem 1.2 to our problem yields the following result: Let $u_h^*(t)$ be a feedback law given by

$$u_h^*(t) = -[\bar{P}_{h2}y(t)][x^0]$$

which we insert into the original dynamics (6.9) to obtain

$$w_{tt}+\Delta^2 w-\rho\Delta w_t = \delta(x-x^0)u_h^*;\quad w|_\Gamma = \Delta w|_\Gamma = 0.$$

Then the corresponding feedback system is uniformly (in h) exponentially stable in the topology of $H^2(\Omega)\times L_2(\Omega)$ and uniformly approximates the original feedback dynamics. This means that the numerical algorithm provides a feedback control which yields uniform (in h) stability results for the original system.

We conclude this section by pointing out that the other examples of [19, Section 3.3] dealing with structurally damped plate problems can be dealt with by a similar approximating scheme.

Example 6.3: Kelvin-Voigt plate equation with point control

Assumption (1.3) $(-A)^{-\gamma}B \in \mathcal{L}(U,Y)$. Again, it is straightforward to verify that assumption (1.3) is satisfied with $\gamma = 1$: From (6.48), we require that

$$(-A)^{-1}Bu = \begin{bmatrix} \rho I & A^{-1} \\ -I & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \delta(x-x^0)u \end{bmatrix} = \begin{bmatrix} A^{-1}\delta(x-x^0) \\ 0 \end{bmatrix} \in H, \tag{S.6.32}$$

i.e., from (6.47) we require that $A^{-1/2}\delta(x-x^0)$. The Sobolev imbedding then yields that (S.6.32) holds true if $n \le 3$.

However, in order to verify assumption (1.3), which requires that γ should be < 1, the most elementary way is to check that assumption (1.3) holds in fact true with $\gamma = \frac{1}{2}$. In this case, we can in fact rely on the direct computation of $(-A)^{-1/2}$ (for simplicity of notation, we take henceforth $\rho = 1$)

$$(-A)^{-1/2} = \begin{bmatrix} (1) & A^{-3/4}(2I+A^{1/2})^{-1/2} \\ (2) & A^{-1/4}(2I+A^{1/2})^{-1/2} \end{bmatrix} \tag{S.6.33}$$

(where the entries $(1) = A^{-3/4}(2I+A^{1/2})^{-1/2}(I+A^{1/2})$ and $(2) = -A^{1/4}(2I+A^{1/2})^{-1/2}$ do not really count in the present analysis), and avoid the domain of fractional powers as in [7].

We need to compute

$$(-A)^{-\frac{1}{2}}Bu = \begin{bmatrix} A^{-\frac{3}{4}}(2I+A^{\frac{1}{2}})^{-\frac{1}{2}}\delta(x-x^0)u \\ A^{-\frac{1}{4}}(2I+A^{\frac{1}{2}})^{-\frac{1}{2}}\delta(x-x^0)u \end{bmatrix}. \qquad (S.6.34)$$

from (S.6.34), we then readily see that $(-A)^{-\frac{1}{2}}Bu \in H = \mathcal{D}(A^{\frac{1}{2}})\times L_2(\Omega)$ provided (#): $^{-\frac{1}{2}}\delta(x-x^0) \in L_2(\Omega)$ But $\mathcal{D}(A^{\frac{1}{2}}) = H^2(\Omega)$ (and, in fact, only $\mathcal{D}(A^{\frac{1}{2}}) \subset H^2(\Omega)$ suffices for the present analysis) so that condition (#) is satisfied provided $\delta(x-x^0) \in [H^2(\Omega)]'$ (duality with respect to $L_2(\Omega)$); i.e , provided $H^2(\Omega) \subset C(\bar{\Omega})$, i.e., by Sobolev embedding provided $> \frac{n}{2}$, or $n < 4$, as desired We have shown: <u>Assumption (1.3)</u> $(-A)^{-\gamma}B \in \mathcal{L}(U,Y)$ <u>holds</u> <u>issue for problem (6.44) with</u> $n \le 3$, <u>and</u> $\gamma = \frac{1}{2}$. The above argument shows some 'leverage. indeed, $\gamma = \frac{1}{2}$ is not the least γ for which assumption (1.3) holds true. Indeed, one can below (see [19]) that <u>Assumption (1.3)</u> $(-A)^{-\gamma}B \in \mathcal{L}(U,H)$ <u>holds true for problem (6.44)</u> <u>povided</u> $\frac{n}{8} < \gamma \le \frac{1}{2}$, $n \le 3$.

Also, the operator A in (6.18) generates an s.c contraction semigroup e^{At} on H, which moreover is analytic here for $t > 0$ (This is a special case of a much more general result [5].) This, along with the requirement $\gamma < 1$ proved above, guarantees that problem (6 14) satisfies the required assumption (ii) of the Introduction, below (1 2).

<u>abilizability Condition (1.5)</u> With A as in (6 18), the semigroup e^{At} is uniformly exponentially) stable in $H/\mathcal{N}(A)$, where $\mathcal{N}(A)$ is the finite-dimensional nullspace of A |, and thus (1 5) is automatically satisfied on this space. For the eigenvalue $\lambda = 0$, apply the same procedure as in the example of Section 6.2.

<u>ctectability Condition (1.6)</u> This is satisfied since in our case R = I

<u>Conclusion</u> Theorem 1.0 applies to problem (6 14) for $n \le 3$

6.3.2. <u>Discrete problem</u>

<u>Approximation Framework</u>. The choice of the spaces V_h and H_h and of the operator A_h is the same as in the case of the example of Section 6.2 (damped plate equation).

<u>Choice of A_h and B_h</u>. We define

$$A_h: H_h \to H_h: \quad A_h \equiv \begin{bmatrix} 0 & Q_h \\ -A_h & -\rho A_h \end{bmatrix};$$

$$B_h: L_2(\Gamma) \to H_h: \quad B_h u = \begin{bmatrix} 0 \\ \mathcal{B}_h u \end{bmatrix},$$

where $(\mathcal{B}_h u, v_h)_{L_2(\Omega)} = v_h(x^0)u$.

Computations of adjoints A_h^* and B_h^*, as previously done yield

$$A_h^* = \begin{bmatrix} 0 & -Q_h \\ A_h & -\rho A_h \end{bmatrix}; \quad B_h^* x_h = x_{h2}(x^0).$$

<u>Approximating control problem</u>. The approximating dynamics is

$$(\ddot{y}_h, \phi_h)_\Omega + (A_h y_h, \phi_h)_\Omega + \rho(A_h \dot{y}_h, \phi_h) = \phi_{h2}(x^0)u$$

with

$$(A_h y_h, \phi_h) = (\Delta y_h, \Delta\phi_h)_\Omega;$$

$$(y_h(0), \phi_h)_\Omega = (y_0, \phi_h)_\Omega; \quad (\dot{y}_h(0), \phi_h)_\Omega = (y_1, \phi_h)_\Omega.$$

The optimal feedback control for the finite-dimensional problem is given by

$$u_h^0(t) = -[\bar{P}_{h2} y_h(t)][x^0],$$

$$P_h y_h = \begin{pmatrix} \overline{P}_{h1} y_h \\ \overline{P}_{h2} y_h \end{pmatrix},$$

and P_h satisfies

$$-(\mathcal{A}_h x_{h2}, \overline{P}_{h1} y_h) + (\mathcal{A}_h x_{h1} - \rho \mathcal{A}_h x_{h2}, \overline{P}_{h2} y_h) - (\mathcal{A}_h \overline{P}_{h1} x_h, y_{h2})$$

$$+ (\overline{P}_{h2} x_h, \mathcal{A}_h y_{h1} - \rho \mathcal{A}_h y_{h2}) + (\mathcal{A}_h x_{h1}, y_{h1}) + (x_{h2}, y_{h2})$$

$$= (\overline{P}_{h2} x_h)(x^0) \overline{P}_{h2} y_h (x^0).$$

<u>Verification of the approximating assumptions</u>. Assumption (1.26a) is satisfied as $R = I$. Assumption (1 27a) follows from the fact that the operator

$$(-A)^{-1} B = \begin{pmatrix} \mathcal{A}^{-1} \delta(x - x^0) \\ 0 \end{pmatrix} : R \to H^2(\Omega) \times L_2(\Omega)$$

is compact, which, in turn, is a consequence of Sobolev imbeddings and compactness of $\mathcal{A}^{-1}$ (indeed, $\mathcal{A}^{-(\frac{1}{2}-\varepsilon)} \delta \in L_2(\Omega)$)

<u>Assumption (A.1)</u>. This follows by applying the arguments of [5] of the continuous case to the finite dimensional operator A_h

<u>Assumption (A.2)</u>. Computing directly, we obtain

$$\|(A_h^{-1} \Pi_h - \Pi_h A^{-1}) x\|_H = \|(Q_h \mathcal{A}^{-1} - \mathcal{A}_h^{-1} \Pi_h) x_2\|_{H^2(\Omega)}$$

by (S.6 22) and (S.6 26) $\qquad \le C\, h^2 \|x_{2h}\|_{L_2(\Omega)} \le C\, h^2 \|x\|_H .$

Thus, we have $s = 2$ in this case

<u>Assumption (A.3)</u> The argument is identical to that of the damped wave equation in the example of Section 6 2

<u>Assumption (A.4)</u>. We compute

$$\|B^* (\Pi_h x - x)\|_U = |(Q_h x_2)(x^0) - x_2(x^0)|_R$$

$$\le C \|Q_h x_2 - x_2\|_{H^{n/2+\varepsilon}(\Omega)} \le C\, h^{2-n/2-\varepsilon} \|x_2\|_{H^2(\Omega)}$$

By the results of [7], $\mathcal{D}(A^*) \subset H^2(\Omega) \times H^2(\Omega)$, so (A.4) follows since

$$2 - \frac{n}{2} - \varepsilon > 2(1 - \frac{n}{8} - \varepsilon)$$

<u>Assumption (A.5)</u>. Coincides with (A.4).

<u>Assumption (A.6)</u>. By Sobolev's imbedding,

$$\|B^* \Pi_h x\|_U = \|x_{h2}(x^0)\|_{L_2(\Omega)} \le C \|x_{h2}\|_{H^{n/2+\varepsilon}(\Omega)}$$

By the results of [7], for $0 < \gamma < \frac{1}{2}$, we have

$$\mathcal{D}(A^{*\gamma}) \subset H^2(\Omega) \times H^{4\gamma}(\Omega),$$

so (A.6) is a consequence of the inequality,

$$4\gamma = 4(\frac{n}{8} + \varepsilon) = \frac{n}{2} + 4\varepsilon > \frac{n}{2} + \varepsilon$$

Thus, we have verified all the assumptions of Theorem 1.1 and the conclusion of Theorem 1.1 yields the desired convergence results which can be listed in an analogous way as in the case of the example of Section 6.2