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ITERATIVE AND PETROV-GALERKIN METHODS FOR SOLVING A SYSTEM OF ONE-DIMENSIONAL NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. Two sequences of supersolutions and subsolutions are constructed. Their limits are the solutions of a system of one-dimensional nonlinear elliptic equations. A Petrov-Galerkin scheme is proposed. The existence of solutions of the resulting discrete system is proved by an iteration which also provides a numerical method.

1. INTRODUCTION

In studying some problems arising in electromagnetism, biology, and some other topics, we have to consider systems of nonlinear elliptic equations and their numerical solutions. The properties of such systems are very different from those of a single equation (see, e.g., Aronson and Weinberger [1], Fife and Tang [4, 5], Grindrod and Sleeman [6], and Guo Ben-yu and Mitchell [7]). Recently, Guo Ben-yu and Miller [8] proposed an iterative method and a Petrov-Galerkin scheme for a single nonlinear elliptic equation. This paper is devoted to generalizing these two methods to a system of nonlinear elliptic equations.

It is not difficult to prove the existence of solutions of such systems following the work of [6]. But we prefer to develop a new constructive proof in §2, which also provides an iterative method. The main idea is to construct sequences of supersolutions and subsolutions, the limits of which are the exact solutions. For each step of the iteration we only have to solve a system of linear elliptic equations by a finite difference scheme or finite element method. If we choose the former, then the whole iteration is quite close to a finite difference method for the original problem in conjunction with a Newton procedure. But the convergence of Newton's method depends on the error between the exact solution and initial value, which is very difficult to estimate. Conversely, it is easier to choose the initial values in our iteration method. Furthermore, the monotonicity of the sequences gives upper bounds and lower bounds of the exact solutions.

In §3, we consider a Petrov-Galerkin method in which test functions are different from the trial functions. Thus, we derive a scheme which is as simple as a finite difference scheme and as accurate as the finite element method. In particular, this scheme is of positive type and thus possesses properties similar

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to those of the original problem. Hence, it is easy to deal with the existence of solutions of the resulting discrete system by an iteration which provides a numerical method for solving such a system. We also estimate the error between the exact solution and the approximate one, using local Green's functions. Finally, we consider further approximations in §4. This method can be generalized to problems with discontinuous coefficients.

2. Iterative method

Let $I = \{x | 0 < x < 1\}$, $\bar{I}$ be the closure of I, and $u = (u_1, u_2, \dots, u_m)^{\mathrm{T}}$ be a vector function of x. The given function

$$f(x, u) \in [C^1(I \times \mathbf{R}^m) \cap C^0(\bar{I} \times \mathbf{R}^m)]^m$$

has components $f_i(x, u)$. Furthermore, let

$$u_i'(x) = \frac{\partial u_i}{\partial x}(x) \quad \text{and} \quad l = \operatorname{diag}(l_1, l_2, \dots, l_m)$$

with

$$l_i u_i(x) = -(a_i(x) u_i'(x))', \qquad 1 \le i \le m,$$

where $a_j(x) \in C^1(\bar{I})$. Assume that there exist positive constants α_0, α_1, and a nonnegative constant α_2 such that

$$\alpha_0 \le a_j(x) \le \alpha_1, \qquad \left|\frac{\partial a_i}{\partial x}(x)\right| \le \alpha_2 \quad \text{for } x \in I, \ 1 \le i \le m.$$

Let $F_{i,j}(x, u) = \frac{\partial f_i}{\partial u_j}(x, u)$ and define

$$Lu(x) = lu(x) + f(x, u(x)).$$

We consider the following problem:

$$\begin{cases} Lu(x) = 0, & x \in I, \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

The solution of such a system is a vector function $u(x) \in [C^2(I) \cap C^1(\bar{I})]^m$ satisfying (2.1). If $u_i(x) \le v_i(x)$ for all $x \in \bar{I}$ and $1 \le i \le m$, we say that $u \le v$. If $u_* \le u \le u^*$, then we say that $u \in \mathbf{K}(u_*, u^*)$. We begin with the maximum principles.

Lemma 2.1. *If* $u \in [C^2(I) \cap C^1(\bar{I})]^m$ *and*

$$\begin{cases} lu(x) \ge 0, & x \in I, \\ u(0) \ge 0, & u(1) \ge 0, \end{cases}$$

then $u(x) \ge 0$ *for* $x \in \bar{I}$. *Similarly, if*

$$\begin{cases} lu(x) \le 0, & x \in I, \\ u(0) \le 0, & u(1) \le 0, \end{cases}$$

then $u(x) \le 0$ *for* $x \in \bar{I}$.

We now introduce the concept of supersolution and subsolution.

Definition 2.1. A vector function $\overline{u} \in [C^2(I) \cap C^1(\overline{I})]^m$ is a *supersolution of* (2.1) if

$$\begin{cases} L\overline{u}(x) \geq 0, & x \in I, \\ \overline{u}(0) \geq 0, & \overline{u}(1) \geq 0. \end{cases}$$

Similarly, $\underline{u} \in [C^2(I) \cap C^1(\overline{I})]^m$ is a *subsolution of* (2.1) if

$$\begin{cases} L\underline{u}(x) \leq 0, & x \in I, \\ \underline{u}(0) \leq 0, & \underline{u}(1) \leq 0. \end{cases}$$

There is no definitive result for the existence of supersolutions and subsolutions. But if $f(x, u^*) \geq 0$ and $f(x, u_*) \leq 0$ for some nonnegative constant vector u^* and nonpositive constant vector u_*, then $\overline{u} \equiv u^*$ and $\underline{u} \equiv u_*$ are supersolution and subsolution of (2.1), respectively. We now turn to the existence of solutions of (2.1).

Theorem 2.1. *Assume that* (2.1) *has a supersolution* $\overline{u}$ *and a subsolution* $\underline{u}$ *such that*

(1) $\underline{u}(x) \leq \overline{u}(x)$ *for* $x \in \overline{I}$;
(2) $|F_{i,i}(x, \eta)| \leq M$ *for* $x \in I$ *and* $\eta \in \mathbf{K}(\underline{u}, \overline{u})$, $1 \leq i \leq m$;
(3) $F_{i,j}(x, \eta) \leq 0$ *for* $x \in I$, $\eta \in \mathbf{K}(\underline{u}, \overline{u})$, *and* $i \neq j$, $1 \leq i, j \leq m$.

Then (2.1) *has a solution in* $\mathbf{K}(\underline{u}, \overline{u})$ *which is the limit of a nonincreasing sequence of supersolutions. Problem* (2.1) *also has a solution in* $\mathbf{K}(\underline{u}, \overline{u})$ *which is the limit of a nondecreasing sequence of subsolutions.*

Proof. We first let $\overline{w}^{(0)} = \overline{u}$ and define a sequence as follows:

$$\begin{cases} (l + M)\overline{w}^{k+1}(x) - M\overline{w}^k(x) + f(x, \overline{w}^k(x)) = 0, & x \in I, \\ \overline{w}^{k+1}(0) = \overline{w}^{k+1}(1) = 0. \end{cases} \tag{2.2}$$

We use induction. Suppose that $\overline{w}^k \in \mathbf{K}(\underline{u}, \overline{u})$ is a supersolution. Clearly, $\overline{w}^{k+1} \in [C^2(I) \cap C^1(\overline{I})]^m$. Let $z^{k+1} = \overline{w}^{k+1} - \overline{w}^k$; then we have from (2.2) that

$$\begin{cases} (l + M)z^{k+1}(x) = -l\overline{w}^k(x) - f(x, \overline{w}^k(x)) = -L\overline{w}^k(x) \leq 0, & x \in I, \\ z^{k+1}(0), \ z^{k+1}(1) \leq 0. \end{cases}$$

Since the maximum principle is also valid for the operator $l + M$, we have $z^{k+1}(x) \leq 0$ and $\overline{w}^{k+1} \leq \overline{w}^k \leq \overline{u}$. Now let

$$F(x, \eta) = \begin{pmatrix} F_{1,1}(x, \eta) & \cdots & F_{1,m}(x, \eta) \\ \vdots & & \vdots \\ F_{m,1}(x, \eta) & \cdots & F_{m,m}(x, \eta) \end{pmatrix}.$$

Then

$$\begin{aligned} &(l + M)(\overline{w}^{k+1}(x) - \underline{u}(x)) \\ &\quad = M(\overline{w}^k(x) - \underline{u}(x)) - L\underline{u}(x) + f(x, \underline{u}(x)) - f(x, \overline{w}^k(x)) \\ &\quad \geq M(\overline{w}^k(x) - \underline{u}(x)) - F(x, \theta^k(x))(\overline{w}^k(x) - \underline{u}(x)), \end{aligned}$$

where $\theta^k \in \mathbf{K}(\underline{u}, \overline{w}^k) \subset \mathbf{K}(\underline{u}, \overline{u})$ and thus

$$|F_{i,i}(x, \theta^k(x))| \leq M, \qquad F_{i,j}(x, \theta^k(x)) \leq 0 \quad \text{for } i \neq j.$$

Therefore,

$$\begin{cases} (l+M)(\overline{w}^{k+1}(x)-\underline{u}(x)) \geq 0\,, & x \in I\,, \\ \overline{w}^{k+1}(0) \geq \underline{u}(0)\,, \quad \overline{w}^{k+1}(1) \geq \underline{u}(1). \end{cases}$$

By the maximum principle, we find that $\underline{u} \leq \overline{w}^{k+1}$, and so $\overline{w}^{k+1} \in \mathbf{K}(\underline{u}, \overline{u})$. Moreover,

$$\begin{aligned} L\overline{w}^{k+1}(x) &= l\overline{w}^{k+1}(x) + f(x, \overline{w}^{k+1}(x)) \\ &= -Mz^{k+1} + f(x, \overline{w}^{k+1}(x)) - f(x, \overline{w}^{k}(x)) \\ &= -Mz^{k+1}(x) + F(x, \theta^{k+1}(x))z^{k+1}(x) \geq 0\,, \end{aligned}$$

where $\theta^{k+1} \in \mathbf{K}(\overline{w}^{k+1}, \overline{w}^{k}) \subset \mathbf{K}(\underline{u}, \overline{u})$, and thus $\overline{w}^{k+1}$ is also a supersolution of (2.1). The above statements ensure that there is a function $\overline{w} \in \mathbf{K}(\underline{u}, \overline{u})$ such that

$$\overline{w}(x) = \lim_{k\to\infty} \overline{w}^{k}(x)\,, \qquad x \in \overline{I}.$$

In order to show that $\overline{w}(x)$ is a solution of (2.1), we introduce a Green's function as follows:

$$G(x, s) = \operatorname{diag}(G_1(x, s),\ G_2(x, s), \dots, G_m(x, s))\,,$$

where

$$\begin{cases} l_iG_i(x, s) = \delta(x, s)\,, & x \in I\,, s \in \overline{I}\,,\ 1 \leq i \leq m\,, \\ G_i(0, s) = G_i(1, s) = 0\,, & s \in \overline{I}\,,\ 1 \leq i \leq m. \end{cases}$$

It can be verified that

$$G_i(x, s) = \begin{cases} \frac{1}{A_i} g_i^{(1)}(s) g_i^{(2)}(x) & \text{if } x \leq s\,, \\ \frac{1}{A_i} g_i^{(1)}(x) g_i^{(2)}(s)\,, & \text{if } s < x\,, \end{cases}$$

where

$$A_i = \left(\int_0^1 \frac{dt}{a_i(t)}\right)^{-1}, \quad g_i^{(1)}(x) = \int_x^1 \frac{dt}{a_i(t)}, \quad g_i^{(2)}(x) = \int_0^x \frac{dt}{a_i(t)}.$$

Then (2.1) is equivalent to

$$u(x) = -\int_0^1 G(x, s) f(s, u(s))\, ds\,, \tag{2.3}$$

while (2.2) is equivalent to

$$\overline{w}^{k+1}(x) = -\int_0^1 G(x, s)[M\overline{w}^{k+1}(s) - M\overline{w}^{k}(s) + f(s, \overline{w}^{k}(s))]\, ds. \tag{2.4}$$

It is easy to show that $\overline{w}^{k}$ converges to $\overline{w}$ uniformly for $x \in \overline{I}$. Letting $k \to \infty$ in (2.4), we see that $\overline{w}$ satisfies (2.3). We also have from (2.3) that $\overline{w} \in [C^2(I) \cap C^1(\overline{I})]^m$ and $\overline{w}(0) = \overline{w}(1) = 0$. Hence $\overline{w}$ is a solution of (2.1).

We next let $\underline{w}^0 = \underline{u}$ and define a sequence as follows:

$$\begin{cases} (l+M)\underline{w}^{k+1}(x) - M\underline{w}^{k}(x) + f(x, \underline{w}^{k}(x)) = 0\,, & x \in I\,, \\ \underline{w}^{k+1}(0) = \underline{w}^{k+1}(1) = 0. \end{cases} \tag{2.5}$$

By an argument similar to that in the previous paragraph, the second assertion is proved. □

The proof of Theorem 2.1 also provides us with an iteration to solve (2.1). For each k, we only have to solve a linear problem by known numerical methods. Fife and Tang [4, 5] also considered (2.1), but with a nonconstructive proof. On the other hand, many researchers constructed the iteration as follows (see, e.g., [6]):

$$\begin{cases} lw^{k+1}(x) + \tilde{f}(x, w^k(x), w^{k+1}(x)) = 0, & x \in I, \\ w^{k+1}(0) = w^{k+1}(1) = 0, \end{cases}$$

with

$$\begin{aligned} &\tilde{f}_i(x, w^k(x), w^{k+1}(x)) \\ &\quad = f_i(x, w_1^k(x), \dots, w_{i-1}^k(x), w_i^{k+1}(x), w_{i+1}^k(x), \dots, w_m^k(x)). \end{aligned}$$

In this case we need two iterations to solve (2.1), which is not so convenient for computation.

If for each step k, we use a finite difference scheme to solve the linear problems (2.2) and (2.5), then the whole iterative method is very close to the same difference scheme approximating (2.1) directly in conjunction with a Newton procedure. But the convergence of such an approach depends on the choice of the initial values. Generally, (2.1) has several solutions. Therefore, the corresponding sequences tend to different exact solutions for different initial values. But it is usually not possible to estimate the errors between the exact solutions and the initial values. On the other hand, it is easier to construct supersolutions and subsolutions. The sequences given by (2.2) and (2.5) tend to fixed solutions, respectively. Furthermore, these sequences are monotonic in k and so provide bounds for the exact solutions and for the error of the approximate ones.

We now consider the uniqueness of the solution.

Lemma 2.2 (Poincaré inequality). *If* $z(0) = 0$ *or* $z(1) = 0$, *then* $\|z\|_{L^2(I)}^2 \le |z|_{H^1(I)}^2$.

Theorem 2.2. *If* $M_1 m < \alpha_0$ *and* $|F_{i,j}(x, \eta)| \le M_1$ *for all* $x \in I$ *and* $\eta \in \mathbf{K}(u_*, u^*)$, *then* (2.1) *has only one solution in* $\mathbf{K}(u_*, u^*)$.

Proof. Let u and $\tilde{u}$ be solutions of (2.1). Let $z = u - \tilde{u}$. Then

$$\begin{cases} lz(x) + F(x, \theta(x))z(x) = 0, & x \in I, \\ z(0) = z(1) = 0, \end{cases} \tag{2.6}$$

where θ lies between u and $\tilde{u}$, and so $\theta \in \mathbf{K}(u_*, u^*)$. Multiplying the above equation by z and integrating by parts, we get

$$\int_0^1 a_i(x)(z_i'(x))^2\,dx + \sum_{j=1}^m \int_0^1 F_{i,j}(x, \theta(x))z_i(x)z_j(x)\,dx = 0, \qquad 1 \le i \le m. \tag{2.7}$$

By Lemma 2.2,

$$\begin{aligned} &\left|\int_0^1 F_{i,j}(x, \theta(x))z_i(x)z_j(x)\,dx\right| \\ &\qquad \le \frac{M_1}{2}(\|z_i\|_{L^2(I)}^2 + \|z_j\|_{L^2(I)}^2) \le \frac{M_1}{2}(|z_i|_{H^1(I)}^2 + |z_j|_{H^1(I)}^2). \end{aligned}$$

By substituting the above estimates into (2.7), we get

$$(\alpha_0 - M_1 m)|z|^2_{H^1(I)} < 0,$$

from which, and the boundary conditions, the conclusion follows. □

We now estimate the error between $\overline{w}^k$ and $\overline{w}$.

Theorem 2.3. *Assume that the conditions of Theorem 2.1 hold and that $|F_{i,j}(x, \eta)| \le M_1$ for all $x \in I$ and $\eta \in \mathbf{K}(\underline{u}, \overline{u})$. Then*

$$\|\overline{w}^k - \overline{w}\|_{L^\infty(I)} \le |\overline{w}^k - \overline{w}|_{H^1(I)} \le \gamma^{k/2}|\overline{w}^0 - \overline{w}|_{H^1(I)},$$

provided that $M_1 m < 2\alpha_0$ and

$$\gamma = \frac{M_1(2m+1)}{4\alpha_0 - 2M_1 m} < 1.$$

Proof. Let $z^k = \overline{w}^k - \overline{w}$. Then

$$\begin{aligned}(l+M)z^{k+1}(x) &= Mz^k(x) + f(x, \overline{w}(x)) - f(x, \overline{w}^k(x)) \\ &= Mz^k(x) - F(x, \theta^k(x))z^k(x),\end{aligned}$$

where θ^k lies between $\overline{w}^k$ and $\overline{w}$, and thus $\theta^k \in \mathbf{K}(\underline{u}, \overline{u})$. By an argument similar to that in the proof of Theorem 2.2, we obtain

$$\begin{aligned}\int_0^1 a_i(x)((z_i^{k+1}(x))')^2\,dx + M\|z_i^{k+1}\|^2_{L^2(I)} - M\int_0^1 z_i^k(x)z_i^{k+1}(x)\,dx \\ + \sum_{j=1}^m \int_0^1 F_{i,j}(x, \theta^k(x))z_i^{k+1}(x)z_j^k(x)\,dx = 0.\end{aligned}$$

Since

$$M\int_0^1 z_i^k(x)z_i^{k+1}(x)\,dx \le M_1\|z_i^{k+1}\|^2_{L^2(I)} + \frac{M_1}{4}|z_i^k|^2_{H^1(I)}$$

and

$$\left|\int_0^1 F_{i,j}(x, \theta^k(x))z_i^{k+1}(x)z_j^k(x)\,dx\right| \le \frac{M_1}{2}(|z_i^{k+1}|^2_{H^1(I)} + |z_j^k|^2_{H^1(I)}),$$

we get

$$(2\alpha_0 - M_1 m)|z^{k+1}|^2_{H^1(I)} < M_1(m + \tfrac{1}{2})|z^k|^2_{H^1(I)},$$

from which the conclusion follows. □

We can similarly estimate the error between $\underline{w}^k$ and $\underline{w}$.

Remark 2.1. If $m = 1$ and $\frac{\partial f}{\partial \eta}(x, \eta) \ge 0$ for all $x \in I$ and $\eta \in \mathbf{R}$, then (2.1) has certainly supersolutions and subsolutions, and any supersolution is not less than any subsolution. So (2.1) has a unique solution in $\mathbf{R}$. We can also estimate the error by the maximum principle (see Guo Ben-Yu and Miller [8]).

3. Petrov-Galerkin method

Another way to solve (2.1) numerically is to discretize (2.1) directly and obtain a discrete nonlinear system. Then we use iteration to solve this discrete system. In this section, we consider a Petrov-Galerkin method, which has been widely used for single equations (see, e.g., Christie et al. [3] and Kuo Pen-yu and Mitchell [9]).

We begin with a weak formulation of (2.1). We seek a solution $u \in [H_0^1(I)]^m$ such that

$$(a_i u_i', v_i') + (f_i, v_i) = 0 \quad \forall v \in [H_0^1(I)]^m, \ 1 \le i \le m, \tag{3.1}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(I)$ and

$$(f_i, v_i) \equiv (f_i(\cdot, u(\cdot)), v_i(\cdot)), \qquad 1 \le i \le m.$$

To discretize (3.1), we introduce a set of mesh points $\{x_p\}_0^N$ such that

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1.$$

For each p, let $I_p = (x_{p-1}, x_p)$, $h_p = x_p - x_{p-1}$, and $h = \max_{1\le p\le N} h_p$. Suppose that there exists a positive constant β such that

$$\frac{\max_{1\le p\le N} h_p}{\min_{1\le p\le N} h_p} \le \beta. \tag{3.2}$$

Let $S_h = \prod_{i=1}^m S_{h,i}$ and $T_h = \prod_{i=1}^m T_{h,i}$ be finite-dimensional spaces of trial and test functions respectively. Then the approximate problem is to find $u_h \in S_h$ such that

$$(a_i u_{h,i}', v_{h,i}') + \int_0^1 f_i(x, u_h(x)) v_{h,i}(x)\,dx = 0 \quad \forall v_h \in T_h, \ 1 \le i \le m. \tag{3.3}$$

Let $\{\varphi_p\}_0^N$ and $\{\psi_p\}_0^N$ be bases for the spaces S_h and T_h, respectively, where

$$\varphi_p = (\varphi_{p,1}, \dots, \varphi_{p,m})^{\mathrm{T}}, \qquad \psi_p = (\psi_{p,1}, \dots, \psi_{p,m})^{\mathrm{T}}.$$

We seek a solution $u_h \in S_h$ of the form

$$u_h(x) = \sum_{p=1}^{N-1} u_h(x_p)\varphi_p(x) \quad \text{for all } x \in \overline{I}.$$

Then (3.3) becomes an integro-difference system of the form

$$\begin{cases} \sum_{p=0}^N (a_i \varphi_{p,i}', \psi_{q,i}') u_{h,i}(x_p) \\ \quad + \int_0^1 f_i(x, u_h(x)) \psi_{q,i}(x)\,dx = 0, \qquad 1 \le i \le m, \ 1 \le q \le N-1, \\ u_h(0) = u_h(1) = 0. \end{cases} \tag{3.4}$$

Now we assume that the following conditions are fulfilled:

(H_1) $\operatorname{supp} \varphi_p \cup \operatorname{supp} \psi_p \subset \overline{I_p \cup I_{p+1}}$ for $1 \le p \le N$,
(H_2) $\varphi_{p,i}(x_q) = \psi_{p,i}(x_q) = \delta_{p,q}$ for $1 \le i \le m$, $0 \le p, q \le N$,
(H_3) $\varphi_{p,i}(x) \ge 0$ and $\sum_{p=1}^N \varphi_{p,i}(x) \equiv 1$ for $x \in \overline{I}$,
(H_4) $\int_{x_{p-1}}^{x_p} \varphi_{q,i}(x)\,dx \le 2$ for $q = p-1, p$,
(H_5) $l_i \psi_{p,i}(x) = 0$ for all $x \notin \{x_p\}_0^N$, $1 \le i \le m$, $1 \le p \le N-1$.

Then, by (H_1) the coefficients for $u_{h,i}(x_p)$ in the qth equation of (3.4) equal zero, unless $|p-q| \le 1$. Furthermore, (H_2) and (H_5) imply that the coefficient for $u_{h,i}(x_{p-1})$, after integrating by parts, is

$$(a_i \varphi_{p-1,i}', \psi_{p,i}') = \int_{x_{p-1}}^{x_p} a_i(x) \varphi_{p-1,i}'(x) \psi_{p,i}'(x)\,dx = -a_i(x_{p-1})\psi_{p,i}'(x_{p-1}+0).$$

The coefficients of $u_{h,i}(x_{p+1})$ and $u_{h,i}(x_p)$ are respectively

$$(a_i\varphi'_{p+1,i}, \psi'_{p,i}) = \int_{x_p}^{x_{p+1}} a_i(x)\varphi'_{p+1,i}(x)\psi'_{p,i}(x)\,dx = a_i(x_{p+1})\psi'_{p,i}(x_{p+1}-0),$$

$$\begin{aligned}(a_i\varphi'_{p,i}, \psi'_{p,i}) &= \int_{x_{p-1}}^{x_{p+1}} a_i(x)\varphi'_{p,i}(x)\psi'_{p,i}(x)\,dx \\ &= a_i(x_p)\psi'_{p,i}(x_p-0) - a_i(x_p)\psi'_{p,i}(x_p+0).\end{aligned}$$

Following Boglaev and Miller [2], we have from (H_2) and (H_5) that

$$\psi_{i,p}(x) = \begin{cases} A_{i,p}\displaystyle\int_{x_{p-1}}^{x} \frac{dt}{a_i(t)}, & x \in I_p, \\ A_{i,p+1}\displaystyle\int_{x}^{x_{p+1}} \frac{dt}{a_i(t)}, & x \in I_{p+1}, \\ 0, & \text{otherwise}, \end{cases}$$

with

$$A_{i,p} = \left(\int_{x_{p-1}}^{x_p} \frac{dt}{a_i(t)}\right)^{-1}.$$

It is easy to see that

$$\begin{aligned} a_i(x_{p-1})\psi'_{p,i}(x_{p-1}+0) &= a_i(x_p)\psi'_{p,i}(x_p-0) = A_{i,p}, \\ a_i(x_{p+1})\psi'_{p,i}(x_{p+1}-0) &= a_i(x_p)\psi'_{p,i}(x_p+0) = -A_{i,p+1}. \end{aligned}$$

We now define

$$l_h = \operatorname{diag}(l_{h,1}, l_{h,2}, \dots, l_{h,m}),$$

$$l_{h,i}u_{h,i}(x_p) = -A_{i,p}u_{h,i}(x_{p-1}) + (A_{i,p} + A_{i,p+1})u_{h,i}(x_p) - A_{i,p+1}u_{h,i}(x_{p+1}),$$

and

$$L_h u_h(x_p) = l_h u_h(x_p) + J_{h,p}(u_h),$$

where

$$\begin{aligned} J_{h,p}(u_h) &= (J_{h,p,1}(u_h), \dots, J_{h,p,m}(u_h))^{\mathrm{T}}, \\ J_{h,p,i}(u_h) &= \int_0^1 f_i(x, u_h(x))\psi_{p,i}(x)\,dx \\ &= A_{i,p+1}\int_{x_p}^{x_{p+1}} \left[f_i(x, u_h(x_p)\varphi_p(x) + u_h(x_{p+1})\varphi_{p+1}(x))\int_x^{x_{p+1}} \frac{dt}{a_i(t)}\right] dx \\ &\quad + A_{i,p}\int_{x_{p-1}}^{x_p} \left[f_i(x, u_h(x_{p-1})\varphi_{p-1}(x) + u_h(x_p)\varphi_p(x))\int_{x_{p-1}}^{x} \frac{dt}{a_i(t)}\right] dx. \end{aligned}$$

Then (3.4) becomes

$$\text{(3.5)} \qquad \begin{cases} L_h u_h(x_p) = 0, & 1 \le p \le N-1, \\ u_h(0) = u_h(1) = 0. \end{cases}$$

It is easy to see that for each i, $l_{h,i}$ is a difference operator of positive type. Thus the following maximum principle holds.

Lemma 3.1. *If*

$$\begin{cases} l_h u_h(x_p) \geq 0, & 1 \leq p \leq N-1, \\ u_h(0) \geq 0, \quad u_h(1) \geq 0, \end{cases}$$

then $u_h(x_p) \geq 0$ *for all* x_p. *Similarly, if*

$$\begin{cases} l_h u_h(x_p) \leq 0, & 1 \leq p \leq N-1, \\ u_h(0) \leq 0, \quad u_h(1) \leq 0, \end{cases}$$

then $u_h(x_p) \leq 0$ *for all* x_p.

We now introduce the concept of supersolution and subsolution.

Definition 3.1. $\overline{u}_h$ is a *supersolution of* (3.5), if

$$\begin{cases} L_h \overline{u}_h(x_p) \geq 0, & 1 \leq p \leq N-1, \\ \overline{u}_h(0) \geq 0, \quad \overline{u}_h(1) \geq 0. \end{cases}$$

Similarly, $\underline{u}_h$ is a *subsolution of* (3.5), if

$$\begin{cases} L_h \underline{u}_h(x_p) \leq 0, & 1 \leq p \leq N-1, \\ \underline{u}_h(0) \leq 0, \quad \underline{u}_h(1) \leq 0. \end{cases}$$

If for some nonnegative constant vector u_h^* and nonpositive constant vector $u_{h,*}$ we have $f(x, u_{h,*}) \leq 0$ and $f(x, u_h^*) \geq 0$, then by (H_3), u_h^* and $u_{h,*}$ are supersolution and subsolution of (3.5), respectively. We now turn to the existence of solutions of (3.5).

Theorem 3.1. *Assume that* (3.5) *has a supersolution* $\overline{u}_h$ *and a subsolution* $\underline{u}_h$ *such that*

(1) $\underline{u}_h(x_p) \leq \overline{u}_h(x_p)$ *for* $0 \leq p \leq N$;
(2) $|F_{i,i}(x, \eta_h)| \leq M$ *for* $x \in I$ *and* $\eta_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$, $1 \leq i \leq m$;
(3) $F_{i,j}(x, \eta_h) \leq 0$ *for* $x \in I$, $\eta_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$, *and* $i \neq j$, $1 \leq i, j \leq m$.

Then (3.5) *has a solution in* $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ *which is the limit of a nonincreasing sequence of supersolutions. Also,* (3.5) *has a solution in* $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ *which is the limit of a nondecreasing sequence of subsolutions.*

Proof. We first let $\overline{w}_h^{(0)} = \overline{u}_h$ and define a sequence as follows:

$$\begin{cases} (l_h + E_h)\overline{w}_h^{k+1}(x_p) - E_h \overline{w}_h^k(x_p) + J_{h,p}(\overline{w}_h^k) = 0, & 1 \leq p \leq N-1, \\ \overline{w}_h^{k+1}(0) = \overline{w}_h^{k+1}(1) = 0, \end{cases} \tag{3.6}$$

where

$$E_h y(x_p) = Mh(y(x_{p-1}) + 2y(x_p) + y(x_{p+1})).$$

We use induction. Suppose that $\overline{w}_h^k \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$ is a supersolution. Let $z_h^{k+1} = \overline{w}_h^{k+1} - \overline{w}_h^k$. Then

$$\begin{cases} (l_h + E_h) z_h^{k+1}(x_p) = -l_h \overline{w}_h^k(x_p) - J_{h,p}(\overline{w}_h^k) \leq 0, & 1 \leq p \leq N-1, \\ z_h^{k+1}(0), \ z_h^{k+1}(1) \leq 0. \end{cases}$$

The maximum principle gives $z_h^{k+1}(x_p) \leq 0$, and so $\overline{w}_h^{k+1} \leq \overline{w}_h^k \leq \overline{u}_h$. We also have

$$\begin{aligned} (l_h + E_h)(\overline{w}_h^{k+1}(x_p) - \underline{u}_h(x_p)) &= E_h(\overline{w}_h^k(x_p) - \underline{u}_h(x_p)) - L_h \underline{u}_h(x_p) \\ &\quad + J_{h,p}(\underline{u}_h) - J_{h,p}(\overline{w}_h^k). \end{aligned} \tag{3.7}$$

We now define

$$
\begin{aligned}
Q_{h,p}(y,\tilde{y}) &= (Q_{h,p,1}(y,\tilde{y}),\ldots,Q_{h,p,m}(y,\tilde{y}))^{\mathrm{T}},\\
Q_{h,p,i}(y,\tilde{y}) &= J_{h,p,i}(y)-J_{h,p,i}(\tilde{y})\\
&= A_{i,p+1}\int_{x_p}^{x_{p+1}}\left\{\sum_{j=1}^{m}F_{i,j}(x,\theta_p)[(y_j(x_p)-\tilde{y}_j(x_p))\varphi_{p,j}(x)\right.\\
&\qquad\left.+(y_j(x_{p+1})-\tilde{y}_j(x_{p+1}))\varphi_{p+1,j}(x)]\int_x^{x_{p+1}}\frac{dt}{a_i(t)}\right\}dx\\
&\quad+A_{i,p}\int_{x_{p-1}}^{x_p}\left\{\sum_{j=1}^{m}F_{i,j}(x,\theta_{p-1})[(y_j(x_{p-1})-\tilde{y}_j(x_{p-1}))\varphi_{p-1,j}(x)\right.\\
&\qquad\left.+(y_j(x_p)-\tilde{y}_j(x_p))\varphi_{p,j}(x)]\int_{x_{p-1}}^{x}\frac{dt}{a_i(t)}\right\}dx,
\end{aligned}
$$

where θ_p lies between $y(x_p)$, $y(x_{p+1})$, $\tilde{y}(x_p)$, and $\tilde{y}(x_{p+1})$. Let

$$
\begin{aligned}
D_{i,j,p-1}(y,\tilde{y}) &= A_{i,p}\int_{x_{p-1}}^{x_p}\left[F_{i,j}(x,\theta_{p-1})\varphi_{p-1,j}(x)\int_{x_{p-1}}^{x}\frac{dt}{a_i(t)}\right]dx,\\
D_{i,j,p}(y,\tilde{y}) &= A_{i,p}\int_{x_{p-1}}^{x_p}\left[F_{i,j}(x,\theta_{p-1})\varphi_{p,j}(x)\int_{x_{p-1}}^{x}\frac{dt}{a_i(t)}\right]dx\\
&\quad+A_{i,p+1}\int_{x_p}^{x_{p+1}}\left[F_{i,j}(x,\theta_p)\varphi_{p,j}(x)\int_x^{x_{p+1}}\frac{dt}{a_i(t)}\right]dx,\\
D_{i,j,p+1}(y,\tilde{y}) &= A_{i,p+1}\int_{x_p}^{x_{p+1}}\left[F_{i,j}(x,\theta_p)\varphi_{p+1,j}(x)\int_x^{x_{p+1}}\frac{dt}{a_i(t)}\right]dx,
\end{aligned}
$$

and

$$
D_q(y,\tilde{y}) = \begin{pmatrix} D_{1,1,q}(y,\tilde{y}) & \cdots & D_{1,m,q}(y,\tilde{y})\\ \vdots & \cdots & \vdots\\ D_{m,1,q}(y,\tilde{y}) & \cdots & D_{m,m,q}(y,\tilde{y})\end{pmatrix}\quad\text{for } q=p-1,p,p+1,
$$

$$
D_q^*(y,\tilde{y}) = \operatorname{diag}(D_{1,1,q}(y,\tilde{y}),\ldots,D_{m,m,q}(y,\tilde{y})).
$$

Then

$$
\begin{aligned}
Q_{h,p}(y,\tilde{y}) &= D_{p-1}(y,\tilde{y})(y(x_{p-1})-\tilde{y}(x_{p-1}))+D_p(y,\tilde{y})(y(x_p)-\tilde{y}(x_p))\\
&\quad+D_{p+1}(y,\tilde{y})(y(x_{p+1})-\tilde{y}(x_{p+1})).
\end{aligned}\tag{3.8}
$$

It is easy to show that

$$
A_{i,p}\geq\frac{\alpha_0}{h_p}.\tag{3.9}
$$

We have from (H_3) that

$$
\left\{\begin{aligned}
|D_{i,i,p-1}| &\leq Mh_p,\\
|D_{i,i,p}| &\leq M(h_p+h_{p+1}),\\
|D_{i,i,p+1}| &\leq Mh_{p+1}.
\end{aligned}\right.\tag{3.10}
$$

Using the above notations and the nonpositivity of $F_{i,j}(x, \theta_p)$ for $i \neq j$, we have from (3.7) that

$$\begin{aligned}
&(l_h + E_h)(\overline{w}_h^{k+1}(x_p) - \underline{u}_h(x_p)) \\
&\quad = E_h(\overline{w}_h^k(x_p) - \underline{u}_h(x_p)) - L_h\underline{u}_h(x_p) - Q_{h,p}(\overline{w}_h^k, \underline{u}_h) \\
&\quad \geq E_h(\overline{w}_h^k(x_p) - \underline{u}_h(x_p)) - \sum_{q=p-1,p,p+1} D_q(\overline{w}_h^k, \underline{u}_h)(\overline{w}_h^k(x_q) - \underline{u}_h(x_q)) \\
&\quad \geq E_h(\overline{w}_h^k(x_p) - \underline{u}_h(x_p)) - \sum_{q=p-1,p,p+1} D_q^*(\overline{w}_h^k, \underline{u}_h)(\overline{w}_h^k(x_q) - \underline{u}_h(x_q)),
\end{aligned}$$

and thus (3.10) leads to

$$(l_h + E_h)(\overline{w}_h^{k+1}(x_p) - \underline{u}_h(x_p)) \geq 0.$$

Therefore, $\underline{u}_h \leq \overline{w}_h^{k+1}$ and $\overline{w}_h^{k+1} \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$. Moreover,

$$\begin{aligned}
L_h\overline{w}_h^{k+1}(x_p) &= l_h\overline{w}_h^{k+1}(x_p) + J_{h,p}(\overline{w}_h^{k+1}) \\
&= -E_h z_h^{k+1}(x_p) + J_{h,p}(\overline{w}_h^{k+1}) - J_{h,p}(\overline{w}_h^k) \\
&= -E_h z_h^{k+1}(x_p) + Q_{h,p}(\overline{w}_h^{k+1}, \overline{w}_h^k) \geq 0.
\end{aligned}$$

Thus, $\overline{w}_h^{k+1}$ is also a supersolution of (3.5). The above argument implies that there exists a function $\overline{w}_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$ such that

$$\overline{w}_h(x_p) = \lim_{k\to\infty} \overline{w}_h^k(x_p) \quad \text{for } 0 \leq p \leq N.$$

Letting $k \to \infty$ in (3.6), we see that $\overline{w}_h$ is a solution of (3.5).

Next, let $\underline{w}_h^{(0)} = \underline{u}_h$ and define a sequence as follows:

$$\begin{cases}
(l_h + E_h)\underline{w}_h^{k+1}(x_p) - E_h\underline{w}_h^k(x_p) + J_{h,p}(\underline{w}_h^k) = 0, & 1 \leq p \leq N-1, \\
\underline{w}_h^{k+1}(0) = \underline{w}_h^{k+1}(1) = 0.
\end{cases}$$

Then the second assertion follows from an argument similar to that above. □

The above statements show that the choice of test functions in this section is also appropriate for rough data. For instance, if $a_i(\tilde{x}+0) \neq a_i(\tilde{x}-0)$, then we take $\tilde{x}$ to be one of the mesh nodes, say $x_p = \tilde{x}$. Then

$$(a_i\varphi'_{p,i}, \psi'_{p,i}) = a_i(\tilde{x}-0)\psi'(\tilde{x}-0) - a_i(\tilde{x}+0)\psi'(\tilde{x}+0).$$

We also avoid integrating the function $a_i(x)Q(x)$, where $Q(x)$ is a polynomial. Besides, such a choice ensures the positivity of the operator $l_{h,i}$, and thus the resulting discrete system keeps properties similar to those of the original problem. These properties play an important role in the proof of the existence of solutions and in error estimations. (For results on a linear problem with rough coefficients, cf. the $\mathscr{L}_2$-method and the Remark on p. 527 of Babuška and Osborn [10].)

We now consider the uniqueness of the solution. Define the following discrete norms:

$$\|z_h\|_\infty = \max_{x\in\bar{I}} |z_h(x)|, \qquad \|z_h\|^2 = \sum_{p=0}^{N} h_p z_h^2(x_p),$$

$$|z_h|_1^2 = \sum_{p=1}^{N} \frac{(z(x_p) - z(x_{p-1}))^2}{h_p}.$$

Lemma 3.2 (see [11]). *If* $z_h(0) = z_h(1) = 0$, *then*

$$\begin{aligned} F(y_h, z_h) := \sum_{p=1}^{N-1} (-y_h(x_p) z_h(x_{p-1}) + y_h(x_p) z_h(x_p) \\ + y_h(x_{p+1}) z_h(x_p) - y_h(x_{p+1}) z_h(x_{p+1})) z_h(x_p) \\ = \sum_{p=1}^{N} y_h(x_p)(z_h(x_p) - z_h(x_{p-1}))^2. \end{aligned}$$

If, in addition, $y_h(x_p) \geq \alpha_0 / h_p$ *for* $1 \leq p \leq N$, *then* $F(y_h, z_h) \geq \alpha_0 |z_h|_1^2$.

The above lemma can be verified directly.

Lemma 3.3. *If* $z_h(0) = 0$ *or* $z_h(1) = 0$, *then* $\|z_h\|^2 \leq |z_h|_1^2$.

Proof. Assume $z_h(0) = 0$. We have

$$z_h(x_p) = \sum_{j=1}^{p} (z_h(x_j) - z_h(x_{j-1})).$$

Thus,

$$z_h^2(x_p) \leq \left(\sum_{j=1}^{p} h_j \right) \left(\sum_{j=1}^{p} \frac{(z_h(x_j) - z_h(x_{j-1}))^2}{h_j} \right) \leq |z_h|_1^2$$

and $\|z_h\|^2 \leq |z_h|_1^2$. □

Theorem 3.2. *If* $4\beta M_1 m < \alpha_0$ *and*

$$|F_{i,j}(x, \eta)| \leq M_1 \quad \text{for all } x \in I \text{ and } \eta \in \mathbf{K}(u_{h,*}, u_h^*),$$

then (3.5) *has only one solution in* $\mathbf{K}(u_{h,*}, u_h^*)$.

Proof. Let u_h and $\tilde{u}_h$ be solutions of (3.5) and $z_h = u_h - \tilde{u}_h$. Then

$$\begin{cases} l_h z_h(x_p) + Q_{h,p}(u_h, \tilde{u}_h) = 0, & 1 \leq p \leq N-1, \\ z_h(0) = z_h(1) = 0. \end{cases}$$

Multiplying the above equation by z_h and summing over all x_p, we obtain from Lemma 3.2, Lemma 3.3, (3.9), and (3.10) that

$$\alpha_0 |z_h|_1^2 \leq -\sum_{p=1}^{N-1} z_h(x_p) Q_{h,p}(u_h, \tilde{u}_h) \leq 4\beta M_1 m \|z_h\|^2 \leq 4\beta M_1 m |z_h|_1^2,$$

from which, and the boundary conditions, the conclusion follows. □

We now estimate the error between $\overline{w}_h^k$ and $\overline{w}_h$.

Theorem 3.3. *Assume that the conditions of Theorem* 3.1 *hold and that* $|F_{i,j}(x, \eta)| \leq M_1$ *for all* $x \in I$ *and* $\eta \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$. *Then*

$$\|\overline{w}_h^k - \overline{w}_h\|_\infty \leq |\overline{w}_h^k - \overline{w}_h|_1 \leq \gamma_1^{k/2} |\overline{w}_h^0 - \overline{w}_h|_1,$$

provided that $2\beta M_1(m+1) < \alpha_0$ *and*

$$\gamma_1 = \frac{2\beta M_1(m+1)}{\alpha_0 - 2\beta M_1(m+1)} < 1.$$

Proof. Let $z_h^k = \overline{w}_h^k - \overline{w}_h$. Then

$$(l_h + E_h)z_h^{k+1}(x_p) = E_h z_h^k(x_p) - Q_{h,p}(\overline{w}_h^k, \overline{w}_h), \qquad 1 \le p \le N-1.$$

Multiplying the above equation by z_h^{k+1} and summing over all x_p, we have

$$\begin{aligned}(\alpha_0 - 2\beta M_1 m)|z_h^{k+1}|_1^2 + \sum_{p=1}^{N-1}(E_h z_h^{k+1}(x_p))z_h^{k+1}(x_p) \\ \le 2\beta M_1 m|z_h^k|_1^2 + \sum_{p=1}^{N-1}(E_h z_h^k(x_p))z_h^{k+1}(x_p).\end{aligned} \tag{3.11}$$

It is easy to show that

$$(E_h z_h^{k+1}(x_p))z_h^{k+1}(x_p) \ge 0,$$

$$\begin{aligned}\sum_{p=1}^{N-1}(E_h z_h^k(x_p))z_h^{k+1}(x_p) &\le 2\beta M_1(\|z_h^k\|^2 + \|z_h^{k+1}\|^2) \\ &\le 2\beta M_1(|z_h^k|_1^2 + |z_h^{k+1}|_1^2)\end{aligned}$$

from which, and (3.11), the conclusion follows. □

We can also estimate the error between $\underline{w}_h^k$ and $\underline{w}_h$ in the same way.

Remark 3.1. If $m = 1$ and $\frac{\partial f}{\partial \eta}(x, \eta) \ge 0$ for all $x \in I$ and $\eta \in \mathbf{R}$, then (3.5) has certainly supersolutions and subsolutions, and any supersolution is not less than any subsolution. So (3.5) has a unique solution in $\mathbf{R}$. We can also estimate the error by the maximum principle (see Guo Ben-yu and Miller [8]).

Finally, we estimate the error between the exact solution u and the approximate solution u_h. To do this, we introduce local Green's functions as follows:

$$G_p(x, s) = \mathrm{diag}(G_{1,p}(x, s), G_{2,p}(x, s), \ldots, G_{m,p}(x, s)),$$

where

$$\begin{cases} l_{h,i}G_{i,p}(x, s) = \delta(x, s), & (x, s) \in I_p \times \overline{I}_p,\ 1 \le p \le N-1,\ 1 \le i \le m, \\ G_{i,p}(x_{p-1}, s) = G_{i,p}(x_p, s) = 0, & s \in \overline{I}_p,\ 1 \le p \le N-1,\ 1 \le i \le m. \end{cases}$$

Clearly,

$$G_{i,p}(x, s) = \begin{cases} \frac{1}{A_{i,p}} g_{i,p}^{(1)}(s) g_{i,p}^{(2)}(x), & x \le s, \\ \frac{1}{A_{i,p}} g_{i,p}^{(1)}(x) g_{i,p}^{(2)}(s), & s < x, \end{cases}$$

where the $A_{i,p}$ are the same as before and

$$g_{i,p}^{(1)}(x) = A_{i,p}\int_x^{x_p} \frac{dt}{a_i(t)}, \qquad g_{i,p}^{(2)}(x) = A_{i,p}\int_{x_{p-1}}^{x} \frac{dt}{a_i(t)}.$$

Thus,

$$u_i(x) = u_i(x_{p-1})g_{i,p}^{(1)}(x) + u_i(x_p)g_{i,p}^{(2)}(x) - \int_{x_{p-1}}^{x_p} G_{i,p}(x, s)f_i(s, u(s))\,ds.$$

By differentiating the above expression for u_i in I_p and $\overline{I}_{p+1}$, and putting $x = x_p$, we get

$$
\begin{aligned}
u_i'(x_p - 0) &= u_i(x_{p-1})g_{i,p}^{(1)'}(x_p) + u_i(x_p)g_{i,p}^{(2)'}(x_p) \\
&\quad - \frac{1}{A_{i,p}} \int_{x_{p-1}}^{x_p} g_{i,p}^{(1)'}(x_p)g_{i,p}^{(2)}(s)f_i(s, u(s))\,ds, \\
u_i'(x_p + 0) &= u_i(x_p)g_{i,p+1}^{(1)'}(x_p) + u_i(x_{p+1})g_{i,p+1}^{(2)'}(x_p) \\
&\quad - \frac{1}{A_{i,p+1}} \int_{x_p}^{x_{p+1}} g_{i,p+1}^{(1)}(s)g_{i,p+1}^{(2)'}(x_p)f_i(s, u(s))\,ds.
\end{aligned}
$$

Since $u \in C^1(I)$, we have $u_i'(x_p - 0) = u_i'(x_p + 0)$ and so

$$
\begin{cases}
l_h u(x_p) + J_p(u) = 0, & 1 \le p \le N-1, \\
u(0) = u(1) = 0,
\end{cases}
$$

where l_h is the same as before, and

$$J_p(u) = (J_{p,1}(u), J_{p,2}(u), \dots, J_{p,m}(u))^{\mathrm{T}},$$

$$
\begin{aligned}
J_{p,i}(u) &= A_{i,p+1} \int_{x_p}^{x_{p+1}} \left(f_i(x, u(x)) \int_x^{x_{p+1}} \frac{dt}{a_i(t)} \right) dx \\
&\quad + A_{i,p} \int_{x_{p-1}}^{x_p} \left(f_i(x, u(x)) \int_{x_{p-1}}^{x} \frac{dt}{a_i(t)} \right) dx.
\end{aligned}
$$

Let $P_h u$ be the piecewise linear interpolant of u corresponding to $\{I_p\}_1^N$ and suppose that $\{\varphi_p\}_0^N$ are the standard piecewise linear basis functions. Then assumptions (H_1)–(H_5) are satisfied and

$$P_h u(x) = \sum_{p=0}^{N} u(x_p)\varphi_p(x).$$

Therefore,

$$\max_{x \in I} |P_h u(x) - u(x)| \le c_1 h^2,$$

where c_1 is a positive constant depending only on $\|u\|_{C^2(I)}$. Hence, there is a positive constant c_2 such that

$$|J_p(u) - J_p(P_h u)| \le c_2 h^3.$$

Now put $z_h(x_p) = u_h(x_p) - u(x_p)$. Then

$$l_h z_h(x_p) + J_{h,p}(u_h) - J_{h,p}(u) = J_p(u) - J_{h,p}(u),$$

and so

$$l_h z_h(x_p) + Q_{h,p}(u_h, u) = J_p(u) - J_{h,p}(u).$$

Theorem 3.4. *Let u and u_h be the solution of* (2.1) *and* (3.1) *in $\mathbf{K}(u_*, u^*)$, respectively. Assume that*

(i) $4\beta M_1 m < \alpha_0$,
(ii) $|F_{i,j}(x, \eta)| \le M_1$ *for all $x \in I$ and $\eta \in \mathbf{K}(u_*, u^*)$*,
(iii) $\{\varphi_p\}_0^N$ *are the standard piecewise linear basis functions.*

Then for some $c_3 > 0$,

$$\|u - u_h\|_\infty \le |u - u_h|_1 \le c_3 h^2.$$

Proof. Analogously to the procedure used for demonstrating Theorem 3.2, one can obtain

$$\alpha_0 |z_h|_1^2 \le 4\beta M_1 m |z_h|_1^2 + c\sum_p h^3 |z_h(x_p)|.$$

One also has that the last term above is bounded by

$$c\sum_p (h_p^{1/2} z_h(x_p))(h_p^{1/2}\beta h^2) \le \varepsilon \sum_p h_p z_h^2(x_p) + \sum_p \left(\frac{\tilde{c}}{4\varepsilon} h_p \beta^2 h^4\right)$$

for any $\varepsilon > 0$, and the result follows. □

4. Further approximations

In general, the integrals in (3.5) cannot be evaluated exactly. One way of overcoming this difficulty is to replace a_i by an approximation. Here we use a piecewise linear approximant $\tilde{a}_i$ to a_i, namely

$$\tilde{a}_i(x) = \frac{1}{h_p}(a_i(x_{p-1})(x_p - x) + a_i(x_p)(x - x_{p-1})), \qquad x \in I_p.$$

Let $\tilde{l} = \operatorname{diag}(\tilde{l}_1, \tilde{l}_2, \dots, \tilde{l}_m)$ and $\tilde{l}_i \tilde{u}_i(x) = -(\tilde{a}_i(x)\tilde{u}_i'(x))'$. Then the corresponding problem is to find $\tilde{u} \in [C^2(I) \cap C^1(\bar{I})]^m$ such that

$$\text{(4.1)} \qquad \begin{cases} \widetilde{L}\tilde{u}(x) = \tilde{l}\tilde{u}(x) + f(x, \tilde{u}(x)) = 0, & x \in I, \\ \tilde{u}(0) = \tilde{u}(1) = 0. \end{cases}$$

Clearly, a_i and $\tilde{a}_i$ satisfy the same conditions as those in §2. Thus, for problem (4.1), we have results similar to those in Theorems 2.1–2.3.

Now let

$$\tilde{l}_h = \operatorname{diag}(\tilde{l}_{h,1}, \tilde{l}_{h,2}, \dots, \tilde{l}_{h,m}),$$

$$\tilde{l}_{h,i}\tilde{u}_{h,i}(x_p) = -\widetilde{A}_{i,p}\tilde{u}_{h,i}(x_{p-1}) + (\widetilde{A}_{i,p} + \widetilde{A}_{i,p+1})\tilde{u}_{h,i}(x_p) - \widetilde{A}_{i,p+1}\tilde{u}_{h,i}(x_{p+1}),$$

and

$$\widetilde{J}_{h,p}(\tilde{u}_h) = (\widetilde{J}_{h,p,1}(\tilde{u}_h), \dots, \widetilde{J}_{h,p,m}(\tilde{u}_h))^{\mathrm{T}},$$

$$\begin{aligned} \widetilde{J}_{h,p,i}(\tilde{u}_h) = {} & \widetilde{A}_{i,p+1}\int_{x_p}^{x_{p+1}} \left[f_i(x, \tilde{u}_h(x_p)\varphi_p(x) + \tilde{u}_h(x_{p+1})\varphi_{p+1}(x)) \int_x^{x_{p+1}} \frac{dt}{\tilde{a}_i(t)} \right] dx \\ & + \widetilde{A}_{i,p}\int_{x_{p-1}}^{x_p} \left[f_i(x, \tilde{u}_h(x_{p-1})\varphi_{p-1}(x) + \tilde{u}_h(x_p)\varphi_p(x)) \int_{x_{p-1}}^{x} \frac{dt}{\tilde{a}_i(t)} \right] dx, \end{aligned}$$

where

$$\widetilde{A}_{i,p} = \left(\int_{x_{p-1}}^{x_p} \frac{dt}{\tilde{a}_i(t)} \right)^{-1}.$$

Then the corresponding Petrov-Galerkin method is to find $\tilde{u}_h \in S_h$ such that

$$\text{(4.2)} \qquad \begin{cases} \widetilde{L}_h \tilde{u}_h(x_p) = 0, & 1 \le p \le N-1, \\ \tilde{u}_h(0) = \tilde{u}_h(1) = 0. \end{cases}$$

We can establish results similar to those in Theorems 3.1–3.3. If, in addition, $\{\varphi_p\}_0^N$ are the standard piecewise linear basis functions, then

$$\|\tilde{u} - \tilde{u}_h\|_\infty \le c_4 h^2. \tag{4.3}$$

On the other hand, if we put $z = u - \tilde{u}$, then

$$\begin{cases} \tilde{l}_i z_i(x) + \sum_{j=1}^m F_{i,j}(x, \theta(x)) z_j(x) \\ \qquad = ((\tilde{a}_i(x) - a_i(x)) u_i'(x))', \qquad x \in I, \ 1 \le i \le m, \\ z(0) = z(1) = 0, \end{cases} \tag{4.4}$$

where θ lies between u and $\tilde{u}$. If $u, \tilde{u} \in \mathbf{K}(u_*, u^*)$ and $|F_{i,j}(x, \eta)| \le M_1$ for all $x \in I$ and $\eta \in \mathbf{K}(u_*, u^*)$, then we have from (4.4) that for an arbitrary positive constant ε,

$$\begin{aligned} (\alpha_0 - M_1 m)|z|^2_{H^1(I)} &\le \left| \sum_{i=1}^m \int_0^1 (\tilde{a}_i(x) - a_i(x)) u_i'(x) z_i'(x)\, dx \right| \\ &\le \varepsilon |z|^2_{H^1(I)} + \frac{1}{4\varepsilon} \sum_{i=1}^m \left| \int_0^1 ((\tilde{a}_i(x) - a_i(x)) u_i'(x))^2\, dx \right| \\ &\le \varepsilon |z|^2_{H^1(I)} + \frac{c_5}{\varepsilon} h^4, \end{aligned}$$

where c_5 is a positive constant depending only on $\|a\|_{H^2(I)}$ and $\|u\|_{H^1(I)}$. Thus, if $M_1 m + \varepsilon < \alpha_0$, then

$$\|u - \tilde{u}\|_{L^\infty(I)} \le |u - \tilde{u}|_{H^1(I)} \le \sqrt{\frac{c_5}{\varepsilon(\alpha_0 - M_1 m - \varepsilon)}} \cdot h^2. \tag{4.5}$$

The integral in (4.2) is still a difficulty for general functions f. To overcome this, we can use piecewise linear interpolation also for $f(x, u(x))$ as follows. Let

$$\tilde{\tilde{J}}_{h,p}(\tilde{\tilde{u}}_h) = (\tilde{\tilde{J}}_{h,p,1}(\tilde{\tilde{u}}_h), \dots, \tilde{\tilde{J}}_{h,p,m}(\tilde{\tilde{u}}_h))^{\mathrm{T}},$$

$$\begin{aligned} \tilde{\tilde{J}}_{h,p,i}(\tilde{\tilde{u}}_h) = \widetilde{A}_{i,p+1} \int_{x_p}^{x_{p+1}} &\Big[(f_i(x_p, \tilde{\tilde{u}}_h(x_p)) \varphi_{p,i}(x) \\ &\quad + f_i(x_{p+1}, \tilde{\tilde{u}}_h(x_{p+1})) \varphi_{p+1,i}(x)) \int_x^{x_{p+1}} \frac{dt}{\tilde{a}_i(t)} \Big]\, dx \\ + \widetilde{A}_{i,p} \int_{x_{p-1}}^{x_p} &\Big[(f_i(x_{p-1}, \tilde{\tilde{u}}_h(x_{p-1})) \varphi_{p-1,i}(x) \\ &\quad + f_i(x_p, \tilde{\tilde{u}}_h(x_p)) \varphi_{p,i}(x)) \int_{x_{p-1}}^{x} \frac{dt}{\tilde{a}_i(t)} \Big]\, dx, \end{aligned}$$

and

$$\tilde{\tilde{L}}_h \tilde{\tilde{u}}_h(x_p) = \tilde{l}_h \tilde{\tilde{u}}_h(x_p) + \tilde{\tilde{J}}_{h,p}(\tilde{\tilde{u}}_h).$$

Then the Petrov-Galerkin method leads to the following problem:

$$\begin{cases} \tilde{\tilde{L}}_h \tilde{\tilde{u}}_h(x_p) = 0, \qquad 1 \le p \le N, \\ \tilde{\tilde{u}}_h(0) = \tilde{\tilde{u}}_h(1) = 0. \end{cases}$$

Let $z_h = \tilde{\tilde{u}}_h - \tilde{u}_h$. Then

$$\tilde{l}_h z_h(x_p) + \tilde{\tilde{Q}}_{h,p}(\tilde{\tilde{u}}_h, \tilde{u}_h) = \widetilde{J}_{h,p}(\tilde{u}_h) - \widetilde{\widetilde{J}}_{h,p}(\tilde{u}_h),$$

where $\tilde{\tilde{Q}}_{h,p}$ is similar to $Q_{h,p}$, but a_i and $A_{i,p}$ are replaced by $\tilde{a}_i$ and $\widetilde{A}_{i,p}$, respectively. It is easy to verify that

$$\|\widetilde{J}_{h,p}(\tilde{u}_h) - \widetilde{\widetilde{J}}_{h,p}(\tilde{u}_h)\| \le c_6 h^3,$$

where c_6 is a positive constant depending only on β, $\max|\partial^2 f/\partial u_i \partial u_j|$, and $|\tilde{u}_h|_1$. So, if $M_1 m < \alpha_0$, then for some $c_7 > 0$, we have

$$\|\tilde{\tilde{u}}_h - \tilde{u}_h\|_\infty \le c_7 h^2. \tag{4.6}$$

By combining (4.3) and (4.5) with (4.6), we conclude that for some $c_8 > 0$,

$$\|u - \tilde{\tilde{u}}_h\|_\infty \le c_8 h^2.$$

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