MATHEMATICS OF COMPUTATION
VOLUME 58, NUMBER 198
APRIL 1992, PAGES 781-792

WEIGHT FORMULAS FOR TERNARY MELAS CODES

GERARD VAN DER GEER, RENÉ SCHOOF, AND MARCEL VAN DER VLUGT

ABSTRACT. In this paper we derive a formula for the frequencies of the weights in ternary Melas codes and we illustrate this formula by computing a table of examples.

1. INTRODUCTION

Let $q = p^m$, where p is a prime, and let α be a generator of the multiplicative group $\mathbf{F}_q^*$. Consider the cyclic code C over $\mathbf{F}_q$ of length $q-1$ with generator polynomial $(X-\alpha)(X-\alpha^{-1})$. The dual code $C^\perp$ is cyclic with zeros $1, \alpha^2, \alpha^3, \ldots, \alpha^{q-3}$, which are zeros of the polynomials

$$\sum_{i=0}^{q-2} (a\alpha^i + b\alpha^{-i})X^i \in \mathbf{F}_q[X]/(X^{q-1}-1) \quad \text{with } a, b \in \mathbf{F}_q.$$

This implies that the code

$$D = \{(ax + b/x)_{x\in\mathbf{F}_q^*} : a, b \in \mathbf{F}_q\}$$

satisfies $D = C^\perp$. The classical Melas code $M(q)$ is defined as the restriction to $\mathbf{F}_p$ of the code C (see [5, 4]). By Delsarte's theorem [4, p. 208] we have

$$\mathrm{Tr}(C^\perp) = (C|_{\mathbf{F}_p})^\perp,$$

where Tr is the trace map from $\mathbf{F}_q$ to $\mathbf{F}_p$. If we substitute $C^\perp = D$ and $C|_{\mathbf{F}_p} = M(q)$ in Delsarte's theorem, we find

$$\{(\mathrm{Tr}(ax + b/x))_{x\in\mathbf{F}_q^*} : a, b \in \mathbf{F}_q\} = M(q)^\perp.$$

To ensure injectivity of the trace map, we require $2m+1 < q$. Then the dual code $M(q)^\perp$ has dimension $2m$.

In [6, 1] we determined the weight distribution of $M(q)^\perp$ for $p = 2$ and 3. Then, by the MacWilliams identities and the Eichler-Selberg trace formula we derived a formula for the number A_i of code words of weight i in $M(q)$ involving traces of Hecke operators on certain spaces of cusp forms [6, Theorem 4.2; 1, Theorem 2.3]. Especially for $p = 3$, this was done in a rather concise way, only announcing results and further illustrations. In this paper we will work out the case $p = 3$ and illustrate the result by computing some weight formulas for ternary Melas codes.

Received December 21, 1989; revised March 4, 1991.
1991 *Mathematics Subject Classification*. Primary 94B05, 11F11.
The second author was supported by the Netherlands Organization of Scientific Research.

An outline of this paper is as follows. In §2 we derive an expression for traces of Hecke operators on $S_k(\Gamma_1(3))$. In §3 we prove the weight distribution theorem for ternary Melas codes. Then, in the next sections, we compute traces of Hecke operators, first for even k, then for odd k. Finally, in §6 we give a table of weight formulas for $M(q)$.

The references on coding theory can be found in the book of MacWilliams and Sloane [4]. For a systematic introduction to cusp forms and Hecke operators we refer to the books by S. Lang [2] and J.-P. Serre [8]. In [6, Theorem 2.2] the reader can find the precise form of the Eichler-Selberg trace formula, as we use it. Our notation in this paper links up with the notation in [6].

2. Traces of Hecke operators on $S_k(\Gamma_1(3))$

For the space of cusp forms $S_k(\Gamma_1(3))$ we have

$$S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), 1) \oplus S_k(\Gamma_0(3), \omega),$$

where 1 is the trivial character on $(\mathbf{Z}/3\mathbf{Z})^*$ and ω is the quadratic character on $(\mathbf{Z}/3\mathbf{Z})^*$. Both characters have conductor 3, and we extend them to $\mathbf{Z}/3\mathbf{Z}$ by defining them 0 on the residue class of 0 modulo 3. Actually,

$$S_k(\Gamma_1(3)) = \begin{cases} S_k(\Gamma_0(3), 1) & \text{for even } k, \\ S_k(\Gamma_0(3), \omega) & \text{for odd } k. \end{cases}$$

Now we can apply the Eichler-Selberg trace formula for $S_k(\Gamma_0(3), \chi)$, expressing traces of Hecke operators in class numbers of binary quadratic forms.

Proposition 2.1. *Let $q = 3^m$ with $m \geq 1$, and denote by $\operatorname{Tr} T_q$ the trace of the Hecke operator T_q acting on the space of cusp forms $S_k(\Gamma_1(3))$. Then*

$$\operatorname{Tr} T_q = \begin{cases} -\sum_t \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q) - 1 & \text{for } k \geq 3, \\ -\sum_t \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q) - 1 + q & \text{for } k = 2. \end{cases}$$

The summation variable t runs over $\{t \in \mathbf{Z}: t^2 < 4q \text{ and } t \equiv 1 \pmod 3\}$. The symbols ρ and $\overline{\rho}$ indicate the zeros of the polynomial $X^2 - tX + q$, and $H(t^2 - 4q)$ is the Kronecker class number of $t^2 - 4q$.

Proof. We start from the Eichler-Selberg trace formula as stated in [6, Theorem 2.2] and employ it for $S_k(\Gamma_0(3), \chi)$, where $\chi = 1$ for even k and $\chi = \omega$ for odd k. In the notation of [6, Theorem 2.2], the contribution of A_1 is 0. As to the contribution of A_2, we notice that $\mu(t, f, n) = \chi(t)$. It follows that

$$A_2 = -\sum_{\substack{t \in \mathbf{Z} \\ t^2 < 4q,\, t \equiv 1 \ (\mathrm{mod}\ 3)}} \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}} H(t^2 - 4q)$$

by adding together terms with $t \equiv 1 \pmod 3$ and $t \equiv 2 \pmod 3$. Furthermore, $A_3 = -1$ in all cases, and $A_4 = q$ for $k = 2$ and $\chi = 1$, while $A_4 = 0$ in the other cases. Altogether, we get the above-mentioned formulas. □

The numbers $(\rho^{k-1} - \overline{\rho}^{k-1})/(\rho - \overline{\rho})$ are symmetric expressions in ρ and $\overline{\rho}$, so they can be written as polynomials $Q_{k-2}(t, q)$ in $t = \rho + \overline{\rho}$ and $q = \rho\overline{\rho}$.

We have $Q_0(t, q) = 1$ and $Q_1(t, q) = t$. From $\rho^{k+1} - \overline{\rho}^{k+1} = (\rho + \overline{\rho})(\rho^k - \overline{\rho}^k) - \rho\overline{\rho}(\rho^{k-1} - \overline{\rho}^{k-1})$ we get the recurrence relation

$$Q_k(t, q) = tQ_{k-1}(t, q) - qQ_{k-2}(t, q) \quad \text{for } k \geq 2. \tag{1}$$

The polynomial Q_k is, as a polynomial in ρ and $\overline{\rho}$, homogeneous of degree k. Therefore, it is also homogeneous of degree k as a polynomial in t and q, provided we assign a weight 1 to the variable t and a weight 2 to the variable q. Note that Q_k is monic in t, and has integer coefficients and terms $q^r t^{k-2r}$, where $0 \leq r \leq [k/2]$. It follows that we can write

$$t^i = \sum_{\substack{j=0 \\ j \text{ even}}}^{i} \lambda_{i,j} Q_{i-j}(t, q) q^{j/2}. \tag{2}$$

The $\lambda_{i,j} \in \mathbf{Z}$ satisfy

$$\lambda_{i,j} = 0 \quad \text{for } j \notin \{0, 1, \ldots, i\} \text{ or } j \text{ odd},$$
$$\lambda_{0,0} = \lambda_{1,0} = 1,$$

while the recurrence relation for Q_k induces the recurrence relation

$$\lambda_{i+1,j} = \lambda_{i,j-2} + \lambda_{i,j}. \tag{3}$$

Now we rewrite the expressions for $\operatorname{Tr} T_q$ on $S_k(\Gamma_1(3))$ in Proposition 2.1 as

$$\operatorname{Tr} T_q = \begin{cases} -\sum_t Q_{k-2}(t, q) H(t^2 - 4q) - 1 & \text{for odd } k \text{ and even } k \geq 4, \\ -\sum_t H(t^2 - 4q) - 1 + q & \text{for } k = 2. \end{cases} \tag{4}$$

From the formula for $\dim S_k(\Gamma_0(N), \chi)$ in [6] we easily derive:

$$\dim S_k(\Gamma_1(3)) = \begin{cases} \dim S_k(\Gamma_0(3), \omega) = [k/3] - 1 & \text{for odd } k, \\ \dim S_k(\Gamma_0(3), 1) = [k/3] - 1 & \text{for even } k \geq 4, \\ 0 & \text{for } k = 2. \end{cases} \tag{5}$$

Because $\dim S_2(\Gamma_1(3)) = 0$, one has

$$\operatorname{Tr} T_2 = -\sum_t H(t^2 - 4q) - 1 + q = 0.$$

3. The weight distribution of ternary Melas codes

Let $q = 3^m$ with $m \geq 2$. In [1] we derived the weight distribution of the dual ternary Melas code $M(q)^\perp$:

> The nonzero weights of $M(q)^\perp$ are $w_t = 2(q - 1 + t)/3$, where $t \in \mathbf{Z}$, $t^2 < 4q$, and $t \equiv 1 \pmod 3$. For $t \neq 1$ the frequency of w_t is $(q-1)H(t^2-4q)$; the weight $w_1 = 2q/3$ has frequency $(q-1)\{H(1-4q) + 2\}$.

Using the MacWilliams identities and the Eichler-Selberg trace formula, we obtain an expression for the weight distribution of $M(q)$. We will elaborate the result announced in [1, Theorem 2.3].

Theorem 3.1. *The number A_i of code words of weight i in the Melas code $M(q)$ is given by*

$$q^2A_i = \binom{q-1}{i}2^i + 2(q-1)\sum_{s=0}^{i}(-1)^s\binom{2q/3}{s}\binom{q/3-1}{i-s}2^{i-s}$$
$$-(q-1)\sum_{j=0}^{i}W_{i,j}(q)(1+\tau_{j+2}(q)),$$

where the polynomials $W_{i,j}(q)$ are defined for $0 \le j \le i$ by

$$W_{0,0}=1, \quad W_{1,0}=0, \quad W_{1,1}=-2,$$
$$(i+1)W_{i+1,j} = -iW_{i,j} - 2qW_{i,j+1} - 2W_{i,j-1} - 2(q-i)W_{i-1,j}$$

(otherwise, the $W_{i,j}$ are 0).

By $\tau_k(q)$ we denote for $k \ge 3$ the trace of the Hecke operator T_q on $S_k(\Gamma_1(3))$. For convenience we let $\tau_2(q) = -q$.

Proof. This proof is a modification of the proof of the analogous theorem in [6]. For $0 \le i \le q-1$, let $P_i(X)$ be the ith Krawtchouk polynomial

$$P_i(X; q-1, 3) = \sum_{s=0}^{i}(-1)^s\binom{X}{s}\binom{q-1-X}{i-s}2^{i-s}. \tag{6}$$

These polynomials satisfy the recurrence relation

$$(i+1)P_{i+1}(X) = (2q-2-i-3X)P_i(X) - 2(q-i)P_{i-1}(X).$$

We define $f_i(X) = P_i(2(q-1+X)/3)$; then

$$f_0(X) = P_0(2(q-1+X)/3) = 1, \qquad f_1(X) = P_1(2(q-1+X)/3) = -2X$$

and the recurrence relation becomes

$$(i+1)f_{i+1}(X) = (-i-2X)f_i(X) - 2(q-i)f_{i-1}(X). \tag{7}$$

It follows that $f_i(X)$ has degree i, and we write

$$f_i(X) = \sum_{k=0}^{i}\pi_i(k)X^k. \tag{8}$$

Now $\pi_0(0)=1$, $\pi_1(0)=0$, $\pi_1(1)=-2$, and from (7) we derive

$$(i+1)\pi_{i+1}(k) = -i\pi_i(k) - 2\pi_i(k-1) - 2(q-i)\pi_{i-1}(k). \tag{9}$$

We define $\pi_i(k) = 0$ for cases other than $0 \le k \le i$. When we apply the MacWilliams identities to $M(q)^\perp$ and $M(q)$, we get

$$q^2A_i = \sum_t \text{frequency}(w_t)P_i(2(q-1+t)/3) + P_i(0),$$

where t runs over $\{t \in \mathbf{Z}\colon t^2 < 4q \text{ and } t \equiv 1 \pmod 3\}$. Using the weight distribution of $M(q)^\perp$ and the polynomials f_i introduced above, we find

$$\frac{q^2}{q-1}A_i = \sum_t H(t^2-4q)f_i(t) + 2f_i(1) + \frac{P_i(0)}{q-1}.$$

From definition (6) we see that $P_i(0) = \binom{q-1}{i}2^i$ and

$$f_i(1) = P_i\left(\frac{2q}{3}\right) = \sum_{s=0}^{i}(-1)^s \binom{2q/3}{s}\binom{q/3-1}{i-s}2^{i-s}.$$

From (8) we obtain

$$\sum_t H(t^2-4q)f_i(t) = \sum_{j=0}^{i}\pi_i(j)\sum_t t^j H(t^2-4q).$$

By formula (2) this becomes

$$\sum_{j=0}^{i}\pi_i(j)\sum_{\substack{k=0\\ k \text{ even}}}^{j}\lambda_{j,k}q^{k/2}\sum_t Q_{j-k}(t,q)H(t^2-4q).$$

Using (4) combined with the fact that, according to (5), $\operatorname{Tr} T_q = 0$ on $S_2(\Gamma_1(3))$, and remembering our convention that $\tau_2(q) = -q$, we get

$$\sum_{j=0}^{i}\pi_i(j)\sum_{\substack{k=0\\ k \text{ even}}}^{j}\lambda_{j,k}q^{k/2}(-1-\tau_{j-k+2}(q)). \tag{10}$$

We define $W_{i,j}(q) = \sum_{k=0,\,k \text{ even}}^{i-j}\pi_i(k+j)\lambda_{k+j,k}q^{k/2}$. By changing the horizontal summation in (10) into a diagonal summation, the expression (10) becomes

$$\sum_{j=0}^{i}W_{i,j}(q)(-1-\tau_{j+2}(q)).$$

Putting all this together, we get the announced formula for q^2A_i.

As to the polynomials $W_{i,j}(q)$, we easily see that $W_{0,0} = 1$, $W_{1,0} = 0$, and $W_{1,1} = -2$. The recurrence relation for $W_{i,j}$ follows by writing out the definition of $(i+1)W_{i+1,j}$ and using the recurrence relations (9) and (3) for $(i+1)\pi_{i+1}(k+j)$ and $\lambda_{k+j,k}$. □

We conclude this section by noticing that to obtain more explicit expressions for A_i, we have to compute the traces of the Hecke operators $\tau_k(q)$. This is the subject of the next two sections.

4. The computation of $\tau_k(q)$ for k even, $k \geq 4$

As always, we take $q = 3^m$ with $m \geq 2$. By convention, we have that $\tau_2(q) = -q$, while for $k \geq 3$ the trace of the Hecke operator T_q acting on the space $S_k(\Gamma_1(3))$ is indicated by $\tau_k(q)$. For even k, the space $S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), 1)$ and the theory of newforms of Atkin and Lehner [2] provides us with a decomposition

$$S_k(\Gamma_0(3), 1) = S_k(\Gamma_0(3))^{\text{new}} \oplus S_k(\Gamma_0(3))^{\text{old}},$$

which is respected by the Hecke operators. The old part is spanned by the forms $f(z)$ and $f(3z)$, where $f(z)$ runs over a basis of simultaneous eigenforms of $S_k(\Gamma_0(1)) = S_k(\mathrm{SL}_2(\mathbf{Z}))$.

Proposition 4.1. *On* $S_k(\Gamma_0(3))^{\text{old}}$ *we have*

$$\begin{aligned} \operatorname{Tr} T_1 &= 2\dim S_k(\mathrm{SL}_2(\mathbf{Z})), \\ \operatorname{Tr} T_3 &= \operatorname{Tr}(T_3 \text{ on } S_k(\mathrm{SL}_2(\mathbf{Z}))), \\ \operatorname{Tr} T_{3^m} &= \operatorname{Tr}(T_{3^m} \text{ on } S_k(\mathrm{SL}_2(\mathbf{Z}))) \\ &\quad - 3^{k-1}\operatorname{Tr}(T_{3^{m-2}} \text{ on } S_k(\mathrm{SL}_2(\mathbf{Z}))) \quad \text{for } m \geq 2. \end{aligned}$$

Proof. The subspace $S_k(\Gamma_0(3))^{\text{old}}$ is a direct sum of 2-dimensional complex vector spaces with basis $\{f(z), f(3z)\}$, where $f(z)$ is a simultaneous eigenform for all T_n in $S_k(\mathrm{SL}_2(\mathbf{Z}))$. The operator T_1 is the identity map, so

$$\operatorname{Tr} T_1 = \dim S_k(\Gamma_0(3))^{\text{old}} = 2\dim S_k(\mathrm{SL}_2(\mathbf{Z})).$$

Let $f(z)=\sum_{m=1}^{\infty} a_m e^{2\pi i m z}$; then by applying the formula for T_n on $S_k(\Gamma_0(3), 1)$ (see [6]) we have

$$T_3(f(z)) = \sum_{m\geq 1} a_{3m} e^{2\pi i m z},$$

while on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ we have

$$T_3(f(z)) = \lambda f(z) = \sum_{m\geq 1} a_{3m} e^{2\pi i m z} + 3^{k-1}\sum_{m\geq 1} a_m e^{3(2\pi i m z)}.$$

For T_3 acting on the 2-dimensional summand $\langle f(z)\rangle \oplus \langle f(3z)\rangle$, we obtain

$$T_3(f(z)) = \lambda f(z) - 3^{k-1} f(3z) \quad \text{and} \quad T_3(f(3z)) = f(z).$$

Then on $\langle f(z)\rangle \oplus \langle f(3z)\rangle$ the operator T_3 has eigenvalues α and β with $\alpha+\beta = \lambda$ and $\alpha\beta = 3^{k-1}$. The eigenvalues of T_3 acting on $S_k(\Gamma_0(3))^{\text{old}}$ are precisely the α and β for all possible eigenvalues λ of T_3 acting on $S_k(\mathrm{SL}_2(\mathbf{Z}))$. We conclude that

$$\operatorname{Tr} T_3 = \sum (\alpha+\beta) = \sum \lambda = \operatorname{Tr}(T_3 \text{ on } S_k(\mathrm{SL}_2(\mathbf{Z}))).$$

From the product formula $T_n \cdot T_m = \sum_{d|m,n} d^{k-1} T_{mn/d^2}$ we derive

$$T_{3^m} = T_3 \cdot T_{3^{m-1}} - 3^{k-1} T_{3^{m-2}} \quad \text{for } m \geq 2 \tag{11}$$

on $S_k(\mathrm{SL}_2(\mathbf{Z}))$. Thus, the eigenvalue λ_{3^m} of T_{3^m} on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ corresponding to λ is

$$\lambda \cdot \lambda_{3^{m-1}} - 3^{k-1} \cdot \lambda_{3^{m-2}}.$$

While $\lambda = \alpha+\beta$ and $3^{k-1} = \alpha\beta$, it follows by induction that the eigenvalue of T_{3^m} on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ corresponding to $\lambda = \alpha+\beta$ is $\sum_{i=0}^{m} \alpha^i\beta^{m-i}$.

Furthermore, it holds that $T_{3^m} = (T_3)^m$ on $S_k(\Gamma_0(3), 1)$, so T_{3^m} has eigenvalues α^m and β^m on $\langle f(z)\rangle \oplus \langle f(3z)\rangle$. Adding up the relation

$$\alpha^m + \beta^m = \sum_{i=0}^{m} \alpha^i\beta^{m-i} - \alpha\beta\sum_{i=0}^{m-2} \alpha^i\beta^{m-2-i}$$

for all pieces of $S_k(\Gamma_0(3))^{\text{old}}$, we obtain the stated result for T_{3^m}, $m \geq 2$. □

Remark 4.2. From the dimension formula [6, Corollary 2.3] we conclude

$$\dim S_2(\mathrm{SL}_2(\mathbf{Z})) = 0,$$

$$\dim S_k(\mathrm{SL}_2(\mathbf{Z})) = \begin{cases} [k/12] & \text{for } k \not\equiv 2 \pmod{12}, \\ [k/12]-1 & \text{for } k \equiv 2 \pmod{12},\ k \geq 4. \end{cases} \tag{12}$$

Next we derive a formula for $\operatorname{Tr} T_q$ on $S_k(\Gamma_0(3))^{\text{new}}$.

Proposition 4.3. *On* $S_k(\Gamma_0(3))^{\text{new}}$ *we have*

$$\operatorname{Tr} T_q = \begin{cases} \dim S_k(\Gamma_0(3))^{\text{new}} \cdot q^{k/2-1} & \text{for } m \text{ even}, \\ q^{k/2-1} & \text{for } m \text{ odd, } k \equiv 2, 6 \pmod{12}, \\ -q^{k/2-1} & \text{for } m \text{ odd, } k \equiv 0, 8 \pmod{12}, \\ 0 & \text{for } m \text{ odd, } k \equiv 4, 10 \pmod{12}. \end{cases}$$

Proof. First we consider T_3. The eigenvalues of T_3 acting on $S_k(\Gamma_0(3))^{\text{new}}$ are $\pm 3^{k/2-1}$ (see [3, Theorem 3]). In order to find the multiplicities of the eigenvalues, we compute

$$\begin{aligned} \operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3))^{\text{new}} &= \operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3), 1) - \operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3))^{\text{old}} \\ &= \operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3), 1) - \operatorname{Tr} T_3 \text{ on } S_k(\mathrm{SL}_2(\mathbf{Z})). \end{aligned}$$

By the Eichler-Selberg formula we find

$$\begin{aligned} &\operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3), 1) \\ &\qquad = -\left\{ \frac{\rho_1^{k-1} - \overline{\rho}_1^{k-1}}{\rho_1 - \overline{\rho}_1} h_w(-11) + \frac{\rho_2^{k-1} - \overline{\rho}_2^{k-1}}{\rho_2 - \overline{\rho}_2} h_w(-8) + 1 \right\}, \end{aligned} \tag{13}$$

where ρ_1, $\overline{\rho}_1$ are the zeros of $X^2 - X + 3$ and ρ_2, $\overline{\rho}_2$ are the zeros of $X^2 - 2X + 3$. Applying the same formula for $\operatorname{Tr} T_3$ on $S_k(\mathrm{SL}_2(\mathbf{Z}))$, we find (13) and the extra terms

$$-\left(\frac{\rho_3^{k-1} - \overline{\rho}_3^{k-1}}{\rho_3 - \overline{\rho}_3} h_w(-3) \right) - \frac{1}{2} \left(\frac{\rho_4^{k-1} - \overline{\rho}_4^{k-1}}{\rho_4 - \overline{\rho}_4} \right) (h_w(-12) + h_w(-3)),$$

where ρ_3, $\overline{\rho}_3$ are the zeros of $X^2 - 3X + 3$ and ρ_4, $\overline{\rho}_4$ are the zeros of $X^2 + 3$. For $\Delta < -4$, the $h_w(\Delta)$ are class numbers and $h_w(-3) = 1/3$.

Note that in the case of $\mathrm{SL}_2(\mathbf{Z})$, the character involved is the principal character, which is 1 on all of $\mathbf{Z}$ and has conductor 1.

Substituting the zeros of $X^2 - 3X + 3$ and $X^2 + 3$, we get

$$\begin{aligned} \operatorname{Tr} T_3 \text{ on } S_k(\Gamma_0(3))^{\text{new}} &= 2.3^{k/2-2}(\sin(k-1)\pi/6 + \sin(k-1)\pi/2) \\ &= \begin{cases} 0 & \text{for } k \equiv 4, 10 \pmod{12}, \\ 3^{k/2-1} & \text{for } k \equiv 2, 6 \pmod{12}, \\ -3^{k/2-1} & \text{for } k \equiv 0, 8 \pmod{12}. \end{cases} \end{aligned} \tag{14}$$

Denoting the multiplicities of the eigenvalues $3^{k/2-1}$ and $-3^{k/2-1}$ by A and B, respectively, we now know $A - B$, while $A + B = \dim S_k(\Gamma_0(3))^{\text{new}}$. Because $T_{3^m} = (T_3)^m$ on $S_k(\Gamma_0(3), 1)$, the eigenvalues of T_{3^m} on $S_k(\Gamma_0(3))^{\text{new}}$ are $(3^{k/2-1})^m$ and $(-3^{k/2-1})^m$, while their multiplicities are known as well. From (14) we easily confirm the required result. □

The dimension of $S_k(\Gamma_0(3))^{\text{new}}$ for even $k \geq 4$ can be computed explicitly. From the decomposition

$$S_k(\Gamma_0(3), 1) = S_k(\Gamma_0(3))^{\text{new}} \oplus S_k(\Gamma_0(3))^{\text{old}}$$

we see that $\dim S_k(\Gamma_0(3))^{\text{new}} = \dim S_k(\Gamma_1(3)) - 2 \dim S_k(\mathrm{SL}_2(\mathbf{Z}))$. Combining

(5) and (12), one finds

$$\dim S_k(\Gamma_0(3))^{\mathrm{new}} = 2[k/12] + \begin{cases} -1 & \text{for } k \equiv 0 \pmod{12}, \\ 1 & \text{for } k \equiv 2, 6, 8 \pmod{12}, \\ 0 & \text{for } k \equiv 4 \pmod{12}, \\ 2 & \text{for } k \equiv 10 \pmod{12}. \end{cases} \tag{15}$$

Conclusion. To calculate $\tau_k(q)$ for k even, $k \geq 4$, and $q = 3^m$, $m \geq 2$, we put Propositions 4.1 and 4.3 and formula (15) for $\dim S_k(\Gamma_0(3))^{\mathrm{new}}$ together. The only quantity left over to compute is the trace of T_q on $S_k(\mathrm{SL}_2(\mathbf{Z}))$. In the next proposition we include a small list of traces of Hecke operators $\tau_k(q)$ for even k.

Proposition 4.4. *The trace $\tau_k(q)$ of the Hecke operator T_q, where $q = 3^m$ with $m \geq 1$, acting on $S_k(\Gamma_1(3))$, is for even k satisfying $4 \leq k \leq 22$ given by the following table*:

k	$\tau_k(q)$ on $S_k(\Gamma_0(3))^{\mathrm{old}}$		$\tau_k(q)$ on $S_k(\Gamma_0(3))^{\mathrm{new}}$	
			m odd	m even
4	0		0	0
6	0		q^2	q^2
8	0		$-q^3$	q^3
10	0		0	$2q^4$
12	$t_{12,m}$	+	$-q^5$	q^5
14	0		q^6	$3q^6$
16	$t_{16,m}$		0	$2q^7$
18	$t_{18,m}$		q^8	$3q^8$
20	$t_{20,m}$		$-q^9$	$3q^9$
22	$t_{22,m}$		0	$4q^{10}$

For $k = 12, 16, 18, 10, 22$, the $t_{k,m}$ are respectively given by $t_{k,0} = 2$, $t_{k,1} = 252, -3348, -4284, 50652, -128844$ and

$$t_{k,m} = t_{k,1} \cdot t_{k,m-1} - 3^{k-1} t_{k,m-1} \quad \textit{for } m \geq 2.$$

Proof. For $k = 4, 6, 8, 10, 14$, the spaces $S_k(\mathrm{SL}_2(\mathbf{Z}))$ are zero, therefore $\operatorname{Tr} T_q$ on $S_k(\Gamma_0(3))^{\mathrm{old}}$ is zero, and our formulas follow easily.

For $k = 12, 16, 18, 20, 22$, the spaces $S_k(\mathrm{SL}_2(\mathbf{Z}))$ are one-dimensional. If λ is the eigenvalue of T_3 on $S_k(\mathrm{SL}_2(\mathbf{Z}))$, we have $\lambda = \alpha + \beta$ and $3^{k-1} = \alpha\beta$, where α and β are the corresponding eigenvalues of T_3 on $S_k(\Gamma_0(3))^{\mathrm{old}}$ (see the proof of Proposition 4.1). Now $t_m = \operatorname{Tr} T_{3^m}$ on $S_k(\Gamma_0(3))^{\mathrm{old}}$ satisfies the recurrence relation

$$t_m = \alpha^m + \beta^m = \lambda t_{m-1} - 3^{k-1} t_{m-2} \quad \text{for } m \geq 2,$$

while $t_1 = \lambda$ and $t_0 = 2$. We calculate $\lambda = \operatorname{Tr} T_3$ on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ by the trace formula. The result is

$$\lambda = -\sum_{t=0}^{3} r_t Q_{k-2}(t, 3) - 1,$$

with $r_0 = \frac{2}{3}$, $r_1 = r_2 = 1$, and $r_3 = \frac{1}{3}$. Combining these observations with Proposition 4.3, we obtain our formulas. □

Note that for $k = 12$, the eigenvalue of T_q on $S_{12}(\mathrm{SL}_2(\mathbf{Z}))$ is $\tau(q)$, where τ is the Ramanujan τ-function. Then $t_m = \tau(q) - 3^{11}\tau(q/9)$ for $q = 3^m$ with $m \geq 2$ and $t_1 = 252$.

5. The computation of $\tau_k(q)$ for odd $k \geq 3$

By (5), we have for odd k that

$$\dim S_k(\Gamma_1(3)) = \dim S_k(\Gamma_0(3), \omega) = [k/3] - 1.$$

Since the action of the character ω on $(\mathbf{Z}/3\mathbf{Z})^*$ differs from the action of the principal character, the space of cusp forms $S_k(\Gamma_0(3), \omega)$ consists entirely of newforms. Therefore, the eigenvalues λ of T_3 acting on $S_k(\Gamma_0(3), \omega)$ have absolute values $3^{(k-1)/2}$ (see [3, Theorem 3]). This implies that the monic polynomial $F_k(X)$ with roots $\lambda/3^{(k-1)/2}$ is reciprocal. So, to determine $F_k(X)$, which has degree $[k/3] - 1$, we only have to know the first $[([k/3] - 1)/2] + 1$ coefficients, provided they are not 0.

Since $T_q = (T_3)^m$ on $S_k(\Gamma_0(3), \omega)$, we have $\operatorname{Tr} T_q = \sum_\lambda \lambda^m$, and from the Newton identities for power sums we can derive some elementary symmetric functions of the eigenvalues λ from $\operatorname{Tr} T_3$, $\operatorname{Tr} T_9$, etc. We only need a few $\operatorname{Tr} T_q$ to fix $F_k(X)$, bearing in mind that $F_k(X)$ is reciprocal. From $F_k(X)$ we obtain the characteristic polynomial of T_3 and from that the eigenvalues λ of T_3. Then we can compute $\tau_k(q) = \sum_\lambda \lambda^m$ for odd $k \geq 3$ and $q = 3^m$ with $m \geq 2$.

Proposition 5.1. *The trace $\tau_k(q)$ of the Hecke operator T_q with $q = 3^m$ and $m \geq 2$, acting on $S_k(\Gamma_0(3), \omega)$, is for $k = 3, 5, 7, 9, 11, 13, 15$, and 17 given by the following table*:

$$\begin{aligned}
\tau_3(q) &= \tau_5(q) = 0,\\
\tau_7(q) &= (-1)^m q^3,\\
\tau_9(q) &= q^4 \cdot \mathrm{Trace}(\alpha_9^m),\\
\tau_{11}(q) &= q^5 \cdot \mathrm{Trace}(\alpha_{11}^m),\\
\tau_{13}(q) &= q^6 \cdot \{1 + \mathrm{Trace}(\alpha_{13}^m)\},\\
\tau_{15}(q) &= q^7 \cdot \mathrm{Trace}(\alpha_{15}^m),\\
\tau_{17}(q) &= q^8 \cdot \mathrm{Trace}(\alpha_{17}^m).
\end{aligned}$$

The α_i are algebraic numbers of absolute value 1 given by

$$\alpha_9 = \frac{5+2\sqrt{-14}}{9}, \quad \alpha_{11} = \frac{-1+4\sqrt{-5}}{9}, \quad \alpha_{13} = \frac{-25+2\sqrt{-26}}{27},$$

$$\alpha_{15} = \frac{61-16\sqrt{91}+4\sqrt{-2002-122\sqrt{91}}}{243},$$

$$\alpha_{17} = \frac{-19+2\sqrt{8089}+2\sqrt{-6583-19\sqrt{8089}}}{243}.$$

In this table, the Trace of an algebraic number is the sum of all its conjugates.

Proof. Since $\dim S_k(\Gamma_0(3), \omega) = 0$ for $k = 3, 5$, we have that $\tau_3(q) = \tau_5(q) = 0$. For the other values of k we compute the eigenvalues of T_3 acting on $S_k(\Gamma_1(3)) = S_k(\Gamma_0(3), \omega)$ in the way indicated above. The trace formula (4) gives us that for weight k:

$$\begin{aligned}\operatorname{Tr} T_3 &= -Q_{k-2}(1, 3)H(-11) - Q_{k-2}(-2, 3)H(-8) - 1\\ &= -Q_{k-2}(1, 3) - Q_{k-2}(-2, 3) - 1,\\ \operatorname{Tr} T_9 &= -1 - \sum_{\substack{t\equiv 1 \pmod 3\\ t^2<36}} Q_{k-2}(t, 9)H(t^2-36).\end{aligned}$$

Using the recurrence relations for the polynomial Q_{k-2} and a small table of class numbers from [7], we get the entries of the table below:

k	$\operatorname{Tr} T_3$	$\operatorname{Tr} T_9$	$F_k(X)$
7	−27	729	$X+1$
9	90	−5022	$X^2 - \frac{10}{9}X + 1$
11	−54	−115182	$X^2 + \frac{2}{9}X + 1$
13	−621	1291059	$(X-1)(X^2 + \frac{50}{27}X + 1)$
15	2196	−1624860	$X^4 - \frac{244}{243}X^3 + \frac{1474}{2187}X^2 - \frac{244}{243}X = 1$
17	−2052	18618660	$X^4 + \frac{76}{243}X^3 - \frac{122}{729}X^2 + \frac{76}{243}X + 1$

The eigenvalues λ of T_3 on $S_k(\Gamma_0(3), \omega)$ are $3^{(k-1)/2} \cdot \alpha$, where α runs over the zeros of $F_k(X)$ and

$$\tau_k(q) = \sum_\lambda \lambda^m = q^{(k-1)/2} \sum_{\{\alpha\,:\,F_k(\alpha)=0\}} \alpha^m,$$

which provides us with our formulas for $k = 7, 9, 11, 13, 15, 17$. □

Note that by computing the trace of more T_{3^m} for $m \geq 3$ we can easily extend Proposition 5.1. Adding $\operatorname{Tr} T_{27}$, for instance, will get us to $k = 23$.

6. Weight formulas for $M(q)$

When we combine Theorem 3.1 with Propositions 4.4 and 5.1, we get explicit formulas for the frequencies A_i of words of weight i in $M(q)$. To obtain these formulas, we used the symbolic manipulation language MACSYMA.

We conclude by giving a table of weight formulas. In this table, Ramanujan's τ-function is denoted by τ and the numbers t_k denote $\mathrm{Trace}(\alpha_k^m)$ as in Proposition 5.1.

TABLE 6.1

Frequencies A_i of small weights i in the Melas codes $M(q)$

$$
\begin{aligned}
A_1 &= A_3 = 0\,,\\
A_2 &= q-1\,,\\
A_4 &= (q-1)(q-3)/2\,,\\
A_5 &= 4(q-1)(q^2+((-1)^m-14)q+36)/15\,,\\
A_6 &= (q-1)(8q^3-165q^2+(1240-68(-1)^m)q-2655)/90\,,\\
A_7 &= 2(q-1)(4q^4-108q^3+(4t_9-18(-1)^m+1215)q^2\\
&\qquad +(399(-1)^m-6744)q+12884)/315\,,\\
A_8 &= (q-1)(16q^5-560q^4+8225q^3-(224t_9-880(-1)^m+66255)q^2\\
&\qquad -(16296(-1)^m-298263)q-517825)/2520\,,\\
A_9 &= (q-1)(16q^6-704q^5+13216q^4\\
&\quad -(160t_9-16t_{11}-216(-1)^m+138656)q^3\\
&\quad +(3816t_9-13776(-1)^m+895209)q^2\\
&\qquad -(3470238-187593(-1)^m)q+5597820)/11340\,,\\
A_{10} &= (q-1)(32q^7-1728q^6+40512q^5-540519q^4\\
&\quad +(6240t_9-720t_{11}-6120(-1)^m+4529826)q^3\\
&\quad +(-110280t_9+360000(-1)^m-24851277)q^2\\
&\quad +(85643448-4448871(-1)^m)q-129806479\\
&\qquad -32(\tau(q)-177147\tau(q/9))/q^2)/113400\,,\\
A_{11} &= (q-1)(32q^8-2080q^7+59520q^6-985920q^5\\
&\quad +(2288t_9+32t_{13}-560t_{11}-440(-1)^m+10453958)q^4\\
&\quad +(-136840t_9+16720t_{11}+110220(-1)^m-74203966)q^3\\
&\quad +(1705506t_9-5122359(-1)^m+358627785)q^2\\
&\quad +(57077625(-1)^m-112429735)q+1617492524\\
&\qquad +880(\tau(q)-177147\tau(q/9))/q^2)/623700\,,\\
A_{12} &= (q-1)(64q^9-4928q^8+168960q^7-3400320q^6\\
&\quad +44564751q^5-(2112t_{13}-33440t_{11}+115808t_9\\
&\quad +398775397+16720(-1)^m)q^4-(664400t_{11}\\
&\quad -5020400t_9+2939640(-1)^m-2486674179)q^3\\
&\quad -(52961436t_9-145879734(-1)^m+10845159710)q^2\\
&\quad +(31412188148-1550485266(-1)^m)q-43190708055\\
&\quad +(\tau(q)-177147\tau(q/9))(1408q^4-46992q^3)/q^5)/7484400.
\end{aligned}
$$

BIBLIOGRAPHY

1. G. van der Geer and M. van der Vlugt, *Artin-Schreier curves and codes*, J. Algebra **139** (1991), 256–272.
2. S. Lang, *Introduction to modular forms*, Grundlehren der Math. Wiss., no. 222, Springer-Verlag, New York, 1976.
3. W. W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315.
4. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland, Amsterdam, 1983.
5. C. M. Melas, *A cyclic code for double error correction*, IBM J. Res. Develop. **4** (1960), 364–366.
6. R. Schoof and M. van der Vlugt, *Hecke operators and the weight distributions of certain codes*, J. Combin. Theory Ser. A **57** (1991), 163–186.
7. R. Schoof, *Non-singular plane cubic curves over finite fields*, J. Combin. Theory Ser. A **46** (1987), 183–211.
8. J.-P. Serre, *Cours d'arithmétique*, Presses Univ. de France, Paris, 1970.

FACULTEIT WISKUNDE EN INFORMATICA, UNIVERSITEIT VAN AMSTERDAM, 1018 TV AMSTERDAM, THE NETHERLANDS
E-mail address: geer@fwi.uva.nl

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, 38050 POVO (TRENTO), ITALY
E-mail address: schoof@itnvax.cineca.it

AFDELING WISKUNDE EN INFORMATICA, UNIVERSITEIT VAN LEIDEN, 2300 RA LEIDEN, THE NETHERLANDS