MATHEMATICS OF COMPUTATION
VOLUME 58, NUMBER 197
JANUARY 1992, PAGES S7–S16

Supplement to

THE EXISTENCE OF EFFICIENT LATTICE RULES FOR MULTIDIMENSIONAL NUMERICAL INTEGRATION

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This supplement contains the proof of Theorem 2 in the main part of the paper. In §4 we establish some auxiliary number–theoretic results that are needed for the proof, and in §5 we complete the proof. The numbering of lemmas and equations is continued from the main part and the references are to the bibliography in the main part.

4. AUXILIARY RESULTS

We need various concepts and results from number theory (see [2] as a general reference). Recall that a function F on the set $\mathbf{N}$ of positive integers is called *multiplicative* if $F(mn) = F(m)F(n)$ whenever $\gcd(m,n) = 1$. We write $\sum_{d|n}$ for a sum over all positive divisors d of $n \in \mathbf{N}$. If F and G are multiplicative functions, then the summatory function $\sum_{d|n} F(d)$ of F and the Dirichlet convolution $\sum_{d|n} F(n/d)G(d)$ of F and G are multiplicative functions of $n \in \mathbf{N}$. Let μ be the Möbius function and note that μ is multiplicative. From now on we abbreviate $\gcd(m,n)$ by (m,n). The symbol p will always denote a prime number. For $n \in \mathbf{N}$ let $e_p(n)$ be the largest exponent such that $p^{e_p(n)} | n$ (if $p \nmid n$, then $e_p(n) = 0$).

Lemma 2. *For $k, m, n \in \mathbf{N}$ we have*

$$B(k,m,n) := \sum_{d|n} \mu\left(\frac{n}{d}\right)(d,k)\left(m, \frac{d}{(d,k)}\right) = \begin{cases} \phi(n) & \text{if } n|km, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $d \in \mathbf{N}$ we have

$$(d,km) = (d,k)\left(\frac{d}{(d,k)}, \frac{k}{(d,k)}m\right) = (d,k)\left(\frac{d}{(d,k)}, m\right) \tag{13}$$

since $d/(d,k)$ and $k/(d,k)$ are coprime. Therefore,

$$B(k,m,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)(d,km).$$

For $b \in \mathbf{N}$ we have

$$B(k,m,p^b) = (p^b, km) - (p^{b-1}, km),$$

hence $B(k,m,p^b) = p^b - p^{b-1} = \phi(p^b)$ if $p^b | km$, and $B(k,m,p^b) = 0$ otherwise. For fixed $k, m \in \mathbf{N}$ we note that (d, km) is a multiplicative function of d, thus $B(k,m,n)$ is a multiplicative function of n as a Dirichlet convolution of multiplicative functions, and so the result follows. □

Lemma 3. *For $k_1, k_2, n_1, n_2 \in \mathbf{N}$ let*

$$T(k_1,k_2,n_1,n_2) = \sum_{d_1|n_1}\sum_{d_2|n_2} \mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)(d_1,k_1)(d_2,k_2)\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right).$$

Put $Q_1 = n_1/(n_1,k_1)$ and $Q_2 = n_2/(n_2,k_2)$. Then

$$T(k_1,k_2,n_1,n_2) = \begin{cases} \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)} & \text{if } Q_1 = Q_2, \\ 0 & \text{if } Q_1 \neq Q_2. \end{cases}$$

Proof. Write T for $T(k_1,k_2,n_1,n_2)$. With the notation in Lemma 2 we get

$$T = \sum_{d_1|n_1} \mu\left(\frac{n_1}{d_1}\right)(d_1,k_1) B\left(k_2, \frac{d_1}{(d_1,k_1)}, n_2\right). \tag{14}$$

According to Lemma 2, we only get a contribution to this sum if n_2 divides $k_2 d_1/(d_1,k_1)$, or equivalently, if Q_2 divides $d_1/(d_1,k_1)$. For any $d_1|n_1$ and any p we have

$$e_p\left(\frac{d_1}{(d_1,k_1)}\right) = \max(e_p(d_1) - e_p(k_1), 0) \le \max(e_p(n_1) - e_p(k_1), 0) = e_p(Q_1),$$

hence $d_1/(d_1,k_1)$ divides Q_1. Thus, if $Q_2 \nmid Q_1$, then there is no contribution to the sum in (14), and so $T = 0$. Since $T = T(k_2,k_1,n_2,n_1)$, it follows that $T = 0$ if $Q_1 \nmid Q_2$. The only remaining case is $Q_2 | Q_1$ and $Q_1 | Q_2$, i.e., $Q_1 = Q_2$. If $Q_1 = Q_2$, then we only get a contribution to the sum in (14) if Q_1 divides $d_1/(d_1,k_1)$, but since $d_1/(d_1,k_1)$ always divides Q_1, we must have $d_1/(d_1,k_1) = Q_1$. Put $a_p = e_p(n_1), b_p = e_p(k_1)$, and

$$m_1 = \prod_{\substack{p \\ a_p > b_p}} p^{a_p}, \quad m_2 = \prod_{\substack{p \\ a_p \le b_p}} p^{a_p},$$

so that $n_1 = m_1 m_2$. Then the conditions $d_1|n_1$ and $d_1/(d_1,k_1) = Q_1$ hold if and only if $e_p(d_1) \le a_p$ and

$$\max(e_p(d_1) - b_p, 0) = \max(a_p - b_p, 0)$$

for all p. These conditions are equivalent to the following: $e_p(d_1) \le a_p$ for all p and $e_p(d_1) = a_p$ whenever $a_p > b_p$. This is in turn equivalent to $d_1 = m_1 d$ with $d | m_2$. We apply this information to (14). We also use Lemma 2 and the fact that $d_1/(d_1,k_1) = Q_1$ implies $(d_1,k_1) = d_1/Q_1$. Then

$$\begin{aligned} T &= \phi(n_2) \sum_{d|m_2} \mu\left(\frac{n_1}{m_1 d}\right) \frac{m_1 d}{Q_1} = \frac{\phi(n_2) n_1}{Q_1} \sum_{d|m_2} \mu\left(\frac{m_2}{d}\right) \frac{d}{m_2} \\ &= \frac{\phi(n_2) n_1}{Q_1} \sum_{d|m_2} \frac{\mu(d)}{d} = \frac{\phi(n_2) n_1}{Q_1} \prod_{p|m_2} \left(1 - \frac{1}{p}\right). \end{aligned}$$

Now $p | m_2$ if and only if $0 < a_p \le b_p$, and the latter is equivalent to $p | n_1$ and $p \nmid Q_1$. Using $\prod_{p|n}(1 - p^{-1}) = \phi(n)/n$ for $n \in \mathbf{N}$, we get

$$\prod_{p|m_2} \left(1 - \frac{1}{p}\right) = \frac{\phi(n_1) Q_1}{n_1 \phi(Q_1)},$$

whence the formula for T in the case $Q_1 = Q_2$. □

Recall that a function G on $\mathbf{N}$ is called *additive* if $G(mn) = G(m) + G(n)$ whenever $(m,n) = 1$. Note that if F is a multiplicative function which only attains positive values, then $\log F$ is additive.

Lemma 4. *Let F be a multiplicative and G an additive function, and put*

$$H(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d) G(d) \quad \textit{for} \quad n \in \mathbf{N},$$

$$J(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d) \quad \textit{for} \quad n \in \mathbf{N}.$$

Then for all $n \in \mathbf{N}$ we have

$$H(n) = \sum_{p|n} H(p^{e_p(n)}) J(n/p^{e_p(n)}). \tag{15}$$

Proof. For $m, n \in \mathbf{N}$ with $(m,n) = 1$ we have

$$\begin{aligned} H(mn) &= \sum_{d|mn} \mu\left(\frac{mn}{d}\right) F(d) G(d) = \sum_{\substack{d_1|m \\ d_2|n}} \mu\left(\frac{mn}{d_1 d_2}\right) F(d_1 d_2) G(d_1 d_2) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu\left(\frac{m}{d_1}\right) \mu\left(\frac{n}{d_2}\right) F(d_1) F(d_2) (G(d_1) + G(d_2)) \\ &= \sum_{d_1|m} \mu\left(\frac{m}{d_1}\right) F(d_1) G(d_1) \sum_{d_2|n} \mu\left(\frac{n}{d_2}\right) F(d_2) + \sum_{d_2|n} \mu\left(\frac{n}{d_2}\right) F(d_2) G(d_2) \sum_{d_1|m} \mu\left(\frac{m}{d_1}\right) F(d_1), \end{aligned}$$

and so

$$H(mn) = H(m)J(n) + H(n)J(m). \tag{16}$$

Now $H(1) = F(1)G(1) = 0$ since $G(1) = 0$, and so (15) holds for $n = 1$ (recall that an empty sum is 0 by convention). We obtain (15) for all $n \in \mathbf{N}$ by proceeding by induction on the number of distinct prime divisors of n and using (16) and the fact that J is multiplicative. □

Lemma 5. *For $k, n \in \mathbf{N}$ put*

$$H(k,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)(d,k)\log\frac{d}{(d,k)}.$$

If there is a unique prime q such that $e_q(n) > e_q(k)$, then

$$H(k,n) = q^{e_q(k)}\phi(n/q^{e_q(n)})\log q.$$

Otherwise we have $H(k,n) = 0$.

Proof. Fix k and note that (d,k) is a multiplicative and $\log(d/(d,k))$ an additive function of d. We have

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)(d,k) = B(k,1,n)$$

with the notation in Lemma 2. Thus, Lemma 4 yields

$$H(k,n) = \sum_{p|n} H(k,p^{e_p(n)})B(k,1,n/p^{e_p(n)}). \tag{17}$$

For $b \in \mathbf{N}$ we obtain

$$H(k,p^b) = (p^b,k)\log\frac{p^b}{(p^b,k)} - (p^{b-1},k)\log\frac{p^{b-1}}{(p^{b-1},k)}.$$

If $b \le e_p(k)$, then $(p^b,k) = p^b$ and $(p^{b-1},k) = p^{b-1}$, hence $H(k,p^b) = 0$. If $b > e_p(k)$, then $(p^b,k) = (p^{b-1},k) = p^{e_p(k)}$, hence $H(k,p^b) = p^{e_p(k)}\log p$. Using also Lemma 2, (17) reduces to

$$H(k,n) = \sum_{p} p^{e_p(k)}\phi(n/p^{e_p(n)})\log p, \tag{18}$$

where the sum runs over all p satisfying the following two conditions: (i) $e_p(n) > e_p(k)$; (ii) $n/p^{e_p(n)}$ divides k. Note that (ii) means $e_{p_1}(n) \le e_{p_1}(k)$ for all primes $p_1 \ne p$. Therefore, (i) and (ii) can hold simultaneously only in the case where there is a unique prime q with $e_q(n) > e_q(k)$, and in this case (i) and (ii) hold for $p = q$ and for no other prime. Thus, if there is a unique prime q with $e_q(n) > e_q(k)$, then the sum in (18) reduces to a single term with $p = q$, and in all other cases the sum in (18) is empty. □

Lemma 6. *For $k, m, n \in \mathbf{N}$ put*

$$E(k,m,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)(d,km)\log(m,\frac{d}{(d,k)}).$$

Then $E(k,m,n) = 0$ if $n \nmid km$. If $n|km$, then

$$E(k,m,n) = \phi(n)\log\left(\prod_p p^{e_p(n)-e_p(k)}\right) + \phi(n)\sum_p \frac{\log p}{p-1},$$

where the product and the sum are extended over all p with $e_p(n) > e_p(k)$.

Proof. Fix k and m and note that (d,km) is a multiplicative and $\log(m, d/(d,k))$ an additive function of d. We have

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)(d,km) = B(km,1,n)$$

with the notation in Lemma 2. Thus, Lemma 4 yields

$$E(k,m,n) = \sum_{p|n} E(k,m,p^{e_p(n)})B(km,1,n/p^{e_p(n)}). \tag{19}$$

For $b \in \mathbf{N}$ we have

$$E(k,m,p^b) = (p^b,km)\log\left(m,\frac{p^b}{(p^b,k)}\right) - (p^{b-1},km)\log\left(m,\frac{p^{b-1}}{(p^{b-1},k)}\right).$$

If $b \le e_p(k)$, then $(p^b,k) = p^b$ and $(p^{b-1},k) = p^{b-1}$, hence $E(k,m,p^b) = 0$. If $b > e_p(km)$, then $(p^b,k) = (p^{b-1},k) = p^{e_p(k)}$ and

$$\left(m,\frac{p^b}{(p^b,k)}\right) = \left(m,\frac{p^{b-1}}{(p^{b-1},k)}\right) = p^{e_p(m)},$$

hence

$$E(k,m,p^b) = p^{e_p(km)}\log p^{e_p(m)} - p^{e_p(km)}\log p^{e_p(m)} = 0.$$

If $e_p(k) < b \le e_p(km)$, then $(p^b,k) = (p^{b-1},k) = p^{e_p(k)}$ and

$$\left(m,\frac{p^b}{(p^b,k)}\right) = p^{b-e_p(k)}, \quad \left(m,\frac{p^{b-1}}{(p^{b-1},k)}\right) = p^{b-1-e_p(k)},$$

hence

$$\begin{aligned} E(k,m,p^b) &= p^b \log p^{b-e_p(k)} - p^{b-1}\log p^{b-1-e_p(k)} \\ &= (p^b - p^{b-1})\log p^{b-e_p(k)} + p^{b-1}(\log p^{b-e_p(k)} - \log p^{b-1-e_p(k)}) \\ &= \phi(p^b)\log p^{b-e_p(k)} + p^{b-1}\log p \\ &= \phi(p^b)(\log p^{b-e_p(k)} + \frac{\log p}{p-1}). \end{aligned}$$

Using also Lemma 2, we conclude that (19) reduces to

$$\begin{aligned} E(k,m,n) &= \sum_p \phi(p^{e_p(n)})(\log p^{e_p(n)-e_p(k)} + \frac{\log p}{p-1})\phi(n/p^{e_p(n)}) \\ &= \phi(n)\sum_p(\log p^{e_p(n)-e_p(k)} + \frac{\log p}{p-1}), \end{aligned}$$

where the sum is over all p satisfying the following two conditions: (i) $e_p(k) < e_p(n) \leq e_p(km)$; (ii) $n/p^{e_p(n)}$ divides km. Note that (i) implies that $p^{e_p(n)}$ divides km. Thus, if there exists a p satisfying (i) and (ii), then we must have $n|km$. Therefore, $E(k,m,n) = 0$ if $n\nmid km$. If $n|km$, then (i) reduces to $e_p(n) > e_p(k)$, whereas (ii) holds automatically. The desired result now follows. □

We write $[m,n]$ for the least common multiple of $m, n \in \mathbf{N}$ and note that $[m,n] = mn/(m,n)$.

Lemma 7. *For $k_1, k_2, n_1, n_2 \in \mathbf{N}$ let*

$$V(k_1,k_2,n_1,n_2) = \\ = \sum_{d_1|n_1}\sum_{d_2|n_2} \mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)(d_1,k_1)(d_2,k_2)\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right)\log\left[\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right].$$

Let Q_1 and Q_2 be as in Lemma 3. Then

$$V(k_1,k_2,n_1,n_2) = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}(\log Q_1 - \sum_{p|Q_1}\frac{\log p}{p-1}) \quad if \quad Q_1 = Q_2.$$

If there is a unique prime q such that $e_q(Q_1) \neq e_q(Q_2)$, then

$$V(k_1,k_2,n_1,n_2) = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_j)}\log q,$$

where $j \in \{1,2\}$ is such that $e_q(Q_j) = \max(e_q(Q_1), e_q(Q_2))$. In all other cases we have $V(k_1,k_2,n_1,n_2) = 0$.

Proof. Using $[m,n] = mn/(m,n)$, we get

(20) $$V(k_1,k_2,n_1,n_2) = V_1(k_1,k_2,n_1,n_2) + V_2(k_1,k_2,n_1,n_2) - V_3(k_1,k_2,n_1,n_2),$$

where

$$V_1(k_1,k_2,n_1,n_2) = \\ = \sum_{d_1|n_1}\sum_{d_2|n_2}\mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)(d_1,k_1)(d_2,k_2)\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right)\log\frac{d_1}{(d_1,k_1)},$$

$$V_2(k_1,k_2,n_1,n_2) = V_1(k_2,k_1,n_2,n_1),$$

$$V_3(k_1,k_2,n_1,n_2) = \\ \sum_{d_1|n_1}\sum_{d_2|n_2}\mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)(d_1,k_1)(d_2,k_2)\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right)\log\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right).$$

Abbreviate $V_1(k_1,k_2,n_1,n_2)$ by V_1. Then

$$\begin{aligned} V_1 &= \sum_{d_1|n_1}\mu\left(\frac{n_1}{d_1}\right)(d_1,k_1)\left(\log\frac{d_1}{(d_1,k_1)}\right)\sum_{d_2|n_2}\mu\left(\frac{n_2}{d_2}\right)(d_2,k_2)\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right) \\ &= \sum_{d_1|n_1}\mu\left(\frac{n_1}{d_1}\right)(d_1,k_1)B\left(k_2, \frac{d_1}{(d_1,k_1)}, n_2\right)\log\frac{d_1}{(d_1,k_1)} \end{aligned}$$

with the notation in Lemma 2. According to Lemma 2, we only get a contribution to the last sum if n_2 divides $k_2 d_1/(d_1,k_1)$, or equivalently, if Q_2 divides $d_1/(d_1,k_1)$. In the proof of Lemma 3 we have seen that $d_1/(d_1,k_1)$ divides Q_1. Therefore, $V_1 = 0$ if $Q_2 \nmid Q_1$. Now let $Q_2|Q_1$. Using again Lemma 2, we obtain

(21) $$V_1 = \phi(n_2)\sum_{d_1}\mu\left(\frac{n_1}{d_1}\right)(d_1,k_1)\log\frac{d_1}{(d_1,k_1)},$$

where the sum is over all $d_1|n_1$ such that Q_2 divides $d_1/(d_1,k_1)$. Now Q_2 divides $d_1/(d_1,k_1)$ if and only if

$$e_p(Q_2) + e_p(k_1) \leq e_p(d_1) \quad \text{for all} \quad p|Q_2.$$

This is equivalent to $t|d_1$ with

(22) $$t = \prod_{p|Q_2} p^{e_p(Q_2)+e_p(k_1)}.$$

We note that

(23) $$t_1 := \frac{n_1}{t} = \prod_{p\nmid Q_2} p^{e_p(n_1)} \prod_{p|Q_2} p^{e_p(n_1)-e_p(k_1)-e_p(Q_2)} = \prod_{p\nmid Q_2} p^{e_p(n_1)} \prod_{p|Q_2} p^{e_p(Q_1)-e_p(Q_2)}$$

is a positive integer. Therefore, the sum in (21) is over all $d_1 = td$ with $d|t_1$, hence

$$V_1 = \phi(n_2)\sum_{d|t_1}\mu\left(\frac{t_1}{d}\right)(td,k_1)\log\frac{td}{(td,k_1)}.$$

By (13) we can write

$$(td,k_1) = (t,k_1)\left(d,\frac{k_1}{(t,k_1)}\right),$$

so that with $k := k_1/(t,k_1)$ we get

$$\begin{aligned} V_1 &= \phi(n_2)(t,k_1)\sum_{d|t_1}\mu\left(\frac{t_1}{d}\right)(d,k)\left(\log\frac{t}{(t,k_1)}+\log\frac{d}{(d,k)}\right)\\ &= \phi(n_2)(t,k_1)B(k,1,t_1)\log\frac{t}{(t,k_1)} + \phi(n_2)(t,k_1)H(k,t_1) =: V_4 + V_5 \end{aligned}$$

with the notation in Lemma 5. By Lemma 2 we have $B(k,1,t_1) \neq 0$ if and only if $t_1|k$. Now from (22),

$$k = \frac{k_1}{(t,k_1)} = \prod_{p\nmid Q_2} p^{e_p(k_1)}, \tag{24}$$

and so a comparison with (23) shows that $t_1|k$ if and only if $e_p(Q_1) = e_p(Q_2)$ for all $p|Q_2$ and $e_p(n_1) \leq e_p(k_1)$ for all $p\nmid Q_2$. But $e_p(n_1) \leq e_p(k_1)$ is equivalent to $e_p(Q_1) = 0$, and so $t_1|k$ if and only if $Q_1 = Q_2$. Thus, $V_4 = 0$ if $Q_1 \neq Q_2$. If $Q_1 = Q_2$, then

$$V_4 = \phi(n_2)(t,k_1)\phi(t_1)\log\frac{t}{(t,k_1)} = \phi(n_2)(t,k_1)\phi(t_1)\log Q_1,$$

and from (22) and (23) we get

$$\begin{aligned} (t,k_1)\phi(t_1) &= \prod_{p|Q_1} p^{e_p(k_1)}\cdot\phi\left(\prod_{p\nmid Q_1} p^{e_p(n_1)}\right)\\ &= \frac{1}{Q_1}\prod_{p|Q_1} p^{e_p(n_1)}\prod_{p\nmid Q_1} p^{e_p(n_1)}\prod_{p|n_1,\,p\nmid Q_1}\left(1-\frac{1}{p}\right) = \frac{n_1}{Q_1}\prod_{p|m_2}\left(1-\frac{1}{p}\right), \end{aligned}$$

where m_2 is as in the proof of Lemma 3. In that proof we have shown that

$$\prod_{p|m_2}\left(1-\frac{1}{p}\right) = \frac{\phi(n_1)Q_1}{n_1\phi(Q_1)},$$

and so it follows that

$$(t,k_1)\phi(t_1) = \frac{\phi(n_1)}{\phi(Q_1)}. \tag{25}$$

Thus, we have

$$V_4 = \begin{cases} \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log Q_1 & \text{if } Q_1 = Q_2,\\ 0 & \text{if } Q_2|Q_1, Q_1 \neq Q_2. \end{cases} \tag{26}$$

Now we consider V_5 for $Q_2|Q_1$. By Lemma 5 we have $H(k,t_1) \neq 0$ if and only if there is a unique prime q such that $e_q(t_1) > e_q(k)$. A comparison of (23) and (24) shows that if $p|Q_2$, then $e_p(t_1) > e_p(k)$ if and only if $e_p(Q_1) - e_p(Q_2) > 0$, and if $p\nmid Q_2$, then $e_p(t_1) > e_p(k)$ if and only if $e_p(n_1) > e_p(k_1)$. Note that $e_p(n_1) > e_p(k_1)$ is equivalent to $e_p(Q_1) > 0$. Therefore, for any p it is true that $e_p(t_1) > e_p(k)$ if and only if $e_p(Q_1) > e_p(Q_2)$. Consequently, the condition that there is a unique prime q such that $e_q(t_1) > e_q(k)$ is equivalent to the existence of a unique prime q (indeed, the same prime q) such that $e_q(Q_1) > e_q(Q_2)$. Thus, if this condition is not met, then $V_5 = 0$. If this condition is met, then the formula for $H(k,t_1)$ in Lemma 5 yields

$$V_5 = \phi(n_2)(t,k_1)q^{e_q(k)}\phi(t_1/q^{e_q(t_1)})\log q.$$

In the case under consideration we have $e_q(Q_1) > e_q(Q_2)$ and $e_p(Q_1) \leq e_p(Q_2)$ for all $p \neq q$, but since $Q_2|Q_1$, it follows that $e_p(Q_1) = e_p(Q_2)$ for all $p \neq q$. By distinguishing the cases $q|Q_2$ and $q\nmid Q_2$, we deduce from (22), (23), and (24) that

$$(t,k_1)q^{e_q(k)} = \prod_{p|Q_1} p^{e_p(k_1)},$$

$$t_1/q^{e_q(t_1)} = \prod_{p\nmid Q_1} p^{e_p(n_1)}.$$

Thus, by the calculation leading to (25) we obtain

$$(t,k_1)q^{e_q(k)}\phi(t_1/q^{e_q(t_1)}) = \prod_{p|Q_1} p^{e_p(k_1)}\cdot\phi\left(\prod_{p\nmid Q_1} p^{e_p(n_1)}\right) = \frac{\phi(n_1)}{\phi(Q_1)},$$

and so

$$V_5 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log q$$

if there is a unique prime q such that $e_q(Q_1) > e_q(Q_2)$. We recall that $V_5 = 0$ if this condition is not satisfied, and we combine this information with (26). Then we get the following formulas for $V_1 = V_4 + V_5$, where we also recall that $V_1 = 0$ if $Q_2\nmid Q_1$:

$$V_1 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log Q_1 \quad \text{if} \quad Q_1 = Q_2,$$

$$V_1 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log q$$

if there is a (unique) prime q such that $e_q(Q_1) > e_q(Q_2)$ and $e_p(Q_1) = e_p(Q_2)$ for all $p \neq q$, and $V_1 = 0$ in all other cases. Using $V_2(k_1,k_2,n_1,n_2) = V_1(k_2,k_1,n_2,n_1)$ and abbreviating $V_2(k_1,k_2,n_1,n_2)$ by V_2, we get the following formulas:

$$V_2 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log Q_1 \quad \text{if} \quad Q_1 = Q_2,$$

$$V_2 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_2)}\log q$$

if there is a (unique) prime q such that $e_q(Q_2) > e_q(Q_1)$ and $e_p(Q_1) = e_p(Q_2)$ for all $p \neq q$, and $V_2 = 0$ in all other cases.

We consider now $V_3(k_1,k_2,n_1,n_2)$, which we abbreviate by V_3. Using (13), we can write

$$\begin{aligned} V_3 &= \sum_{d_1|n_1} \mu\left(\frac{n_1}{d_1}\right)(d_1,k_1) \sum_{d_2|n_2} \mu\left(\frac{n_2}{d_2}\right)\left(d_2, \frac{k_2 d_1}{(d_1,k_1)}\right) \log\left(\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right) \\ &= \sum_{d_1|n_1} \mu\left(\frac{n_1}{d_1}\right)(d_1,k_1) E\left(k_2, \frac{d_1}{(d_1,k_1)}, n_2\right) \end{aligned}$$

with the notation in Lemma 6. According to Lemma 6, we only get a contribution to the last sum if n_2 divides $k_2 d_1/(d_1,k_1)$, or equivalently, if Q_2 divides $d_1/(d_1,k_1)$. In the proof of Lemma 3 we have shown that $d_1/(d_1,k_1)$ divides Q_1. Therefore, $V_3 = 0$ if $Q_2 \nmid Q_1$. From the definition of V_3 at the beginning of the proof of Lemma 7 we see that $V_3 = V_3(k_2,k_1,n_2,n_1)$, thus $V_3 = 0$ if $Q_1 \nmid Q_2$. Hence $V_3 = 0$ if $Q_1 \neq Q_2$, and it remains to consider the case $Q_1 = Q_2$. In this case we only get a contribution to the last sum if $d_1/(d_1,k_1) = Q_1$, and noting that then $(d_1,k_1) = d_1/Q_1$, we obtain

$$V_3 = \frac{1}{Q_1}E(k_2,Q_1,n_2)\sum_{d_1}\mu\left(\frac{n_1}{d_1}\right)d_1,$$

where the sum is over all $d_1|n_1$ with $d_1/(d_1,k_1) = Q_1$. In the proof of Lemma 3 we have shown that these conditions on d_1 are equivalent to $d_1 = m_1 d$ with $d|m_2$. Therefore,

$$\begin{aligned} V_3 &= \frac{m_1}{Q_1}E(k_2,Q_1,n_2)\sum_{d|m_2}\mu\left(\frac{m_2}{d}\right)d \\ &= \frac{m_1}{Q_1}E(k_2,Q_1,n_2)\sum_{d|m_2}\mu(d)\frac{m_2}{d} \\ &= \frac{n_1}{Q_1}E(k_2,Q_1,n_2)\prod_{p|m_2}\left(1-\frac{1}{p}\right). \end{aligned}$$

The last product was evaluated in the proof of Lemma 3, hence

$$V_3 = \frac{\phi(n_1)}{\phi(Q_1)}E(k_2,Q_1,n_2).$$

By Lemma 6 we have

$$E(k_2,Q_1,n_2) = \phi(n_2)\log\left(\prod_p p^{e_p(n_2)-e_p(k_2)}\right) + \phi(n_2)\sum_p \frac{\log p}{p-1},$$

where the product and the sum run over all p with $e_p(n_2) > e_p(k_2)$. Note that $e_p(n_2) > e_p(k_2)$ if and only if $p|Q_2 = Q_1$, and that

$$\prod_p p^{e_p(n_2)-e_p(k_2)} = \prod_{p|Q_2} p^{e_p(Q_2)} = Q_2 = Q_1.$$

Thus, we have shown

$$V_3 = \frac{\phi(n_1)\phi(n_2)}{\phi(Q_1)}\left(\log Q_1 + \sum_{p|Q_1}\frac{\log p}{p-1}\right) \quad \text{if} \quad Q_1 = Q_2$$

and $V_3 = 0$ if $Q_1 \neq Q_2$. Using (20) and the formulas for V_1, V_2, and V_3 above, we get the desired result. □

Lemma 8. *For $k_1,k_2,n_1,n_2 \in \mathbf{N}$ with $n_2|n_1$ let*

$$Y(k_1,k_2,n_1,n_2) = \sum_{d_1|n_1}\sum_{d_2|n_2}\left|\mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)\right|d_1 d_2\left[\frac{d_1}{(d_1,k_1)}, \frac{d_2}{(d_2,k_2)}\right].$$

Then we have

$$Y(k_1,k_2,n_1,n_2) \leq \frac{n_1^3 n_2^2}{\phi(n_1)\phi(n_2)}.$$

Proof. It is clear that $[d_1/(d_1,k_1), d_2/(d_2,k_2)]$ divides n_1; therefore,

$$\begin{aligned} Y(k_1,k_2,n_1,n_2) &\leq n_1\sum_{d_1|n_1}\sum_{d_2|n_2}\left|\mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)\right|d_1 d_2 \\ &= n_1^2 n_2 F(n_1)F(n_2) \end{aligned}$$

with

$$F(n) = \sum_{d|n}\frac{|\mu(d)|}{d} \quad \text{for} \quad n \in \mathbf{N}.$$

Now F is a multiplicative function as the summatory function of a multiplicative function. For $b \in \mathbf{N}$ we have

$$F(p^b) = \frac{p+1}{p} \leq \frac{p}{p-1} = \frac{p^b}{\phi(p^b)},$$

hence $F(n) \leq n/\phi(n)$ for all $n \in \mathbf{N}$. This implies the desired result. □

5. PROOF OF THEOREM 2

We fix the dimension $s \geq 2$ and the invariants $n_1, n_2 \in \mathbb{N}$ with $n_1 > 1, n_2 > 1$, and $n_2 | n_1$. For the family $\mathcal{L} = \mathcal{L}(s; n_1, n_2)$ we then have

$$M(\mathcal{L}) = \frac{1}{\operatorname{card}(\mathcal{L})} \sum_{L \in \mathcal{L}} R_1(L) = \frac{1}{\operatorname{card}(\mathcal{L})} \sum_{L \in \mathcal{L}} \sum_{\mathbf{h} \in E(L)} r(\mathbf{h})^{-1},$$

where $E(L)$ is as in Definition 3. Interchanging the order of summation, we get

$$M(\mathcal{L}) = \frac{1}{\operatorname{card}(\mathcal{L})} \sum_{\mathbf{h} \in C_s^*(N)} A(\mathbf{h}) r(\mathbf{h})^{-1},$$

where $N = n_1 n_2$ and $A(\mathbf{h})$ is the number of $L \in \mathcal{L}$ with $\mathbf{h} \in L^\perp$. Since $A(\mathbf{0}) = \operatorname{card}(\mathcal{L})$, we obtain

$$M(\mathcal{L}) = \frac{1}{\operatorname{card}(\mathcal{L})} \sum_{\mathbf{h} \in C_s(N)} A(\mathbf{h}) r(\mathbf{h})^{-1} - 1. \tag{27}$$

We write $e(t) = e^{2\pi\sqrt{-1}t}$ for real t. Then by [17, Theorem 1] we have for $\mathbf{h} \in \mathbf{Z}^s$:

$$\frac{1}{N} \sum_{\mathbf{x} \in X(L)} e(\mathbf{h} \cdot \mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{h} \in L^\perp, \\ 0 & \text{if } \mathbf{h} \notin L^\perp. \end{cases}$$

Therefore,

$$A(\mathbf{h}) = \frac{1}{N} \sum_{L \in \mathcal{L}} \sum_{\mathbf{x} \in X(L)} e(\mathbf{h} \cdot \mathbf{x}).$$

For $L \in \mathcal{L}$ the node set $X(L)$ has the special form described in §3. Using also the special form of $\mathbf{z}_1$ and $\mathbf{z}_2$ in (12), we get

$$\sum_{\mathbf{x} \in X(L)} e(\mathbf{h} \cdot \mathbf{x}) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} e\left(\mathbf{h} \cdot \left(\frac{k_1}{n_1}\mathbf{z}_1 + \frac{k_2}{n_2}\mathbf{z}_2\right)\right) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} e\left(\sum_{i=1}^{2} \sum_{j=i}^{s} \frac{k_i}{n_i} h_j z_i^{(j)}\right),$$

where $\mathbf{h} = (h_1, \ldots, h_s)$. It follows that

$$\begin{aligned} A(\mathbf{h}) &= \frac{1}{N} \sum_{z_1^{(1)}, \ldots, z_1^{(s)} \in Z_1} \sum_{z_2^{(2)}, \ldots, z_2^{(s)} \in Z_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} e\left(\sum_{i=1}^{2} \sum_{j=i}^{s} \frac{k_i}{n_i} h_j z_i^{(j)}\right) \\ &= \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{z_1^{(1)}, \ldots, z_1^{(s)} \in Z_1} \sum_{z_2^{(2)}, \ldots, z_2^{(s)} \in Z_2} \prod_{i=1}^{2} \prod_{j=i}^{s} e\left(\frac{k_i}{n_i} h_j z_i^{(j)}\right) \\ &= \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \prod_{i=1}^{2} \prod_{j=i}^{s} \left(\sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h_j z\right)\right) \\ &= \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \prod_{j=1}^{s} \prod_{i=1}^{\min(j,2)} \left(\sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h_j z\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\mathbf{h} \in C_s(N)} \frac{A(\mathbf{h})}{r(\mathbf{h})} &= \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{h_1, \ldots, h_s \in C(N)} \frac{1}{r(h_1) \ldots r(h_s)} \prod_{j=1}^{s} \prod_{i=1}^{\min(j,2)} \left(\sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h_j z\right)\right) \\ &= \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \prod_{j=1}^{s} \left(\sum_{h \in C(N)} \frac{1}{r(h)} \prod_{i=1}^{\min(j,2)} \left(\sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h z\right)\right)\right). \end{aligned} \tag{28}$$

Now we use

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

and we obtain

$$\begin{aligned} \sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h z\right) &= \sum_{z=0}^{n_i-1} e\left(\frac{k_i}{n_i} h z\right) \sum_{d|(z,n_i)} \mu(d) = \sum_{d|n_i} \mu(d) \sum_{\substack{z=0 \\ d|z}}^{n_i-1} e\left(\frac{k_i}{n_i} h z\right) \\ &= \sum_{d|n_i} \mu(d) \sum_{a=0}^{(n_i/d)-1} e\left(\frac{k_i}{n_i} h d a\right) = \sum_{d|n_i} \mu\left(\frac{n_i}{d}\right) \sum_{a=0}^{d-1} e\left(\frac{k_i}{d} h a\right), \end{aligned}$$

where in the last step we changed the summation variable from d to n_i/d. The last inner sum has the value d if $d | k_i h$ and the value 0 otherwise. Thus,

$$\sum_{z \in Z_i} e\left(\frac{k_i}{n_i} h z\right) = \sum_{\substack{d|n_i \\ d|k_i h}} \mu\left(\frac{n_i}{d}\right) d.$$

If $h = 0$, then the sum on the left-hand side trivially has the value $\phi(n_i)$. Therefore, we get

(29)

$$\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{\min(j,2)} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right) = \prod_{i=1}^{\min(j,2)} \phi(n_i) + \sum_{h\in C^*(N)} \frac{1}{|h|} \prod_{i=1}^{\min(j,2)} \left(\sum_{\substack{d|n_i\\ d|k_ih}} \mu\left(\frac{n_i}{d}\right) d\right).$$

Now let $j \geq 2$. Then (29) yields

$$\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{2} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right) = \tag{30}$$
$$= \phi(n_1)\phi(n_2) + \sum_{d_1|n_1} \sum_{d_2|n_2} \mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right) d_1d_2 \sum_{\substack{h\in C^*(N)\\ d_1|k_1h, d_2|k_2h}} \frac{1}{|h|}.$$

For $i = 1, 2$ we have $d_i|k_ih$ if and only if $d_i/(d_i, k_i)$ divides h. Therefore, the conditions $d_1|k_1h$ and $d_2|k_2h$ hold simultaneously if and only if the least common multiple $W(d_1, d_2, k_1, k_2) := [d_1/(d_1, k_1), d_2/(d_2, k_2)]$ divides h. Thus,

$$\sum_{\substack{h\in C^*(N)\\ d_1|k_1h, d_2|k_2h}} \frac{1}{|h|} = \sum_{\substack{h\in C^*(N)\\ W(d_1,d_2,k_1,k_2)|h}} \frac{1}{|h|} = \frac{1}{W(d_1,d_2,k_1,k_2)} S\left(\frac{N}{W(d_1,d_2,k_1,k_2)}\right)$$
$$= \frac{1}{W(d_1,d_2,k_1,k_2)}\left(2\log N + C - 2\log W(d_1,d_2,k_1,k_2) + \varepsilon\left(\frac{N}{W(d_1,d_2,k_1,k_2)}\right)\right)$$

by Lemma 1. Since $[m, n] = mn/(m, n)$, it follows from (30) that

$$\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{2} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right) = \phi(n_1)\phi(n_2) + (2\log N + C)T(k_1,k_2,n_1,n_2)$$
$$- 2V(k_1,k_2,n_1,n_2) + \sum_{d_1|n_1}\sum_{d_2|n_2} \mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right) \frac{d_1d_2}{W(d_1,d_2,k_1,k_2)} \varepsilon\left(\frac{N}{W(d_1,d_2,k_1,k_2)}\right)$$

with the notation in Lemmas 3 and 7. Since $|\varepsilon(m)| < 4/m^2$ by Lemma 1, we get

$$\left|\sum_{d_1|n_1}\sum_{d_2|n_2} \mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right) \frac{d_1d_2}{W(d_1,d_2,k_1,k_2)} \varepsilon\left(\frac{N}{W(d_1,d_2,k_1,k_2)}\right)\right| \leq$$
$$\leq \frac{4}{N^2} \sum_{d_1|n_1}\sum_{d_2|n_2} \left|\mu\left(\frac{n_1}{d_1}\right)\mu\left(\frac{n_2}{d_2}\right)\right| d_1d_2W(d_1,d_2,k_1,k_2)$$
$$= \frac{4}{N^2} Y(k_1,k_2,n_1,n_2) \leq \frac{4n_1}{\phi(n_1)\phi(n_2)}$$

according to Lemma 8. It follows that

$$\left|\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{2} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right)\right| \leq \tag{31}$$
$$\leq |\phi(n_1)\phi(n_2) + (2\log N + C)T(k_1,k_2,n_1,n_2) - 2V(k_1,k_2,n_1,n_2)| + \frac{4n_1}{\phi(n_1)\phi(n_2)}.$$

Let $Q_1 = n_1/(n_1, k_1)$ and $Q_2 = n_2/(n_2, k_2)$. If $Q_1 = Q_2$, then using

$$0 \leq \log Q_1 - \sum_{p|Q_1} \frac{\log p}{p-1} \leq \log n_1,$$

we obtain from Lemmas 3 and 7,

$$|\phi(n_1)\phi(n_2) + (2\log N + C)T(k_1,k_2,n_1,n_2) - 2V(k_1,k_2,n_1,n_2)| \leq \phi(n_1)\phi(n_2) + \frac{2\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log N.$$

If there is a unique prime q such that $e_q(Q_1) \neq e_q(Q_2)$, then using the fact that $q|n$ implies $\phi(n) \geq q - 1 > \log q$, we obtain from Lemmas 3 and 7,

$$|\phi(n_1)\phi(n_2) + (2\log N + C)T(k_1,k_2,n_1,n_2) - 2V(k_1,k_2,n_1,n_2)| \leq \phi(n_1)\phi(n_2).$$

In all other cases we have $T(k_1,k_2,n_1,n_2) = V(k_1,k_2,n_1,n_2) = 0$ by Lemmas 3 and 7, and so the last inequality holds trivially. By [2, Theorem 328] we have

$$\frac{n}{\phi(n)} < c\log\log(n+1) \quad \text{for all} \quad n \geq 2 \tag{32}$$

with an absolute constant $c > 0$. In the following we denote by c a positive absolute constant which may have different values in different occurrences. In view of (31) we then get

$$\left|\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{2} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right)\right| < \phi(n_1)\phi(n_2) + \frac{c\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log N \tag{33}$$

if $Q_1 = Q_2$, where we also used (32) and the fact that $Q_1|n_1$ implies $\phi(Q_1) \leq \phi(n_1)$. From (31) and (32) we get

$$\left|\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{2} \left(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\right)\right| < \phi(n_1)\phi(n_2) + c\,\log\log(n_1+1) \tag{34}$$

if $Q_1 \neq Q_2$.

Now we consider the expression on the left-hand side of (29) for $j = 1$. If $k_1 = n_1$, then it is clear from Lemma 1 that

$$(35)\qquad \Big|\sum_{h\in C(N)} \frac{1}{r(h)} \sum_{z\in Z_1} e\left(\frac{k_1}{n_1}hz\right)\Big| < c\phi(n_1)\log N.$$

If $1 \le k_1 < n_1$, then we apply (34) with $k_2 = n_2$ to obtain

$$(36)\qquad \Big|\sum_{h\in C(N)} \frac{1}{r(h)} \sum_{z\in Z_1} e\left(\frac{k_1}{n_1}hz\right)\Big| < \phi(n_1) + \frac{c\log\log(n_1+1)}{\phi(n_2)}.$$

Now we combine the information in (33), (34), (35), and (36) to obtain bounds for the product

$$\prod_{j=1}^{s}\Big|\sum_{h\in C(N)} \frac{1}{r(h)} \prod_{i=1}^{\min(j,2)} \Big(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\Big)\Big|,$$

which we abbreviate by $\prod$. If $k_1 = n_1$ and $k_2 = n_2$, then $Q_1 = Q_2 = 1$, and so it follows from (33) and (35) that

$$(37)\qquad \prod < c_s\phi(n_1)^s\phi(n_2)^{s-1}(\log N)^s,$$

where here and in the sequel, c_s denotes a positive constant which depends only on s and which may have different values in different occurrences. If $k_1 = n_1$ and $1 \le k_2 < n_2$, then $Q_1 = 1$ and $Q_2 > 1$, hence it follows from (34) and (35) that

$$(38)\qquad \begin{aligned}\prod &< c(\phi(n_1)\phi(n_2) + c\log\log(n_1+1))^{s-1}\phi(n_1)\log N\\ &= c\phi(n_1)^s\phi(n_2)^{s-1}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)^{s-1}\log N < c_s\phi(n_1)^s\phi(n_2)^{s-1}\log N,\end{aligned}$$

where we used (32) in the last step. If $1 \le k_1 < n_1$ and $Q_1 = Q_2$, then (33) and (36) yield

$$(39)\qquad \begin{aligned}\prod &< \left(\phi(n_1) + \frac{c\log\log(n_1+1)}{\phi(n_2)}\right)\left(\phi(n_1)\phi(n_2) + \frac{c\phi(n_1)\phi(n_2)}{\phi(Q_1)}\log N\right)^{s-1}\\ &= \phi(n_1)^s\phi(n_2)^{s-1}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)\left(1+\frac{c\log N}{\phi(Q_1)}\right)^{s-1}.\end{aligned}$$

If $1 \le k_1 < n_1$ and $Q_1 \ne Q_2$, then (34) and (36) yield

$$(40)\qquad \begin{aligned}\prod &< \left(\phi(n_1) + \frac{c\log\log(n_1+1)}{\phi(n_2)}\right)(\phi(n_1)\phi(n_2) + c\log\log(n_1+1))^{s-1}\\ &= \phi(n_1)^s\phi(n_2)^{s-1}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)^s\\ &< \phi(n_1)^s\phi(n_2)^{s-1}\left(1+\frac{c_s\log\log(n_1+1)}{\phi(n_1)}\right) < \phi(n_1)^s\phi(n_2)^{s-1}\left(1+\frac{c_s\log N}{n_1}\right),\end{aligned}$$

where we also used (32). From (27), (28), (37), (38), (39), and (40) and the fact that $\mathrm{card}(\mathcal{L}) = \phi(n_1)^s\phi(n_2)^{s-1}$ we obtain

$$\begin{aligned}M(\mathcal{L})+1 &\le \frac{1}{\mathrm{card}(\mathcal{L})N}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\prod_{j=1}^{s}\Big|\sum_{h\in C(N)}\frac{1}{r(h)}\prod_{i=1}^{\min(j,2)}\Big(\sum_{z\in Z_i} e\left(\frac{k_i}{n_i}hz\right)\Big)\Big|\\ &< c_s\frac{(\log N)^s}{N} + c_s(n_2-1)\frac{\log N}{N} + \frac{1}{N}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)\cdot\\ &\quad\cdot\sum_{k_1=1}^{n_1-1}\sum_{\substack{k_2=1\\Q_1=Q_2}}^{n_2}\left(1+\frac{c\log N}{\phi(Q_1)}\right)^{s-1}\\ &\quad+\frac{1}{N}\left(1+\frac{c_s\log N}{n_1}\right)\sum_{k_1=1}^{n_1-1}\sum_{\substack{k_2=1\\Q_1\ne Q_2}}^{n_2} 1\\ &< c_s\frac{(\log N)^s}{N} + c_s\frac{\log N}{n_1} + \frac{1}{N}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)\sum_{k_1=1}^{n_1}\sum_{\substack{k_2=1\\Q_1=Q_2}}^{n_2}\left(1+\frac{c\log N}{\phi(Q_1)}\right)^{s-1}\\ &\quad+\frac{1}{N}\left(1+\frac{c_s\log N}{n_1}\right)\sum_{k_1=1}^{n_1}\sum_{\substack{k_2=1\\Q_1\ne Q_2}}^{n_2} 1.\end{aligned}$$

Consider the condition $Q_1 = Q_2$, i.e., $n_1/(n_1,k_1) = n_2/(n_2,k_2)$, with $1 \le k_1 \le n_1$ and $1 \le k_2 \le n_2$. If d is the common value of Q_1 and Q_2, then necessarily $d|n_2$ and $(n_2,k_2) = n_2/d$, and the last condition holds for exactly $\phi(d)$ values of k_2. Furthermore, $(n_1,k_1) = n_1/d$, and this condition holds for exactly $\phi(d)$ values of k_1. Thus, for fixed d we have $Q_1 = Q_2 = d$ for exactly $\phi(d)^2$ choices of k_1, k_2. The total number of solutions k_1, k_2 of $Q_1 = Q_2$ is given by

$$g(n_2) := \sum_{d|n_2}\phi(d)^2.$$

Therefore, we get

$$\begin{aligned}M(\mathcal{L})+1 &< c_s\frac{(\log N)^s}{N} + c_s\frac{\log N}{n_1} + \frac{1}{N}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)\sum_{d|n_2}\phi(d)^2\left(1+\frac{c\log N}{\phi(d)}\right)^{s-1}\\ &\quad+\frac{1}{N}\left(1+\frac{c_s\log N}{n_1}\right)(n_1n_2 - g(n_2)),\end{aligned}$$

and since $n_1n_2 = N$, we obtain
(41)
$$M(\mathcal{L}) < c_s\frac{(\log N)^s}{N} + c_s\frac{\log N}{n_1} - \frac{g(n_2)}{N} + \frac{1}{N}\left(1+\frac{c\log\log(n_1+1)}{\phi(n_1)\phi(n_2)}\right)\sum_{d|n_2}\phi(d)^2\left(1+\frac{c\log N}{\phi(d)}\right)^{s-1}$$

Now

$$\begin{aligned}\sum_{d\mid n_2}\phi(d)^2\left(1+\frac{c\log N}{\phi(d)}\right)^{s-1} &= \sum_{d\mid n_2}\phi(d)^2\sum_{m=0}^{s-1}\binom{s-1}{m}\frac{c^m(\log N)^m}{\phi(d)^m}\\ &\le g(n_2)+c_s(\log N)\sum_{d\mid n_2}\phi(d)+\sum_{m=2}^{s-1}c_s(\log N)^m\sum_{d\mid n_2}1\\ &\le g(n_2)+c_s n_2\log N+c_s\tau(n_2)(\log N)^{s-1},\end{aligned}$$

where $\tau(n)$ is the number of positive divisors of $n \in \mathrm{N}$. Together with (41) we get

$$\begin{aligned}M(\mathcal{L}) <& c_s\frac{(\log N)^s}{N}+c_s\frac{\log N}{n_1}+c_s\tau(n_2)\frac{(\log N)^{s-1}}{N}+\\ &+\frac{c\log\log(n_1+1)}{N\phi(n_1)\phi(n_2)}\left(g(n_2)+c_s n_2\log N+c_s\tau(n_2)(\log N)^{s-1}\right).\end{aligned}$$

Since

$$g(n_2)=\sum_{d\mid n_2}\phi(d)^2\le\phi(n_2)\sum_{d\mid n_2}\phi(d)=\phi(n_2)n_2,$$

it follows that

$$M(\mathcal{L}) < c_s\frac{(\log N)^s}{N}+c_s\frac{\log N}{n_1}+c_s\tau(n_2)\frac{(\log N)^{s-1}}{N}. \tag{42}$$

By [2, Theorem 315] we have $\tau(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$. In particular, $\tau(n_2) \le c_s n_2^{1/(s-1)}$. Suppose that with this value of c_s we had

$$\tau(n_2) > \log N \quad\text{and}\quad \tau(n_2) > c_s^{s-1}\frac{n_2}{(\log N)^{s-2}}.$$

Then,

$$\tau(n_2)=\tau(n_2)^{(s-2)/(s-1)}\tau(n_2)^{1/(s-1)} > (\log N)^{(s-2)/(s-1)}c_s\frac{n_2^{1/(s-1)}}{(\log N)^{(s-2)/(s-1)}}=c_s n_2^{1/(s-1)},$$

a contradiction. Thus we have either $\tau(n_2) \le \log N$ or $\tau(n_2) \le c_s^{s-1}n_2/(\log N)^{s-2}$. This means that the last term on the right-hand side of (42) can be incorporated into the other two terms on the right-hand side of (42). Hence,

$$M(\mathcal{L}) < c_s\frac{(\log N)^s}{N}+c_s\frac{\log N}{n_1},$$

which completes the proof of Theorem 2.